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CLONE PROPERTIES OF TOPOLOGICAL SPACES

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Dedicated to Professor Jiří Rosický on the occasion of his 60th birthday

Abstract. Clone properties are the properties expressible by the first order sentence of the clone language. The present paper is a contribution to the field of problems asking when distinct sentences of the language determine distinct topological properties. We fully clarify the relations among the rigidity, the fix-point property, the image-determining property and the coconnectedness.

1. Introduction

The monograph [16] investigates clones of topological spaces. Let us recall that the clone clo$X$ of a topological space X is the category, the objects of which are all finite powers X^n of X, $n \in \omega$ $(= \{0,1,2,\dots\})$, and whose morphisms are all continuous mappings between them. Viewed as an abstract category (with the product projections specified and enumerated), it forms an abstract clone in the sense of [12, 15, 19]. Any abstract clone determines the first order language, *the clone language*. Such a language $\mathcal{L}$ has ω sorts of variables (for clo$X$, the variables of the n-th sort range through all continuous maps $X^n \to X$); for every n, $\mathcal{L}$ has n constants of the sort n (for clo$X$, the constants of the sort n are the product projections $\pi_i^{(n)}\colon X^n \to X$, $i \in n = \{0,1,\dots,n-1\}$ sending any $(x_0,\dots,x_{n-1})$ to x_i); the equality $=$ is the unique predicate of $\mathcal{L}$ but $\mathcal{L}$ has infinitely many operations S_m^n, $m,n \in \omega$ (for clo$X$, S_m^n is a substitution acting on m-tuples $g_0,\dots,g_{m-1}$ of continuous maps $X^n \to X$ and a continuous map $f\colon X^m \to X$, i.e. $S_m^n(f;g_0,\dots,g_{m-1})$ is the map $h\colon X^n \to X$, given by

$$h(x_0,\dots,x_{n-1}) = f\big(g_0(x_0,\dots,x_{n-1}),\dots,g_{m-1}(x_0,\dots,x_{n-1})\big)\,.$$

Sentences of the first order language $\mathcal{L}$ applied to clones of topological spaces are topological properties: if clo$X$ satisfies a sentence of $\mathcal{L}$ then cloX' also satisfies it for every space X' homeomorphic to X. Properties of X expressible by sentences of

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$\mathcal{L}$ are called *clone properties of* X. The variables of the zero-th sort range through maps $X^0 \to X$, i.e. through points of X. Clearly, the following properties of X are clone properties:

(R) *rigidity:* every continuous map $f\colon X \to X$ is either the identity or a constant; expressing it as a sentence of $\mathcal{L}$, one gets

$$(\forall x^{(1)})\big((x^{(1)} = \pi_0^{(1)}) \vee (\exists x^{(0)})(\forall y^0)(S_1^0(x^{(1)}; y^{(0)}) = x^{(0)})\big)\,,$$

where $x^{(1)}$ is a variable of the first sort, $x^{(0)}, y^{(0)}$ variables of the zero-th sort and the identity $\pi_0^{(1)}$ is the constant of $\mathcal{L}$ of the first sort;

(FP) *fix-point property:* every continuous map $f\colon X \to X$ has a fix-point, i.e. $f(x) = x$ for some $x \in X$; expressed as a sentence of $\mathcal{L}$,

$$(\forall x^{(1)})(\exists x^{(0)})(S_1^0(x^{(1)}; x^{(0)}) = x^{(0)})\,;$$

(ID) *image-determining property:* for every continuous maps $f, g\colon X \to X$, $\mathrm{Im}\ f = \mathrm{Im}\ g$ implies $f = g$; expressed as a sentence of $\mathcal{L}$,

$$\begin{aligned}(\forall x^{(1)})(\forall y^{(1)})\big([(\forall x^{(0)})(\exists y^{(0)})(S_1^0(x^{(1)}; x^{(0)}) = S_1^0(y^{(1)}; y^{(0)}))\wedge \\ (\forall u^{(0)})(\exists v^{(0)})(S_1^0(y^{(1)}; u^{(0)}) = S_1^0(x^{(1)}; v^{(0)})] \\ \Rightarrow x^{(1)} = y^{(1)}\big)\,;\end{aligned}$$

(CC) *coconnectedness:* every continuous map $f\colon X \times X \to X$ depends on at most one coordinate; expressed as a sentence of $\mathcal{L}$,

$$(\forall x^{(2)})(\exists x^{(1)})\big[\big(x^{(2)} = S_1^2(x^{(1)}; \pi_0^{(2)})\big) \vee \big(x^{(2)} = S_1^2(x^{(1)}; \pi_1^{(2)})\big)\big]\,.$$

The present paper is focused on these four properties and fully clarifies their mutual relationship. However, the question of when distinct sentences of $\mathcal{L}$ determine distinct topological properties or, more precisely, within which classes of topological spaces this is so, leads to a rich field of problems with only a few results. Moreover, such relationship certainly depends on the class in question. For example, by [7, 8], a Hausdorff space X is rigid if and only if every continuous map $f\colon X \times X \to X$ is either a projection or constant (expressed as a sentence of $\mathcal{L}$,

$$\begin{aligned}(\forall x^{(2)})\big[(\pi_0^{(2)} = x^{(2)}) \vee (\pi_1^{(2)} = x^{(2)})\vee\big((\exists x^{(0)})(\forall y^{(0)})(\forall z^{(0)}) \\ (S_2^0(x^{(2)}; y^{(0)}, z^{(0)}) = x^{(0)})\big]\big)\,,\end{aligned}$$

however this is not so within the class of all topological spaces because of the Sierpiński space (i.e. the space $\{a, b\}$ where $\emptyset$, $\{a\}$, $\{a, b\}$ are just all open subsets) which is rigid but it does not satisfy this sentence.

Let us mention that the term "clone" was used by P. Hall in [6] first time and that clones form an important notion of universal algebra, see [12, 15, 17]. In category theory, abstract clones live under the name (finitary) algebraic theories used by W. F. Lawvere (see [10, 11]) for the categorical approach to universal algebra.

