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# METRICALLY REGULAR SQUARE OF METRICALLY REGULAR BIPARTITE GRAPHS OF DIAMETER D = 6

Vladimír Vetchý

Abstract. The present paper deals with the spectra of powers of metrically regular graphs. We prove that there is only one table of the parameters of an association scheme so that the corresponding metrically regular bipartite graph of diameter $D = 6$ (7 distinct eigenvalues of the adjacency matrix) has the metrically regular square. The results deal with the graphs of the diameter $D < 6$ see [7] and [8].

## 1. Introduction and Notation

The theory of *metrically regular graphs* originates from the theory of *association schemes* first introduced by R.C. Bose and Shimamoto [2]. All graphs will be undirected, without loops and multiple edges.

**1.1. Definition [1].** Let $X$ be a finite set, $n := |X| \geq 2$. For an arbitrary natural number $D$ let $\mathbf{R}= \{R_0, R_1, \dots, R_D\}$ be a system of binary relations on $X$. A pair $(X,\mathbf{R})$ will called *an association scheme* with $n$ classes if and only if it satisfies the axioms $A1 - A4$:

A1. The system $\mathbf{R}$ forms a partition of the set $X^2$ and $R_0$ is the diagonal relation, i.e. $R_0 = \{(x, x); x \in X\}$.

A2. For each $i \in \{0, 1, \dots, D\}$ it holds $R_i^{-1} \in \mathbf{R}$.

A3. For each $i, j, k \in \{0, 1, \dots, D\}$ it holds
$(x, y) \in R_k \wedge (x_1, y_1) \in R_k \Rightarrow p_{ij}(x, y) = p_{ij}(x_1, y_1)$,
where $p_{ij}(x, y) = |\{z; (x, z) \in R_i \wedge (z, y) \in R_j\}|$.
Then define $p^k_{ij} := p_{ij}(x, y)$ where $(x, y) \in R_k$.

A4. For each $i, j, k \in \{0, 1, \dots, D\}$ it holds $p^k_{ij} = p^k_{ji}$.

The set $X$ will be called the *carrier* of the association scheme $(X,\mathbf{R})$. Especially, $p^k_{i0} = \delta_{ik}$, $p^0_{ij} = v_i\delta_{ij}$, where $\delta_{ij}$ is the Kronecker-Symbol and $v_i := p^0_{ii}$, and define $P_j := (p^k_{ij})$, $0 \leq i, j, k \leq D$.

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Given a graph $G = (X, E)$ of diameter $D$ we may define $R_k = \{(x, y); d(x, y) = k\}$, where $\mathrm{d}(x, y)$ is the distance from the vertex $x$ to the vertex $y$ in the standard graph metric. If $(X,\mathbf{R})$, $\mathbf{R} = \{R_0, R_1, \dots, R_D\}$, gives rise to an association scheme, the graph is called *metrically regular* and the $p_{ij}^k$ are said to be its *parameters* or its *structural constants*. Especially, metrically regular graphs with the diameter $D = 2$ are called *strongly regular.*

**1.2. Definition.** Let $G = (X, E)$ be an undirected graph without loops and multiple edges. *The second power* (or *the square)* of $G$ is the graph $G^2 = (X, E)$ with the same vertex set $X$ and in which different vertices are adjacent if and only if there is at least one path of the length 2 or 1 in $G$ between them.

**1.3. Definition.** Let $G$ be a graph with an adjacency matrix $A$. The characteristic polynomial $|\lambda I - A|$ of the adjacency matrix $A$ is called the *characteristic polynomial* of $G$ and denoted by $P_G(\lambda)$. The eigenvalues of $A$ and the spectrum of $A$ are called the *eigenvalues* and the *spectrum* of $G$, respectively. If $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ are the eigenvalues of $G$, the whole spectrum is denoted by $S_p(G)$ and $\lambda_1$ is called the *index* of $G$.

