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MORE ON THE GIRTH OF GRAPHS ON WEYL GROUPS

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Abstract. The girth of graphs on Weyl groups, with no restriction on the associated root system, is determined. It is shown that the girth, when it is defined, is 3 except for at most four graphs for which it does not exceed 4.

1. Introduction

We investigate here the girth of graphs on Weyl groups with arbitrary root system. We have studied elsewhere the planarity and connectivity of such graphs. The concept of these graphs come from the method employed in proving the truth of Verma's conjecture on Weyl's dimension polynomial in connection with the irreducible representations of algebraic Chevalley groups and their Lie algebras [1,2]. A certain matrix was defined there which imposes a new partial order on Weyl groups. This matrix is explored in [3] to study the representations of algebraic Chevalley groups. The same matrix is the weighted incidence matrix for our definition of graphs on Weyl groups.

Let E be the Euclidean space of dimension n with a positive definite symmetric bilinear form $(,)$. For $\alpha \in E$ a reflection R_α is given by $xR_\alpha = x - (x, \alpha^v)\alpha$ where $\alpha^v = 2\alpha/(\alpha, \alpha)$ and $x \in E$. Here the reflecting hyperplane P_α is given by $P_\alpha = \{x \in E | (x, \alpha) = 0\}$. Let Φ be a root system in E. By this we mean Φ is a finite subset of E of nonzero vectors which span E and have the following properties: $\alpha \in \Phi$ implies $K\alpha \in \Phi$ iff $K = \pm 1$; Φ is invariant under R_α, $\alpha \in \Phi$; $\alpha, \beta \in \Phi$ implies (β, α^v) is an integer. The group generated by the reflections R_α, $\alpha \in \Phi$ is called the Weyl group of Φ and written as $W(\Phi)$ or W. Let $\Lambda = \{\alpha_1, \dots, \alpha_n\}$ be the simple roots of Φ i.e. $\alpha_1 \dots \alpha_n$ span E and every root $\beta \in \Phi$ can be written as $\beta = \sum_{i=1}^{n} K_{\alpha_i}\alpha_i$ where K_{α_i} are either all non-negative or all non-positive integers. The hyperplanes $P_\alpha, \alpha \in \Phi$ partition E into finitely many regions. The connected components of $E - \underset{\alpha}{U} P_\alpha$ are called the Weyl chambers. The fundamental Weyl chamber $C(\Lambda)$ has the property that $(x, \alpha_i) > 0$ for $x \in E$ and for $i = 1, 2, \dots, n$. We call a root system Φ irreducible if it cannot be written as union of two nonempty

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subsets of Φ such that any root of one subset is orthogonal to every root of other subset. It can be shown that $W(\Phi)$ is generated by the reflections $R_i = R_{\alpha_i}$, $i = 1, \ldots, n$ where Φ is an arbitrary root system. Define the fundamental weights $\lambda_1, \ldots, \lambda_n$ by $(\lambda_i, \alpha_j^\vee) = \delta_{ij}$ where δ_{ij} is kronecker delta. We have $\lambda_i R_j = \lambda_i - \delta_{ij}\alpha_j$. Since $R_1, \ldots, R_n$ generate W every element of W is of the form $R_{i_1} R_{i_2} \ldots R_{i_K}$. Two such expression may represent the same element in W. The minimum number of generators R_i required to express $\sigma \in W$ is called the length of σ, written $\ell(\sigma)$. In fact Weyl groups are a special case of Coxeter groups and all the results of Coxeter groups are applicable to it. It can be easily shown that for $\alpha, \beta \in \Delta$ and $\alpha \pm \beta$, we have $(\alpha, \beta^\vee)(\beta, \alpha^\vee) = 0, 1, 2$ or 3. The graph having the n vertices and ith vertex joined to jth vertex by $(\alpha_i, \alpha_j^\vee)(\alpha_j, \alpha_i^\vee)$ number of edges gives a Coxeter graph of Φ. The Coxeter graphs give the complete classification of irreducible root systems and therefore also of Weyl groups. In fact the order of $R_i R_j$ in W is $2, 3, 4$ or 6 according as the vertex i is joined to vertex j by $0, 1, 2$ or 3 edges. For details of Coxeter groups, root system etc. refer to [4,5]. For relevant concepts and definitions see [1].