Clearly, clone properties can be examined for any kind of structures forming a category with finite products. Hence the investigation of clone properties is in fact a categorical field of problems. It admits in a way to compare the "semantics" of the language $\mathcal{L}$ determined by various categories with finite products. Thus, the investigation of the clone properties for the category Top of the topological spaces and for its full subcategories can be regarded only as a first step in a broader categorical program.

We use category theory and general topology in a standard way, see e.g. [1] and [4].

Looking at the clone properties (R), (FP), (ID), one can see immediately that in Top (and in any concrete [over Set] category with concrete finite products and all constants) the property (R) implies the properties (FP) and (ID). Also, every rigid space X with card $X \geq 3$ is coconnected (see [7, 8, 16]). Are there any other relations among these properties? The aim of the present paper is just to answer this question. In Section 2, we show that none of the properties (FP), (ID) and (CC) implies the disjunction of the two others; in Section 3, we show that the conjunction of any pair of the properties does not imply the remaining one and that the conjunction of all the three properties (FP), (ID) and (CC) still does not imply the rigidity. These results are mostly proved for the category of all metrizable spaces (with the exception in 3.1), in several cases even for the category of all compact metrizable spaces. In Section 4, we examine very natural "n-variants" of the properties (R), (ID), (CC) and their dependence on n. While this is known for (R) and (CC), only partial results were obtained for (ID), see the *Problem* at the end of the paper.

2. The disjunctions

In this section we show that none of the properties (FP), (ID), (CC) implies the disjunction of the remaining two.

2.1. **In the category of all compact metrizable spaces, the implication (FP) $\Longrightarrow$ (ID) $\vee$ (CC) fails.** The space X_1 which is the witness against the implication is e.g. the interval $\langle 0,1\rangle$ of real numbers which is well-known to have the fix-point property (see [2]), but it has neither the image determining property nor it si coconnected, evidently.

2.2. **In the category of all metrizable spaces, the implication (ID) $\Longrightarrow$ (FP) $\vee$ (CC) fails.** A space X_2 being the witness against the implication is e.g. the metric space constructed in [18]. Since none of the properties (FP), (ID), (CC) is mentioned in [18], we recall briefly the properties of X_2 and we prove that it has all the properties required here. Some auxiliary notions and statements have to be presented first.

2.2.1.

Definition ([18])**.** Let X be a topological space and $B \subseteq X$ its closed subset. We say that X is *B-semirigid* if, for every continuous map $f\colon X \to X$, f is either the identity or constant or $f(X) \subseteq B$.

Remark. Clearly, if $B = \emptyset$, then X is rigid. If $X\backslash B \neq \emptyset$, then X is connected.

Proposition (Proposition II.5 in [18]). *Let X be a B-semirigid space such that* $\operatorname{card}(X \setminus B) \geq 3$. *Then for every natural number n, every continuous map $f\colon X^n \to X$ is either a projection or constant or $f(X^n) \subseteq B$.*

2.2.2. *The construction of the metric space $X_2 = (P, \rho^{(2)})$* proceeds as follows: we start from a free groupoid (P, b) on a set G_0 of generators with card $G_0 = 2^{\aleph_0}$, where $b\colon P \times P \to P$ is the groupoid operation. Describing it in more detail,

(1) $P = \bigcup\limits_{k=0}^{\infty} G_k$, where G_0 is the set of generator and, for $k \geq 1$, $G_k = G_{k-1} \cup b(G_{k-1} \times G_{k-1})$;

(2) $b\colon P \times P \to P$ is a bijection of $P \times P$ onto the set $P \setminus G_0$ with
$b^{-1}(G_1 \setminus G_0) = G_0 \times G_0$ and
$b^{-1}(G_k \setminus G_{k-1}) = (G_{k-1} \times G_{k-1}) \setminus (G_{k-2} \times G_{k-2})$ for $k > 1$.

Let us denote $P \setminus G_0$ by B. In [18], a metric $\rho^{(2)}$ is constructed (named only ρ in [18]) such that
(a) $X_2 = (P, \rho^{(2)})$ is B-semirigid and
(b) b is an isometry of $X_2 \times X_2$ onto $(B, \rho^{(2)} \mid B)$.

2.2.3. *X_2 satisfies our requirements.* In fact,
(α) X_2 is not coconnected because b is a homeomorphism of $X_2 \times X_2$ into X_2, and hence it depends on both coordinates;
(β) X_2 does not have the fix-point property: choose $a \in P$ and denote by $v_a\colon X_2 \to X_2 \times X_2$ the map $v_a(x) = (x, a)$ for all $x \in P$. Then $b \circ v_a\colon X_2 \to X_2$ is a continuous map which has no fix-point, as it follows from (1) and (2) in 2.2.2.

2.2.4. Before the proof that X_2 has also the (ID)-property, let us present the following

Remark. Given $n \in \omega, n \geq 1$ and a continuous map $f\colon X_2^n \to X_2$, denote by $h(f)$ the smallest k such that $G_k \cap \operatorname{Im} f \neq \emptyset$. Since X_2 is B-semirigid, f is a constant map with the value in G_0 or a projection $\pi_i^{(n)}$ whenever $h(f) = 0$. If $h(f) > 0$, necessarily $f(X_2) \subseteq B$ and we can form

$$f_0 = \pi_0^{(2)} \circ b^{-1} \circ f \quad \text{and} \quad f_1 = \pi_1^{(2)} \circ b^{-1} \circ f\,.$$

Hence $f = b \circ (f_0 \dot{\times} f_1)$ where $\circ$ denotes the composition of maps and $\dot{\times}$ denotes the fibered product, i.e. $(f_0 \dot{\times} f_1)(x) = (f_0(x), f_1(x))$. Moreover, by (1) and (2) in 2.2.2, $h(f_0) < h(f)$ and $h(f_1) < h(f)$.

2.2.5. *Now, we prove that X_2 has the* (ID)*-property.* Let continuous maps $f, g\colon X_2 \to X_2$ be given. Then $\operatorname{Im} f = \operatorname{Im} g$ implies $h(f) = h(g)$. We proceed by induction in $h(f)$.