Define $(0, 1)$-matrices $A_0, \dots, A_D$ by $A_0 = I$ and $(A_i)_{jk} = 1$ if and only if the distance from the vertex $j$ to the vertex $k$ in $G$ is $\mathrm{d}(j, k) = i$. Using these notations it follows:

**1.4. Theorem [3].** *For a metrically regular graph $G$ with diameter $D$ and for any real numbers $r_1, \dots, r_D$ the distinct eigenvalues of $\sum_{i=1}^D r_i A_i$ and $\sum_{i=1}^D r_i P_i$ are the same. In particular the distinct eigenvalues of a metrically regular graph are the same as those of $P_1$.*

**1.5. Theorem [6].** *A metrically regular graph with diameter $D$ has $D+1$ distinct eigenvalues.*

**1.6. Theorem [5].** *The number of components of a regular graph $G$ is equal to the multiplicity of its index.*

**1.7. Theorem [4, p.87].** *A graph containing at least one edge is bipartite if and only if its spectrum, considered as a set of points on the real axis, is symmetric with respect to the zero point.*

**1.8. Theorem [4, p.82].** *A strongly connected digraph $G$ with the greates eigenvalue $r$ has no odd cycles if and only if $-r$ is also an eigenvalue of $G$.*

**1.9. Theorem [7].** *For every $k \in N$, $k \geq 2$ there is one and only one metrically regular bipartite graph $G = (X, E)$ with diameter $D = 3$, $n = |X| = 2k + 2$, so that $G^2$ is a strongly regular graph. Its nonzero structural constants are:*

| | | | | |
|---|---|---|---|---|
| $p_{01}^1 = 1$ | $p_{02}^2 = 1$ | $p_{03}^3 = 1$ | $v_0 = 1$ | $\lambda_1 = k = m_3$ |
| $p_{12}^1 = k - 1$ | $p_{11}^2 = k - 1$ | $p_{12}^3 = k$ | $v_1 = k$ | $\lambda_2 = 1$ |
| $p_{23}^1 = 1$ | $p_{13}^2 = 1$ | $m_1 = 1$ | $v_2 = k$ | $\lambda_3 = -1$ |
| $m_2 = k$ | $p_{22}^2 = k - 1$ | $m_4 = 1$ | $v_3 = 1$ | $\lambda_4 = -k$ |

**1.10. Theorem [7].** *There is only one table of the parameters of an association scheme so that the corresponding metrically regular bipartite graph with 5 distinct eigenvalues has the strongly regular square. The table of the nonzero parameters is following:*

| | | | | |
|---|---|---|---|---|
| $p^1_{01}=1$ | $p^2_{02}=1$ | $p^3_{03}=1$ | $p^4_{04}=1$ | $v_0=1$ |
| $p^1_{12}=3$ | $p^2_{11}=2$ | $p^3_{12}=3$ | $p^4_{13}=4$ | $v_1=4=v_3$ |
| $p^1_{23}=3$ | $p^2_{13}=2$ | $p^3_{14}=1$ | $p^4_{22}=6$ | $\lambda_3=0$ |
| $p^1_{34}=1$ | $p^2_{22}=4$ | $p^3_{23}=3$ | $\lambda_4=-2$ | $m_3=6=v_2$ |
| $\lambda_1=4$ | $p^2_{24}=1$ | $\lambda_2=2$ | $m_4=4$ | $\lambda_5=-4$ |
| $m_1=1$ | $p^2_{33}=2$ | $m_2=4$ | $v_4=1$ | $m_5=1$ |

*The realization of this table is the 4-dimensional unit cube.*

**1.11. Theorem [8].** *There are only four tables of the parameters of association schemes for $k \in \{1,2,4,10\}$ so that the corresponding metrically regular bipartite graphs with 6 distinct eigenvalues have the metrically regular square. The nonzero structural constants of the graphs are following:*

| | |
|---|---|
| $p^i_{i0}=p^1_{45}=p^2_{35}=p^3_{25}=p^4_{15}=1$ | $v_0=v_5=1$ |
| $p^2_{11}=p^2_{44}=p^3_{14}=k$ | $v_1=v_4=2k+1$ |
| $p^2_{13}=p^2_{24}=p^3_{12}=p^3_{34}=k+1$ | $v_2=v_3=2(2k+1)$ |
| $p^1_{12}=p^1_{34}=p^4_{13}=p^4_{24}=2k$ | $\lambda_1=2k+1=-\lambda_6$ |
| $p^5_{14}=2k+1$ | $\lambda_2=k+1=-\lambda_5$ |
| $p^1_{23}=p^4_{22}=p^4_{33}=2k+2$ | $\lambda_3=1=-\lambda_4$ |
| $p^2_{22}=p^2_{33}=p^3_{23}=3k$ | $p^5_{23}=2(2k+1)$ |

*The realization of the table for $k=2$ is the 5-dimensional unit cube.*

**1.12. Remark.** Theorems 1.9., 1.10. and 1.11. show that for $k=3,4,5$ the $k$-dimensional unit cubes have the metrically regular square.