For $\sigma \in W$ define $I_\sigma = \{i | 1 \leq i \leq n, \ell(\sigma R_i) < \ell(\sigma)\}$ where $\ell(\sigma)$ is the length of σ. Let $\delta_\sigma = \sum_{i \in I_\sigma} \lambda_i$ and $\varepsilon_\sigma = \delta_\sigma \sigma^{-1}$. Let $D(x)$ for $x \in E$ be the Weyl's dimension polynomial for Φ. We call a point $x \in E$ W-regular iff $D(x) \neq 0$ which is equivalent to saying that x lies in the interior of a Weyl Chamber of W. Let σ_0 be the unique element of W with maximum length. We can define a relation $\rightarrow$ on W as follows: for σ, $\tau \in W$, define $\sigma \rightarrow \tau$ iff $-\varepsilon_{\sigma\sigma_0} + \varepsilon_\tau$ is W-regular. We construct a graph on W by using this relation. The vertices of this graph are elements of W and for $\sigma, \tau \in W$ with $\sigma \neq \tau$, the unordered pair (σ, τ) is an edge iff either $\sigma \rightarrow \tau$ or $\tau \rightarrow \sigma$. It is shown in [2] that at most one of $\sigma \rightarrow \tau$ or $\tau \rightarrow \sigma$ holds for $\sigma \neq \tau$. This graph structure depends on the root system as well. We denote this graph by $\Gamma(W(\Phi))$ or sometimes by $\Gamma(W)$ or $\Gamma(\Phi)$ depending upon the context. We have studied the planarity of such graphs in [6]. The girth of $\Gamma(W(\Phi))$, written $g(\Gamma(W(\Phi)))$, for irreducible root system Φ has been investigated in [7]. Here we shall generalize the result on girth of $\Gamma(W(\Phi))$ for arbitrary root system Φ.

2. Some lemmas on $\Gamma(W)$

Recall that the girth of graph is defined as the length of the smallest cycle, if any, in the graph [8]. We shall write $g(\Gamma(\Phi))$ or $g(\Gamma(W))$ for $g(\Gamma(W(\Phi)))$. We have proved the following result on $g(\Gamma(W))$ in [7]. The graph $\Gamma(W)$ when W is of type A_1, A_2, A_3 or B_2 does not have a cycle. In the remaining cases $g(\Gamma(W)) = 3$ except when W is of type $A_n (n \geq 4)$ and G_2. Further $g(\Gamma(G_2)) = 4$ and $g(\Gamma(A_n)) \leq 4$ for $n \geq 4$. The proof depends upon showing that $g(\Gamma(A_4)) = g(\Gamma(G_2)) = 4$, $g(\Gamma(B_3)) = g(\Gamma(C_3)) = g(\Gamma(D_4)) = 3$ by exhibiting the cycles from the respective graphs and then appealing to the following lemma proved in [6] which is valid for arbitrary root systems.

Lemma 1. *Let J be a subset of $I = \{1, 2, \ldots, n\}$. Let W_J be the (Weyl) subgroup generated by R_J, $j \in J$. Then for σ, $\tau \in W_J$, (σ, τ) is an edge in $\Gamma(W_J)$ iff (σ, τ) is an edge in $\Gamma(W)$.*