$h(f) = h(g) = 0:$ Then f is either the identity or a constant and analogously g. Hence $\operatorname{Im} f = \operatorname{Im} g$ implies immediately $f = g$.

The induction step: If $h(f) = h(g) > 0$, then $f = b \circ (f_0 \dot{\times} f_1)$ and $g = b \circ (g_0 \dot{\times} g_1)$. Since b is one-to-one, $\operatorname{Im} f = \operatorname{Im} g$ implies $\operatorname{Im}(f_0 \dot{\times} f_1) = \operatorname{Im}(g_0 \dot{\times} g_1)$ which in turn implies Im $f_0 =$ Im g_0 and Im $f_1 =$ Im g_1. By the induction hypothesis, $f_0 = g_0$ and $f_1 = g_1$. Hence $f = g$.

2.3. **In the category of compact metrizable spaces, the implication** (CC) $\Rightarrow$ (FP) $\vee$ (ID) **fails.** A compact metrizable space X_3 which witnesses against the implication is constructed just below.

2.3.1. Let $\mathcal{C}$ be a Cook continuum, i.e. a compact metric non-degenerate continuum such that for every subcontinuum K of $\mathcal{C}$ and every continuous map $f\colon K \to \mathcal{C}$, either f is constant or $f(x) = x$ for all $x \in K$. (Such a continuum was constructed in [3], for a more detailed description of the construction, see also [13].) Let $\mathcal{C}_0 = \mathcal{C} \times \{0\}$, $\mathcal{C}_1 = \mathcal{C} \times \{1\}$, $\mathcal{C}_2 = \mathcal{C} \times \{2\}$ be three copies of $\mathcal{C}$ and let

$$v_i\colon \mathcal{C} \to \mathcal{C}_i$$

be the map sending any $x \in \mathcal{C}$ to $x^{(i)} = (x, i)$. Let us choose two distinct points a, b in $\mathcal{C}$. Our space X_3 is obtained from the disjoint union of $\mathcal{C}_0$, $\mathcal{C}_1$ and $\mathcal{C}_2$ by the identification of $b^{(0)}$ with $a^{(1)}$ (let us denote the obtained point by t_1), $b^{(1)}$ with $a^{(2)}$ (let us denote the obtained point by t_2) and $b^{(2)}$ with $a^{(0)}$ (let us denote the obtained point by t_0). Clearly, we get a compact metrizable space (our X_3) which does not have the fix-point property because the map

$$g\colon X_3 \to X_3$$

sending any $x^{(i)}$ to $x^{(j)}$ with $j = (i+1)\text{mod}3$ moves all points. The space X_3 also does not have the image-determining property because $X_3 = \text{Im } g = \text{Im } \mathbf{1}$ for the identity map $\mathbf{1}$.

It remains to show that X_3 is coconnected. The proof is presented in 2.3.2–2.3.4 below.

2.3.2. To prove that X_3 is coconnected, we describe the monoid of all continuous selfmaps $X_3 \to X_3$ first. It will be proved in several lemmas below. The following classical statement will be used (see [9]):

$$(*)\qquad \begin{cases} \text{If } O \text{ is an open subset of a continuum } H \text{ such that } O \neq \emptyset, \text{ and} \\ H \setminus O \neq \emptyset, \text{ then the closure of every component of } O \text{ intersects} \\ \text{the boundary of } O. \end{cases}$$

Lemma. *Let $f\colon \mathcal{C} \to X_3$ be a continuous map. Then either f is constant or $f = v_i$ for some $i \in \{0, 1, 2\}$.*

Proof. Let us denote $T = \{t_0, t_1, t_2\}$ the distinguished points of X_3 (i.e. $b^{(i)} = a^{(j)} = t_j$ with $j = (i+1)\text{mod}3$, $i \in \{0, 1, 2\}$), put $U_i = \mathcal{C}_i \setminus T$, $O_i = f^{-1}(U_i)$, $i \in \{0, 1, 2\}$.

a) Let us suppose that all the three sets O_0, O_1, O_2 are non-empty. We deduce a contradiction. Choose a component, say C_i, of O_i. Then, by $(*)$, the closure $\bar{C}_i$ of C_i intersects its boundary, choose $c_i \in \bar{C}_i \cap (\mathcal{C} \setminus O_i)$. Since $O_i = f^{-1}(U_i)$, the closure $\bar{O}_i$ of O_i in $\mathcal{C}$ is mapped by f into the closure of U_i in X_3 which is precisely $\mathcal{C}_i$. Hence f maps c_i into $\{a^{(i)}, b^{(i)}\}$, hence the subcontinuum $\bar{C}_i$of $\mathcal{C}$ into

$\mathcal{C}_i$. Since both $\mathcal{C}$ and $\mathcal{C}_i$ are copies of the Cook continuum, f restricted to $\bar{C}_i$ is either constant or $f(x) = x^{(i)}$ for all $x \in \bar{C}_i$. However the former case cannot happen because $f(\bar{C}_i)$ contains a point in U_i and also a point in $\{a^{(i)}, b^{(i)}\}$. Hence $f(x) = x^{(i)}$ for all $x \in \bar{C}_i$. Since $f(c_i) \in \{a^{(i)}, b^{(i)}\}$, necessarily $c_i = a$ or $c_i = b$ depending on whether $f(c_i) = a^{(i)}$ or $f(c_i) = b^{(i)}$. Hence the three elements c_0, c_1, c_2 are not all distinct because they are in the set $\{a, b\}$.

(i) Let us suppose that $c_i = a = c_j$ for $i \neq j$. This is a contradiction because $f(a) = a^{(i)}$ and $f(a) = a^{(j)}$ while $a^{(i)} \neq a^{(j)}$.
(ii) Analogously, $c_i = b = c_j$ with $i \neq j$ is impossible.
(iii) The equation $a^{(i)} = b^{(j)}$ with $j = (i-1)\mathrm{mod}3$ is valid in X_3. Hence $c_i = c_j$ is possible with $i \neq j$ but only in the case $j = (i-1)\mathrm{mod}3$ and $c_i = a,\ c_j = b$. Put $s = (i+1)\mathrm{mod}3$ so that $\{j, i, s\} = \{0, 1, 2\}$. Then neither $c_s = a$ nor $c_s = b$, by (i) and (ii).