Further, we use some of the known relations from the theory of associations schemes [1]

$$v_i = \sum_j p^k_{ij} \tag{1.1}$$

$$v_i p^i_{jk} = v_j p^j_{ik} \tag{1.2}$$

## 2. Main result

**2.1. Theorem.** *There is only one table of the parameters of an association scheme with 6 classes so that the corresponding metrically regular bipartite graph of diameter $D = 6$ (7 distinct eigenvalues of the adjacency matrix) has the metrically regular square.*

**Proof.** Let $\lambda_1 > \lambda_2 > \cdots > \lambda_7$ be the distinct eigenvalues of a metrically regular bigraph $G$ and $m_1, m_2, \dots, m_7$ are the corresponding multiplicities. As $G$ is a bipartite graph it holds according to Theorems 1.7. and 1.8.:

$$p_{ij}^k = 0, \quad i+j+k \equiv 1 \ (mod\ 2), \quad i,j,k \in \{1,2,\dots,6\} \tag{2.1}$$

$$S_p(G) = \left\{ \begin{matrix} \lambda_1 & \lambda_2 & \lambda_3 & 0 & -\lambda_3 & -\lambda_2 & -\lambda_1 \\ 1 & m_2 & m_3 & m_4 & m_3 & m_2 & 1 \end{matrix} \right\} \tag{2.2}$$

According to Theorem 1.4 it holds for these eigenvalues

$$|\lambda I - P_1| = 0 \tag{2.3}$$

So we obtain

$$\begin{gathered} \lambda^7 - \lambda^5(\lambda_1 + p_{12}^1 p_{11}^2 + p_{13}^2 p_{12}^3 + p_{14}^3 p_{13}^4 + p_{15}^4 p_{14}^5 + p_{16}^5 p_{15}^6) + \\ +\lambda^3[p_{12}^1 p_{11}^2 (p_{14}^3 p_{13}^4 + p_{15}^4 p_{14}^5 + p_{16}^5 p_{15}^6) + p_{13}^2 p_{12}^3 (p_{15}^4 p_{14}^5 + p_{16}^5 p_{15}^6) + \\ + p_{14}^3 p_{13}^4 p_{16}^5 p_{15}^6 + \lambda_1 (p_{13}^2 p_{12}^3 + p_{14}^3 p_{13}^4 + p_{15}^4 p_{14}^5 + p_{16}^5 p_{15}^6)] - \\ -\lambda[p_{12}^1 p_{11}^2 p_{14}^3 p_{13}^4 p_{16}^5 p_{15}^6 + \lambda_1 p_{13}^2 p_{12}^3 (p_{15}^4 p_{14}^5 + p_{16}^5 p_{15}^6) + \lambda_1 p_{14}^3 p_{13}^4 p_{16}^5 p_{15}^6]. \end{gathered}$$

Because of (2.2) we get

$$\begin{gathered} \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \\ = \lambda_1 + p_{12}^1 p_{11}^2 + p_{13}^2 p_{12}^3 + p_{14}^3 p_{13}^4 + p_{15}^4 p_{14}^5 + p_{16}^5 p_{15}^6. \end{gathered} \tag{2.4}$$

$$\begin{gathered} \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2 = \\ = p_{12}^1 p_{11}^2 (p_{14}^3 p_{13}^4 + p_{15}^4 p_{14}^5 + p_{16}^5 p_{15}^6) + p_{13}^2 p_{12}^3 (p_{15}^4 p_{14}^5 + p_{16}^5 p_{15}^6) + \\ + p_{14}^3 p_{13}^4 p_{16}^5 p_{15}^6 + \lambda_1 (p_{13}^2 p_{12}^3 + p_{14}^3 p_{13}^4 + p_{15}^4 p_{14}^5 + p_{16}^5 p_{15}^6). \end{gathered} \tag{2.5}$$

$$\begin{gathered} \lambda_1^2\lambda_2^2\lambda_3^2 = \\ = p_{12}^1 p_{11}^2 p_{14}^3 p_{13}^4 p_{16}^5 p_{15}^6 + \lambda_1 [p_{13}^2 p_{12}^3 (p_{15}^4 p_{14}^5 + p_{16}^5 p_{15}^6) + p_{14}^3 p_{13}^4 p_{16}^5 p_{15}^6]. \end{gathered}$$