We conclude that at least one of the sets O_0, O_1, O_2 is empty.

b) Let us suppose that O_0 is empty, the other cases are analogous. Hence f maps $\mathcal{C}$ into the two copies $\mathcal{C}_1$ and $\mathcal{C}_2$ glued together at $b^{(1)} = a^{(2)}$. Clearly, the map $h\colon \operatorname{Im} f \to \mathcal{C}_2$ which is identical on $\operatorname{Im} f \cap \mathcal{C}_2$ and collapses the points in $\operatorname{Im} f \cap \mathcal{C}_1$ onto $b^{(1)} = a^{(2)}$ is continuous. Hence $h \circ f$ is a continuous map of $\mathcal{C}$ into $\mathcal{C}_2$ so that either $(h \circ f)(x) = x^{(2)}$ for all $x \in \mathcal{C}$ or $h \circ f$ is constant. In the former case, $h \circ f = v_2$ hence $h \circ f = f$ so that $f = v_2$.

If $h \circ f$ is a constant with the value distinct from $b^{(1)} = a^{(2)}$, then also $h \circ f = f$ and f is constant. Let us suppose that $h \circ f$ is a constant with the value $a^{(2)} = b^{(1)}$. In this case f is in fact a map of $\mathcal{C}$ into $\mathcal{C}_1$ hence f is either constant or $f = v_1$. □

2.3.3.

Lemma. *Let $f\colon X_3 \to X_3$ be a continuous map. Then either f is constant or f is the identity* $\mathbf{1}$ *or $f = g$ or $f = g \circ g$ (where g is as* 2.3.1).

Proof. Given continuous map $f\colon X_3 \to X_3$, let us investigate $f_i\colon \mathcal{C} \to X_3$ defined by $f_i = f \circ v_i$, where v_i is as in 2.3.2. By Lemma of 2.3.2, every f_i is either constant or some v_j, $i, j \in \{0, 1, 2\}$. If some f_i is constant with the value in $X_3 \setminus \{t_0, t_1, t_2\}$, where t_i are as in 2.3.1-2, then also the remaining f_j must be constant with the same value so that f is constant. If f_i is a constant with the value in $\{t_0, t_1, t_2\}$, say $t_j = a^{(j)}$, then necessarily either f_{i+1} (where the addition is always mod3) is a constant with the same value (and then f is constant because f_{i+2} maps a and b on the same point in X_3 hence it must be constant) or $f_{i+1} = f_{j+1}$. Then necessarily $f_{i+2}(b) = a^{(j)}$ and $f_{i+2}(a) = b^{(j)}$ so that f_{i+2} cannot be some v_s, $s \in \{0, 1, 2\}$ which is a contradiction to 2.3.2.

We conclude that if some of the maps f_0, f_1, f_2 is constant, then f must be constant.

In the remaining cases, necessarily $f_0 = v_j, f_1 = v_{j+1}$ and $f_2 = v_{j+2}$. If $j = 0$, then $f = \mathbf{1}$; if $j = 1$, then $f = g$; and if $j = 2$, then $f = g \circ g$. □

2.3.4. Finally, we show that X_3 is coconnected. Let a continuous map $f\colon X_3 \times X_3 \to X_3$ be given. For every c, d denote by ${}_cf, f_d$ the functions $X_3 \to X_3$ defined by

$${}_cf(x) = f(c,x)\,, \quad f_d(x) = f(x,d) \quad \text{for all} \quad x \in X_3\,.$$

We discuss the following cases:

A) **There exists $c \in X_3$ such that ${}_cf$ is non constant.**
Then, by Lemma of 2.3.3, either ${}_cf = \mathbf{1}$ or ${}_cf = g$ or ${}_cf = g \circ g$. Hence $f(c,x)$ is either x or $g(x)$ or $(g \circ g)(x)$. Choose d arbitrarily in $Y = X_3 \setminus \{c, g(c), (g \circ g)(c), g^{-1}(c), (g \circ g)^{-1}(c)\}$ so that $Z = \emptyset$ whenever

$$Z = \{c, g(c), (g \circ g)(c)\} \cap \{d, g(d), (g \circ g)(d)\}\,.$$

Let us suppose that f_d is non-constant for some $d \in Y$. Then either $f_d = \mathbf{1}$ or $f_d = g$ or $f_d = g \circ g$. Since ${}_cf(d) = f(c,d) = f_d(c)$, we get $f(c,d) \in Z$, but Z is empty. Hence f_d is constant for every d from the dense subset Y with the value $f(c,d)$. Thus $f(x,d) = f(c,d)$ on a dense subset $X_3 \times Y$ hence on the whole $X_3 \times X_3$. Consequently f does not depend on the first coordinate.

B) **There exists $d \in X_3$ such that f_d is non-constant.**
Then, analogously, f does not depend on the second coordinate.

C) **The cases A) and B) do not hold**, i.e. for every $c, d \in X_3$, ${}_cf$ and f_d are constants. Then necessarily f is also constant hence it does not depend on any coordinate.

3. The conjunctions

In this section, we show that no pair of the properties (FP), (ID), (CC) implies the remaining one and that the conjunction of these three properties still does not imply rigidity.

3.1. **In the category of Tychonoff spaces, the implication** ((CC) $\wedge$ (ID)) $\Rightarrow$ (FP) **fails.** A space Y_1 which is a witness against the implication is the space Y constructed in [14]. In that paper, spaces X and Y with the same monoid of all continuous selfmaps are constructed such that X is not coconnected and Y *is* coconnected. The space X is the metrizable space used in 2.2 of this paper (and named X_2 there) to show a space which has the image determining property but neither (FP) nor it is coconnected. If $Y = Y_1$ has the same monoid of all continuous selfmaps as $X = X_2$, then Y_1 also has (ID) but not (FP). Since it is coconnected, we see that (ID) $\wedge$ (CC) $\Rightarrow$ (FP) fails.

Problem. The space Y is a Tychonoff space and we do not know whether the implication fails also in the category of all metrizable spaces.

3.2. **In the category of all metrizable spaces, the implication** ((FP) $\wedge$ (ID)) $\Rightarrow$ (CC) **fails.** A space Y_2 which is a witness against the implication is the metric space constructed (for other reasons) in Section 4 of [21] (and named

$X_3 = (P, \rho^{(3)})$ there). We recall briefly the construction and show that the space has all the properties required here.

3.2.1. Let P be a set and B its subset such that card $B = \operatorname{card}(P \setminus B) = 2^{\aleph_0}$. Denote $G = P \setminus B$ and choose an element $b_0 \in B$ fix. Let g be a map $P \times P \to P$ which maps $(P \times P) \setminus (G \times G)$ onto $\{b_0\}$ and $G \times G$ bijectively onto $B \setminus \{b_0\}$. In [21], a metric ρ on P is constructed such that

$$g \text{ is a quotient map of } (P, \rho) \times (P, \rho) \text{ onto } (B, \rho \mid B)\,,$$

where the quotient and the product are performed in the category Metr of all metric spaces of diameter at most one and all their distance-non-increasing maps such that the space $Y = (P, \rho)$ satisfies 3.2.2 below. [In fact, a more general construction is presented in [21]: the map g is a map $P^n \to P$ for a given natural number $n \geq 2$. We use the construction only for $n = 2$.]

3.2.2. In [21], for every $m \in \omega$, all the continuous maps $Y^m \to Y$ are proved to be precisely

$$F \cup \{g \circ (f_0 \dot{\times} f_1) \mid f_i \in F\} \cup \{c_{b_0}\}\,, \quad \text{with}$$
$$F = \{\pi_0^{(m)}, \dots, \pi_{m-1}^{(m)}\} \cup \{c_x \mid x \in G\}$$

where $\pi_i^{(m)}$ are the products projections $P^m \to P$ and c_y are the constant maps $P^m \to P$ with the value y.

This result summarizes rather long reasoning and it is presented in Remark in 4.9 of [21], p. 395. Unfortunately, there are two misprints in its formulation:

a) in F, $\{x \mid x \in G\}$ is printed instead of $\{c_x \mid x \in G\}$;

b) the expression $g \circ (f_0 \dot{\times} \dots \dot{\times} f_{m-1})$ is printed instead of $g \circ (f_0 \dot{\times} \dots \dot{\times} f_{n-1})$ where n is the natural number such that g is the map $P^n \to P$ [as already mentioned in 3.2.1, we use only the case $n = 2$, hence we have the expression $g \circ (f_0 \dot{\times} f_1)$ in the above description of the continuous maps $Y^m \to Y$].

3.2.3. Now, we choose $m = 1$ and describe the monoid M of all continuous selfmaps $Y \to Y$:

M consists of all constant maps, the identity, the map $\bar{g}$ sending any $y \in Y$ to $g(y, y)$ and all the maps $g(x, -)$ and $g(-, x)$ with $x \in G$.

This follows easily from 3.2.2.

3.2.4. We put $Y_2 = Y$ and show that Y_2 has all the required properties.

(ID): if $f, g \in M$ then $\operatorname{Im} f = \operatorname{Im} g$ implies $f = g$ evidently.

(FP): if $f \in M$ is non-constant, then $f(b_0) = b_0$. This follows from the fact that $g\colon P \times P \to P$ sends $(P \times P) \setminus (G \times G)$ to b_0.

Finally, Y_2 is not coconnected because g is one-to-one on the subset $G \times G$.

3.3. **In the category of all compact metrizable spaces, the implications** (FP) $\wedge$ (CC) $\Rightarrow$ (ID) **and** (FP) $\wedge$ (CC) $\wedge$ (ID) $\Rightarrow$ (R) **fail.**

3.3.1. The spaces which are witnesses against the implications follow immediately from [13]. In this monograph, an almost full embedding $F\colon \mathrm{Graph}_c \to M\,\mathrm{Top}$ is constructed. Here, Graph_c is the category of all connected directed graphs (i.e. all (X,R), where X is a set, $R \subseteq X \times X$ and, for every $x,y \in X$ there exist $x_0 = x, \ldots, x_n = y$ such that $(x_i, x_{i+1}) \in R \cup R^{-1}$ for $i = 0, \ldots, n-1$) and all compatible maps (i.e. a map $f\colon X \to X'$ is a morphism $(X,R) \to (X',R')$ iff $(x,y) \in R$ implies $(f(x), f(y)) \in R'$) and $M\,\mathrm{Top}$ is the category of all metrizable topological spaces and all continuous maps. The almost full embeddability of F means that F is a one-to-one functor such that, for every $(X,R), (X',R')$, a continuous map

$$g\colon F(X,R) \to F(X',R')$$

is non-constant iff $g = F(f)$ for some compatible map $f\colon (X,R) \to (X',R')$.

3.3.2. In the construction of F, a very specific compact metric space T with three distinguished points t_1, t_2, t_3 is used (for the construction of T and t_1, t_2, t_3, see [13], pp. 223-4). Given a connected graph (X,R), the space $F(X,R)$ is obtained by the "arrow construction", i.e. any arrow $r = (x,y) \in R$ is replaced by a copy of T, say $T^{(r)}$ (with the distinguished points $t_1^{(r)}, t_2^{(r)}, t_3^{(r)}$), i.e. in the disjoint union of all $T^{(r)}$, $r \in R$, we identify (these identifications are performed in the category Metr so that the resulting space $F(X,R)$ is metrizable):

$t_1^{(r)}$ with $t_1^{(s)}$ iff r and s start in the same vertex (i.e. $r = (x,y), s = (x,z)$);
$t_2^{(r)}$ with $t_2^{(s)}$ iff r and s terminate in the same vertex (i.e. $r = (x,y)$, $s = (z,y)$);
$t_1^{(r)}$ with $t_2^{(s)}$ iff r starts in the vertex in which s terminates;

and all the points $t_3^{(r)}$, $r \in R$, are identified together.