If $A$ resp. $A_2$ denotes the adjacency matrix of a metrically regular bigraph $G$ resp. its square $G^2$ it is easy to see that

$$A_2 = \frac{1}{p_{11}^2} A^2 + A - \frac{\lambda_1}{p_{11}^2} I. \tag{2.6}$$

and according to (2.6) we get the eigenvalues of $G^2$ in the form

$$\mu_i = \frac{\lambda_i^2 + p_{11}^2 \lambda_i - \lambda_1}{p_{11}^2}, \quad i \in \{1, \ldots, 7\}. \tag{2.7}$$

Because of $p_{11}^2(\mu_1 - \mu_i) = p_{11}^2(\lambda_1 - \lambda_i)(\lambda_1 + \lambda_i + p_{11}^2) > 0$ it holds $\mu_1$ is the index of $G^2$.

As the diameter of $G^2$ is $D = 3$ we obtain according to *Theorem 1.5.* that the graph $G^2$ has 4 distinct eigenvalues. So it must hold one the following posibilities

**1.** $\mu_i = \mu_j = \mu_k = \mu_m; \quad i, j, k, m \in \{2, \ldots, 7\}$.
Because of (2.7) we get
$-p_{11}^2 = \lambda_i + \lambda_j = \lambda_i + \lambda_k = \lambda_i + \lambda_m = \lambda_j + \lambda_k = \lambda_j + \lambda_m = \lambda_k + \lambda_m$
and we obtain a contradiction with $\lambda_s \neq \lambda_t$ for $s \neq t$; $s, t \in \{2, \ldots, 7\}$.

**2.** $\mu_i = \mu_j = \mu_k, \mu_m = \mu_n; \quad i, j, k, m, n \in \{2, \ldots, 7\}$.
Because of (2.7) we get
$-p_{11}^2 = \lambda_i + \lambda_j = \lambda_i + \lambda_k = \lambda_j + \lambda_k, -p_{11}^2 = \lambda_m + \lambda_n$.
So, we again obtain a contradiction with $\lambda_s \neq \lambda_t$ for $s \neq t$; $s, t \in \{2, \ldots, 7\}$.

**3.** $\mu_i = \mu_j, \mu_k = \mu_m, \mu_s = \mu_t; \quad i, j, k, m, s, t \in \{2, \ldots, 7\}$

| | | | | |
|---|---|---|---|---|
| $\mu_2 = \mu_j$ | implies | $\lambda_2 + \lambda_j = -p_{11}^2$, | so | $j \in \{7\}$. |
| $\mu_3 = \mu_k$ | implies | $\lambda_3 + \lambda_k = -p_{11}^2$, | so | $k \in \{6, 7\}$. |
| $\mu_4 = \mu_m$ | implies | $\lambda_4 + \lambda_m = -p_{11}^2$, | so | $m \in \{5, 6, 7\}$. |
| $\mu_5 = \mu_n$ | implies | $\lambda_5 + \lambda_n = -p_{11}^2$, | so | $n \in \{4, 5, 6, 7\}$. |
| $\mu_6 = \mu_s$ | implies | $\lambda_6 + \lambda_s = -p_{11}^2$, | so | $s \in \{3, 4, 5, 6, 7\}$. |
| $\mu_7 = \mu_t$ | implies | $\lambda_7 + \lambda_t = -p_{11}^2$, | so | $t \in \{2, 3, 4, 5, 6, 7\}$. |

So, it must hold $\mu_2 = \mu_7, \mu_3 = \mu_6, \mu_4 = \mu_5$ and according to (2.2) we obtain

$$\lambda_2 = \lambda_1 - p_{11}^2,\ \lambda_3 = \lambda_2 - p_{11}^2,\ \lambda_4 = \lambda_3 - p_{11}^2.$$

So, we get the spectrum of G in the form

$$S_p(G) = \left\{ \begin{array}{ccccccc} 3p_{11}^2, & 2p_{11}^2, & p_{11}^2, & 0, & -p_{11}^2, & -2p_{11}^2, & -3p_{11}^2 \\ 1, & m_2, & m_3, & m_4, & m_3, & m_2, & 1 \end{array} \right\} \tag{2.8}$$