The functor F is defined on morphisms by a simple rule: for a compatible map $f\colon (X,R) \to (X',R')$, $g = F(f)$ sends any $u^{(r)} \in T^{(r)}$ to $u^{(s)} \in T^{(s)}$ whenever $r = (x,y)$, $s = \big(f(x), f(y)\big)$. For the proof that F is almost full, see [13], pp. 225–229 (Theorem 7.5).

3.3.3. Every space $F(X,R)$ has the fix-point property: if $g\colon F(X,R) \to F(X,R)$ is a nonconstant continuous map, then $g = F(f)$ for a compatible map $f\colon (X,R) \to (X,R)$, i.e. g is obtained by the construction described in 3.3.2. Hence $g(t) = t$ for the point t obtained by the identifications of all $t_3^{(r)}$, $r \in R$.

Every space $F(X,R)$ is coconnected, see [20].

3.3.4. Clearly, if a graph (X,R) is finite, then the space $F(X,R)$ is compact. To find counterexample Y_3 to the implication

$$(\mathrm{FP}) \wedge (\mathrm{CC}) \Longrightarrow (\mathrm{ID})$$

it suffices to put $Y_3 = F(X,R)$, where (X,R) is the graph with two vertices, say x,y, and two arrows, say $r = (x,y)$ and $s = (y,x)$. If $e\colon \big(\{x,y\}, \{r,s\}\big) \to \big(\{x,y\}, \{r,s\}\big)$ is a compatible map exchanging the vertices x and y, then $F(e)$ has the same image as the identity map but it is distinct.

3.3.5. To find a counterexample Y_4 to the implication

$$(\mathrm{FP}) \wedge (\mathrm{CC}) \wedge (\mathrm{ID}) \Longrightarrow (\mathrm{R}),$$

it suffices to put $Y_4 = F(Y, S)$, where (Y, S) is a graph with two vertices, say x and y, and two arrows, say $u = (x, y)$ and $v = (y, y)$. The compatible maps of (Y, S) into itself are just the identity and the constant map c to y. Hence the identity and the map $F(c)$ are just the non-constant continuous selfmaps of Y_4 hence Y_4 has the (ID) property but it is not rigid. It has (FP) and it is (CC), see 3.3.3.

4. The n-variants of the properties

4.1. While the investigation of (FP) of an n-th power X^n of a space X, $n \in \omega$, offers an interesting field of problems, the properties (R), (ID), (CC) are satisfied on X^n with $n \geq 2$ only trivially, as shown just below.

Proposition. *Let $n \geq 2$ and let X be a space such that X^n is rigid* (*or it has* (ID) *or it is* (CC)). *Then* card $X \leq 1$.

Proof. a) Let X^n be rigid, let $p\colon n \to n$ be a non-identical permutation, $\tilde{p}\colon X^n \to X^n$ the map which permutes the coordinates $\tilde{p}(x_0, \dots, x_{n-1}) = (x_{p(0)}, \dots, x_{p(n-1)})$. If card $X \geq 2$, the $\tilde{p}$ is neither the identity nor constant.

b) Let X^n have (ID), let p and $\tilde{p}$ be as in a). Then the identity $\mathbf{1}\colon X^n \to X^n$ and $\tilde{p}$ satisfy Im $\mathbf{1} = X^n =$ Im $\tilde{p}$, but $\tilde{p} \neq \mathbf{1}$ whenever card $X \geq 2$.

c) Let X^n be (CC). Define $f\colon X^n \times X^n \to X^n$ by

$$f((x_0, \dots, x_{n-1}), (y_0, \dots, y_{n-1})) = (x_0, \dots, x_{n-2}, y_{n-1}).$$

If card $X \geq 2$, then f depends on both coordinates. In fact, choose $a, b \in X$, $a \neq b$. Then $c^{(1)} = \big((a, \dots, a), (a, \dots, a)\big)$ and $c^{(2)} = \big((a, \dots, a), (a, \dots, b)\big)$ differ only in the second coordinate and $f(c^{(1)}) = (a, \dots, a) \neq (a, \dots, a, b) = f(c^{(2)})$ so that f depends on the second coordinate. And $d^{(1)} = \big((b, a, \dots, a),\ (a, \dots, a)\big)$ and $d^{(2)} = c^{(1)}$ differ only in the first coordinate and $f(d^{(1)}) \neq f(d^{(2)})$ so that f depends also on the first coordinate; hence X^n is not coconnected. $\square$

4.2. However, there are clone properties which can be regarded as "reasonable" n-variants of the properties (R), (ID), (CC), $n \in \omega$, namely the following ones:

n-(R): A space X is n-rigid if every continuous map $f\colon X^n \to X$ is either a projection or constant.

n-(ID): A space X has the n-image-determining property if for any continuous maps $f, g\colon X^n \to X$ with Im $f =$ Im g there exists a permutation $p\colon n \to n$ such that $f = g \circ \tilde{p}$ (where $\tilde{p}$ is as in 4.1).

n-(CC): A space X is n-coconnected if every continuous map $X^n \to X$ depends on at most one coordinate.

One can verify that all the above properties are really clone properties though e.g. it is not possible to quantify permutations $n \to n$ in the clone language $\mathcal{L}$. In the corresponding sentence of $\mathcal{L}$, we have to go through all the possibilities; in 2-(ID), the phrase “there exists a permutation $p\colon 2 \to 2$ such that $f = g \circ \tilde{p}$” has to be described as

$$f = g \quad \text{or} \quad f(x, y) = g(y, x) \quad \text{for every} \quad x, y \in X\,.$$

Hence the sentence is rather long for large n. We omit the description of the sentences.

Clearly, 1-(R) is just rigidity, 1-(ID) is (ID), while 1-(CC) is the empty condition and (CC) is just 2-(CC).

4.3. Do the properties really depend on n? For (CC) and (R), the answer is negative with the exception $n = 1$.

Proposition. *Let $m, n \in \omega$ and let $2 \le m, n$. Then a space is m-*(R) *if and only if it is n-*(R)*.*

Proof. If a space X is not the Sierpiński space (see Introduction), then n-(R) is just the rigidity, by [7, 8, 16]. And the Sierpiński space is neither n-(R) nor m-(R) for any $m, n \ge 2$. $\square$

Proposition. *Let $m, n \in \omega$ and let $2 \le m, n$. Then a space is m-*(CC) *if and only if it is n-*(CC)*.*

Proof. ([19]). $\square$

4.4. Concerning n-(ID), we are able to prove only a weaker statement.

Proposition. *Let $n, m \in \omega$ and let $n \le m$. If a space has m-*(ID)*, then it has n-*(ID)*.*

Proof. Let a space X have $(n+1)$-(ID). We prove that it has n-(ID).

Let $f, g\colon X^n \to X$ be given with $\operatorname{Im} f = \operatorname{Im} g$. Let $v\colon X^{n+1} \to X^n$ be the map omitting the last coordinate, i.e. $v(x_0, \dots, x_n) = (x_0, \dots, x_{n-1})$. Denote $\bar{f} = f \circ v$, $\bar{g} = g \circ v$. Then $\operatorname{Im} \bar{f} = \operatorname{Im} \bar{g}$, hence there exists a permutation $p\colon (n+1) \to (n+1)$ such that $\bar{f} = \bar{g} \circ \tilde{p}$ where, for $x \in X^{n+1}$, $\tilde{p}((x_0, \dots, x_n)) = (x_{p(0)}, \dots, x_{p(n)})$. If $p(n) = n$, we are ready because p can be restricted to $q\colon n \to n$ and, clearly, $f = g \circ \tilde{q}$.

Thus, let us suppose that $p(n) \ne n$. Since $\bar{f}$ does not depend on the n-th coordinate, $\bar{g}$ does not depend on the $(p^{-1}(n))$-th coordinate. But $\bar{g}$ also does not depend on the n-th coordinate. Hence $\bar{g} = \bar{g} \circ \tilde{t}$ where $t\colon (n+1) \to (n+1)$ is the transposition exchanging n and $p^{-1}(n)$ and $\tilde{t}(x) = x \circ t$ for every $x \in X^{n+1}$. Hence $\bar{f} = \bar{g} \circ \tilde{p} = \bar{g} \circ \tilde{t} \circ \tilde{p}$. The permutation $t \circ p$ leaves n fixed, hence we can form its restriction $q\colon n \to n$, so that $f = g \circ \tilde{q}$ again. $\square$

4.5. We present a sufficient condition for a space to have n-(ID).

Proposition. *Let X be a coconnected space which has* (ID)*. Then it has n-*(ID) *for all $n \in \omega$.*

Proof. Let continuous maps $f, g\colon X^n \to X$ with $\operatorname{Im} f = \operatorname{Im} g$ be given. If $n = 1$, then $f = g$ because X has (ID). Hence let us suppose $n > 1$. Since X is coconnected, there exist $i, j \in n$ and $\bar{f}, \bar{g}\colon X \to X$ such that $f = \bar{f} \circ \pi_i^{(n)}$ and

$g = \bar{g} \circ \pi_j^{(n)}$, where $\pi_i^{(n)}, \pi_j^{(n)} \colon X^n \to X$ are the product projections. Clearly, $\operatorname{Im} f = \operatorname{Im} \bar{f}$ and $\operatorname{Im} g = \operatorname{Im} \bar{g}$ so that $\operatorname{Im} \bar{f} = \operatorname{Im} \bar{g}$. Since X has (ID), necessarily $\bar{f} = \bar{g}$. Hence $f = g \circ \tilde{p}$ where $p \colon n \to n$ is a permutation sending j to i. □

Remark. The sufficient condition in the Proposition is not necessary. Our spaces X_2 in 2.2, Y_1 in 3.1, Y_2 in 3.2 and Y_4 in 3.3 all have the (ID)-property. The spaces Y_1 and Y_4 are coconnected hence they have the n-(ID) property for all $n \in \omega$, by the Proposition. The spaces X_2 and Y_2 are not coconnected, but they still have the n-(ID) property for all $n \in \omega$. This will be proved just below.

4.6. We are going to show that the space Y_2 in 3.2 has m-(ID) for all $m \in \omega$. As mentioned in 3.2.2, every continuous map $X^m \to X$ is either constant or a projection $\pi_i^{(m)}$ or $g \circ (\pi_i^{(m)} \dot{\times} \pi_j^{(m)})$ or $g \circ (\operatorname{const}_a \dot{\times} \pi_i^{(m)})$ or $g \circ (\pi_i^{(m)} \dot{\times} \operatorname{const}_a)$ for some $i, j \in m$ and $a \in G$. If $f, h \colon X^m \to X$ are continuous maps then $\operatorname{Im} f = \operatorname{Im} h$ occurs only in the following cases:

- α) both f, h are constant - then $f = h$;
- β) $f = \pi_i^{(m)}$, $h = \pi_j^{(m)}$;
- γ) $f = g \circ (\operatorname{const}_a \dot{\times} \pi_i^{(m)})$ and $h = g \circ (\operatorname{const}_a \dot{\times} \pi_j^{(m)})$;
- δ) $f = g \circ (\pi_i^{(m)} \dot{\times} \operatorname{const}_a)$ and $h = g \circ (\pi_j^{(m)} \dot{\times} \operatorname{const}_a)$;
- ε) $f = g \circ (\pi_i^{(m)} \dot{\times} \pi_\ell^{(m)})$ and $h = g \circ (\pi_j^{(m)} \dot{\times} \pi_k^{(m)})$ and $i \neq \ell$ iff $j \neq k$.

In the cases β), γ), δ), it suffices to choose a permutation $p \colon m \to n$ which sends j to i, in the case ε) p has to send j to i and k to ℓ. Then always $f = g \circ \tilde{p}$.

4.7. The space X_2 in 2.2 also has the m-(ID) property for all $m \in \omega$. We prove it just below. We keep the notation of 2.2 and use the notions and the claims presented there. We only omit the subscript 2 in X_2 and write X instead of X_2.