On the other hand if $G^2$ is metrically regular, the parameters of $G^2$ are

$$\begin{array}{lllllllll}
(2.9) & {}^2p^1_{11} & = & & 2p^1_{12} & & = & p^2_{11} & + & p^2_{22} \\
(2.10) & {}^2p^1_{12} & = & & p^1_{23} & & = & p^2_{13} & + & p^2_{24} \\
(2.11) & {}^2p^1_{22} & = & & 2p^1_{34} & & = & p^2_{33} & + & p^2_{44} \\
(2.12) & {}^2p^1_{23} & = & & p^1_{45} & & = & p^2_{35} & + & p^2_{46} \\
(2.13) & {}^2p^1_{33} & = & & 2p^1_{56} & & = & p^2_{55} & + & p^2_{66} \\
(2.14) & {}^2p^2_{11} & = & & 2p^3_{12} & & = & & p^4_{22} & \\
(2.15) & {}^2p^2_{12} & = & p^3_{14} & + & p^3_{23} & = & p^4_{13} & + & p^4_{24} \\
(2.16) & {}^2p^2_{13} & = & & p^3_{25} & & = & p^4_{15} & + & p^4_{26} \\
(2.17) & {}^2p^2_{22} & = & & 2p^3_{34} & & = & p^4_{33} & + & p^4_{44} \\
(2.18) & {}^2p^2_{23} & = & p^3_{36} & + & p^3_{45} & = & p^4_{35} & + & p^4_{46} \\
(2.19) & {}^2p^2_{33} & = & & 2p^3_{56} & & = & p^4_{55} & + & p^4_{66} \\
(2.20) & {}^2p^3_{12} & = & p^5_{14} & + & p^5_{23} & = & & p^6_{24} & \\
(2.21) & {}^2p^3_{13} & = & p^5_{16} & + & p^5_{25} & = & p^6_{15} & + & p^6_{26} \\
(2.22) & {}^2p^3_{22} & = & & 2p^5_{34} & & = & p^6_{33} & + & p^6_{44} \\
(2.23) & {}^2p^3_{23} & = & p^5_{36} & + & p^5_{45} & = & p^6_{35} & + & p^6_{46} \\
(2.24) & {}^2p^3_{33} & = & & 2p^5_{56} & & = & p^6_{55} & + & p^6_{66}
\end{array}$$

From (1.1) (i=1, k=1) and (2.8) we get $\lambda_1 = 1 + p^1_{12}$, so

$$p^1_{12} = 3p^2_{11} - 1. \tag{2.25}$$

(1.2) (i=1, j=2, k=1) implies $\lambda_1 p^1_{12} = v_2 p^2_{11}$ and

$$v_2 = 3(3p^2_{11} - 1). \tag{2.26}$$

From (1.1) (i=2, k=1) we obtain $v_2 = p^1_{12} + p^1_{23}$ and

$$p^1_{23} = 2(3p^2_{11} - 1), \tag{2.27}$$

so (1.2) (i=1, j=2, k=3) it implies $\lambda_1 p^1_{23} = v_2 p^2_{13}$ and

$$p^2_{13} = 2p^2_{11}. \tag{2.28}$$

The relation (1.1) (i=1, k=6) gives $\lambda_1 = p^6_{15}$, so

$$p^6_{15} = 3p^2_{11} \tag{2.29}$$

and from (1.1) (i=6, k=1) we get

$$v_6 = p^1_{56}. \tag{2.30}$$

The relations (2.9) and (2.25) give

$$p_{22}^2 = 5p_{11}^2 - 2 \tag{2.31}$$

and from (2.10), (2.27) and (2.28) we obtain

$$p_{24}^2 = 2(2p_{11}^2 - 1). \tag{2.32}$$

From (1.2) (i=2, j=4, k=2), (2.14), (2.26) and (2.32) we get

$$v_2 p_{24}^2 = v_4 p_{22}^4 = v_4 2p_{12}^3,$$

so

$$3(3p_{11}^2 - 1)(2p_{11}^2 - 1) = v_4 p_{12}^3. \tag{2.33}$$

(1.2) (i=2, j=3, k=1) implies $v_2 p_{13}^2 = v_3 p_{12}^3$, so from (2.26) and (2.28) it follows

$$3(3p_{11}^2 - 1)(2p_{11}^2) = v_3 p_{12}^3. \tag{2.34}$$

(2.33) and (2.34) give $\dfrac{v_3}{v_4} = \dfrac{2p_{11}^2}{2p_{11}^2 - 1}$, so

$$v_3 = 2p_{11}^2 t, \quad v_4 = (2p_{11}^2 - 1)t, \quad t \in N. \tag{2.35}$$

The relations (1.1) (i=1; k=3,4,5), (2.4), (2.8), (2.25), (2.28) and (2.29) imply

$$14(p_{11}^2)^2 = 3p_{11}^2 + (3p_{11}^2 - 1)p_{11}^2 + 2p_{11}^2 p_{12}^3 + (3p_{11}^2 - p_{12}^3)p_{13}^4 + \\ +(3p_{11}^2 - p_{13}^4)p_{14}^5 + 3p_{11}^2(3p_{11}^2 - p_{14}^5),$$

so

$$2p_{11}^2(p_{11}^2 - 1 - p_{12}^3) = p_{13}^4(3p_{11}^2 - p_{12}^3) - p_{13}^4 p_{14}^5. \tag{2.36}$$

From (1.1) (i=1; k=3,4,5), (2.5), (2.8), (2.25), (2.28) and (2.29) we get

$$49(p_{11}^2)^4 = \\ = (3p_{11}^2 - 1)p_{11}^2[(3p_{11}^2 - p_{12}^3)p_{13}^4 + (3p_{11}^2 - p_{13}^4)p_{14}^5 + (3p_{11}^2 - p_{14}^5)3p_{11}^2] + \\ +2p_{11}^2 p_{12}^3[(3p_{11}^2 - p_{13}^4)p_{14}^5 + (3p_{11}^2 - p_{14}^5)3p_{11}^2] + (3p_{11}^2 - p_{12}^3)p_{13}^4(3p_{11}^2 - p_{14}^5)3p_{11}^2 + \\ +3p_{11}^2[2p_{11}^2 p_{12}^3 + (3p_{11}^2 - p_{12}^3)p_{13}^4 + (3p_{11}^2 - p_{13}^4)p_{14}^5 + (3p_{11}^2 - p_{14}^5)3p_{11}^2],$$

and we obtain

$$2p_{11}^2[(3p_{11}^2 - 1)p_{12}^3 - (p_{11}^2 - 2)(p_{11}^2 + 1)] = p_{12}^3 p_{13}^4 p_{14}^5. \tag{2.37}$$

From (2.37) $+p_{12}^3$(2.36) we obtain

$$2p_{11}^2[-(p_{12}^3)^2+2(2p_{11}^2-1)p_{12}^3-(p_{11}^2-2)(p_{11}^2+1)]=p_{13}^4p_{12}^3(3p_{11}^2-p_{12}^3).$$

Because of $D=6$ it implies $p_{12}^3\neq 0$. (1.1) (i=1, k=3) gives $p_{12}^3<3p_{11}^2$ and we get

$$p_{13}^4=2p_{11}^2\frac{(p_{12}^3)^2-2(2p_{11}^2-1)p_{12}^3+(p_{11}^2-2)(p_{11}^2+1)}{(p_{12}^3)^2-3p_{11}^2p_{12}^3}. \tag{2.38}$$

From (1.2) (i=3, j=4, k=1) we get $v_3p_{14}^3=v_4p_{13}^4$ and from (1.1) (i=1, k=3), (2.35) we obtain

$$2p_{11}^2t(3p_{11}^2-p_{12}^3)=(2p_{11}^2-1)tp_{13}^4,$$

so

$$p_{13}^4=2p_{11}^2\frac{3p_{11}^2-p_{12}^3}{2p_{11}^2-1}. \tag{2.39}$$

The relations (2.38) and (2.39) give the equation

$$(p_{12}^3)^3-(4p_{11}^2+1)(p_{12}^3)^2+[(p_{11}^2)^2+8p_{11}^2-2]p_{12}^3+$$
$$+[2(p_{11}^2)^3-3(p_{11}^2)^2-3p_{11}^2+2]=0$$

and

$$[p_{12}^3-(p_{11}^2+1)][(p_{12}^3)^2-3p_{11}^2p_{12}^3-2(p_{11}^2)^2+5p_{11}^2-2]=0.$$