4.7.1. Let us recall that a map $h \colon A^m \to A$ depends on the i-th coordinate whenever there exist points $a = (a_0, \dots, a_{m-1})$ and $x = (x_0, \dots, x_{m-1})$ in A^m which differ precisely in the i-th coordinate (i.e. $a_i \neq x_i$ and $a_j = x_j$ for all $j \in m$, $j \neq i$) such that $h(a) \neq h(x)$.

Lemma. *Let $f \colon X^m \to X$ be a continuous map. If f depends on the i-th coordinate, then $f(a) \neq f(x)$ whenever $a, x \in X^m$ differ in the i-th coordinate* (*regardless of any of their other coordinates*).

Proof. Let $h(f)$ be as in 2.2.4. We proceed by induction in $h(f)$.

$h(f) = 0$**:** Since X is B-semirigid, f is either constant or a projection $\pi_j^{(m)}$. It depends on i precisely when $f = \pi_i^{(m)}$, hence the Lemma is satisfied.

The induction step: since $h(f) > 0$, f is of the form $f = b \circ (f_0 \dot{\times} f_1)$ with $h(f_j) < h(f)$ for $j = 0, 1$, see 2.2.4. Since f depends on the i-th coordinate, there exist $a, c \in X^m$ such that $a_i \neq c_i$ and $a_j = c_j$ for all $j \neq i$ such that $f(a) \neq f(c)$. Then either $f_0(a) \neq f_0(c)$ or $f_1(a) \neq f_1(c)$ hence either f_0 or f_1 depends on the i-th coordinate, say f_0. By the induction hypothesis, $f_0(a) \neq f_0(c)$ for arbitrary $a, c \in X^m$ with $a_i \neq c_i$. Since b is one-to-one, then necessarily $f(a) \neq f(c)$. □

4.7.2.

Proposition. *Let* $f, g\colon X^m \to X$ *be continuous maps with* $\operatorname{Im} f = \operatorname{Im} g$*. Then* $f = g \circ \tilde{p}$ *for a suitable permutation* $p\colon m \to m$*.*

Proof. $\operatorname{Im} f = \operatorname{Im} g$ implies $h(f) = h(g)$. We proceed by induction in $h(f)$.

$h(f) = 0$**:** Then either f and g are constant maps with the same value or $f = \pi_i^{(m)}$ and $g = \pi_j^{(m)}$. For any permutation $p\colon m \to m$ sending j to i, we get $f = g \circ \tilde{p}$.

The induction step: If $h(f) > 0$, we get

$$f = b \circ (f_0 \dot{\times} f_1) \quad \text{and} \quad g = b \circ (g_0 \dot{\times} g_1)$$

with $h(f_0), h(f_1), h(g_0), h(g_1)$ less than $h(f)$ by 2.2.4. By the induction hypothesis, there exist permutations $p_0, p_1\colon m \to m$ such that $f_0 = g_0 \circ \tilde{p}_0$ and $f_1 = g_1 \circ \tilde{p}_1$. Let I_0 (resp. I_1) be the set of all $i \in m$ for which g_0 (resp. g_1) depends on the i-th coordinate.

A) We show that $p_0(i) = p_1(i)$ for every $i \in I_0 \cap I_1$: Thus, let $i \in I_0 \cap I_1$ be given. Since $\operatorname{Im} f = \operatorname{Im} g$, for every $x \in X^m$ there exists $a \in X^m$ such that $f(x) = g(a)$. We choose $x = (x_0, \dots, x_{m-1})$ such that all $x_0, \dots, x_{m-1}$ are distinct. Since $f(x) = g(a)$, necessarily $f_0(x) = g_0(a)$ hence, by the induction hypothesis, $f_0(x_0, \dots, x_{m-1}) = g_0(x_{p_0(0)}, \dots, x_{p_0(m-1)})$ which is precisely $g_0(a_0, \dots, a_{m-1})$. Let us suppose that $x_{p_0}(i) \neq a_i$. Then, by 4.7.1, $(g_0 \circ \tilde{p}_0)(x) \neq g_0(a)$ which is not the case. We conclude that $x_{p_0(i)} = a_i$. Analogously $f_1(x) = g_1(a)$, hence, by the same reasoning, $x_{p_1(i)} = a_i$. Hence $x_{p_0(i)} = a_i = x_{p_1(i)}$. Our choice of x with all its coordinates distinct implies $p_0(i) = p_1(i)$.

B) Now, we show that $p_0(i) \neq p_1(j)$ whenever $i \in I_0 \setminus I_1$ and $j \in I_1$: Since $\operatorname{Im} f = \operatorname{Im} g$, for every $a \in X^m$ there exists $x \in X^m$ such that $f(x) = g(a)$. We choose $a = (a_0, \dots, a_{m-1})$ such that $a_0 \dots, a_{m-1}$ are all distinct. By the same reasoning as in A) we conclude that $x_{p_0(i)} = a_i$ and $x_{p_1(j)} = a_j$. By our choice of a, a_i is distinct from a_j hence $x_{p_0(i)} \neq x_{p_1(j)}$ so that necessarily $p_o(i) \neq p_1(j)$.

C) The case that $i \in I_0$ and $j \in I_1 \setminus I_0$ is quite analogous to B).

We define a permutation

$$p\colon n \to n$$

by

$$\begin{aligned} p(i) &= p_0(i) \quad \text{for all} \quad i \in I_0\,, \\ p(i) &= p_1(i) \quad \text{for all} \quad i \in I_1 \end{aligned}$$

and the elements of $m \setminus (I_0 \cup I_1)$ are sent onto $m \setminus (p_0(I_0) \cup p_1(I_1))$ to get a permutation $m \to m$. Then $f_0 = g_0 \circ \tilde{p}_0 = g_0 \circ \tilde{p}$ and $f_1 = g_1 \circ \tilde{p}_1 = g_1 \circ \tilde{p}$ hence $f = g \circ \tilde{p}$. □

4.8.

Problem. All the four spaces with the image-determining property which are presented in this paper have m-(ID) for all $m \in \omega$. Does there exist a space with (ID) which has not 2-(ID)?

More generally. In which categories of topological spaces is it true that any space satisfies n-(ID) if and only if it satisfies m-(ID) for $m > n$?

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