Because of $0<p_{12}^3<\lambda_1=3p_{11}^2$ it must hold

$$0<17(p_{11}^2)^2-20p_{11}^2+8<9(p_{11}^2)^2$$

and

$$(2p_{11}^2-1)(p_{11}^2-2)<0.$$

But there are no $p_{12}^3\in\mathbf{N}$ for $p_{11}^2=1$ and it must hold

$$p_{12}^3=p_{11}^2+1 \tag{2.40}$$

(1.1) (i=1, k=3) and (2.40) give

$$p_{14}^3=2p_{11}^2-1 \tag{2.41}$$

and from (2.39) and (2.40) it follows

$$p_{13}^4=2p_{11}^2. \tag{2.42}$$

The relations (1.1) (i=1, k=4) and (2.42) give

$$p^4_{15} = p^2_{11} \tag{2.43}$$

and from (2.33) and (2.34) we obtain

$$v_3 = 18p^2_{11} - 24 + \frac{24}{p^2_{11}+1}, \tag{2.44}$$

and

$$v_4 = 18p^2_{11} - 33 + \frac{36}{p^2_{11}+1}. \tag{2.45}$$

Substituting (2.40) and (2.42) in (2.37) we obtain

$$p^5_{14} = 2p^2_{11} + 1 \tag{2.46}$$

and from (1.1) (i=1, k=5) it follows

$$p^5_{16} = p^2_{11} - 1. \tag{2.47}$$

Because diameter of $G$ is $D = 6$ it holds $p^5_{16} > 0$, so

$$p^2_{11} > 1. \tag{2.48}$$

From (1.2) (i=4, j=5, k=1) we get $v_4 p^4_{15} = v_5 p^5_{14}$ so, from (2.43), (2.45) and (2.47) we obtain

$$v_5 = \frac{3(3p^2_{11}-1)(2p^2_{11}-1)p^2_{11}}{(p^2_{11}+1)(2p^2_{11}+1)} \tag{2.49}$$

and according to $v_3, v_4, v_5 \in \mathbf{N}$, (2.44), (2.45), (2.48) and (2.49) imply

$$p^2_{11} = 2. \tag{2.50}$$

The relations (1.1), (1.2), (2.1) - (2.50) give the following table of the nonzero parameters of a graph G:

| | | | | | |
|---|---|---|---|---|---|
| $p^1_{10}=1$ | $p^2_{20}=1$ | $p^3_{30}=1$ | $p^4_{40}=1$ | $p^5_{50}=1$ | $p^6_{60}=1$ |
| $p^1_{12}=5$ | $p^2_{11}=2$ | $p^3_{12}=3$ | $p^4_{13}=4$ | $p^5_{14}=5$ | $p^6_{15}=6$ |
| $p^1_{23}=10$ | $p^2_{13}=4$ | $p^3_{14}=3$ | $p^4_{15}=2$ | $p^5_{16}=1$ | $p^6_{24}=15$ |
| $p^1_{34}=10$ | $p^2_{22}=8$ | $p^3_{23}=9$ | $p^4_{22}=6$ | $p^5_{23}=10$ | $p^6_{33}=20=v_3$ |
| $p^1_{45}=5$ | $p^2_{24}=6$ | $p^3_{25}=3$ | $p^4_{24}=8$ | $p^5_{25}=5$ | $\lambda_6=-4$ |
| $p^1_{56}=1$ | $p^2_{33}=12$ | $p^3_{34}=9$ | $p^4_{26}=1$ | $p^5_{34}=10$ | $m_6=6$ |
| $\lambda_1=6$ | $p^2_{35}=4$ | $p^3_{36}=1$ | $p^4_{33}=12$ | $\lambda_4=0$ | $\lambda_7=-6$ |
| $m_1=1$ | $p^2_{44}=8$ | $p^3_{45}=3$ | $p^4_{35}=4$ | $m_4=20$ | $m_7=1=v_6$ |
| $\lambda_2=4$ | $p^2_{46}=1$ | $\lambda_3=2$ | $p^4_{44}=6$ | $\lambda_5=-2$ | $v_1=6=v_5$ |
| $m_2=6$ | $p^2_{55}=2$ | $m_3=15$ | $v_0=1$ | $m_5=15$ | $v_2=15=v_4$ |

The realization of this table is the 6-dimensional unit cube. □

With respect to Theorems 1.9.- 1.11. and 2.1. it would be reasonable to conjecture:

*There is only one table of parameters of an association scheme with $2k$ classes ($k \geq 2$) so that the corresponding metrically regular bipartite graph of diameter $D = 2k$ has a metrically regular square. The realization of this table is the $2k$-dimensional unit cube.*

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