Let Φ be an arbitrary root system. Then Φ is, in general, union of irreducible root systems. If Φ is union of root systems $\Phi_1, \Phi_2, \ldots, \Phi_k$ we write $\Phi = \Phi_1 \times \Phi_2 \cdots \times \Phi_k$. The Dynkin diagram of Φ is disjoint union of Dynkin diagrams of $\Phi_1, \ldots, \Phi_k$ which are also the connected components if $\Phi_i, i = 1, \ldots, k$ are irreducible root systems. Let Φ be union of root systems Φ_1 and Φ_2 where Φ_1, Φ_2 are not necessarily irreducible root systems. Then $\Phi = \Phi_1 \times \Phi_2$ and if $\Phi_1 = \Phi_2$ we write $\Phi = (\Phi_1)^2$. If W, W_1 and W_2 are Weyl groups of Φ, Φ_1 and Φ_2 respectively then $W = W_1 \times W_2$, the direct product of W_1 and W_2. Therefore if $\varrho \in W$ then $\varrho = \sigma\tau$ where σ, τ are unique with $\sigma \in W_1$ and $\tau \in W_2$. Further $\delta_\varrho = \delta_\sigma \oplus \delta_\tau$ (direct sum) since $I_\varrho = I_\sigma U I_\tau$ (disjoint union). Therefore $\varepsilon_\varrho = \varepsilon_\sigma \oplus \varepsilon_\tau$ i.e. $\varepsilon_{\sigma\tau} = \varepsilon_\sigma \oplus \varepsilon_\tau$ (direct sum). Suppose δ_1 and δ_2 are the sum of the fundamental weights of Φ_1 and Φ_2 respectively. Then $\delta = \delta_1 \oplus \delta_2$ (direct sum). If $\sigma_\circ$, $\sigma'_\circ$ and $\sigma''_\circ$ are the unique elements of maximal length in W, W_1 and W_2 respectively then $\sigma_\circ = \sigma'_\circ\sigma''_\circ$. With above notations we have the

Lemma 2. *Suppose σ_1, $\sigma_2 \in W_1$ and τ_1, $\tau_2 \in W_2$. Then $\sigma_1 \rightarrow \sigma_2$ holds in W_1 and $\tau_1 \rightarrow \tau_2$ holds in W_2 iff $\sigma_1\tau_1 \rightarrow \sigma_2\tau_2$ holds in W.*

Proof. We have the following series of equalities.

$$\begin{aligned}
&-\varepsilon_{\sigma_1\tau_1\sigma_\circ} + \varepsilon_{\sigma_2\tau_2} = (\delta - \delta_{\sigma_1\tau_1})(\sigma_1\tau_1)^{-1} + \varepsilon_{\sigma_2\tau_2}\\
&= (\delta_1 + \delta_2 - \delta_{\sigma_1} - \delta_{\tau_1})\sigma_1^{-1}\tau_1^{-1} + (\varepsilon_{\sigma_2} \oplus \varepsilon_{\tau_2})\\
&= ((\delta_1 - \delta_{\sigma_1})\sigma_1^{-1} \oplus (\delta_2 - \delta_{\tau_1})\tau_1^{-1}) + (\varepsilon_{\sigma_2} \oplus \varepsilon_{\tau_2})\\
&= (-\varepsilon_{\sigma_1\sigma'_\circ} + \varepsilon_{\sigma_2}) \oplus (-\varepsilon_{\tau_1\sigma''_\circ} + \varepsilon_{\tau_2}),
\end{aligned}$$

where we have used the fact that $\lambda_i R_j = \lambda_i$ for $j \neq i$. Recall that for $\sigma, \tau \in W$, $\sigma \rightarrow \tau$ iff $-\varepsilon_{\sigma\sigma_\circ} + \varepsilon_\tau$ is W-regular which is same as $-\varepsilon_{\sigma\sigma_\circ} + \varepsilon_\tau$ lies in the interior of a Weyl chamber. This shows that $-\varepsilon_{\sigma_1\tau_1\sigma_\circ} + \varepsilon_{\sigma_2\tau_2}$ is W-regular iff $-\varepsilon_{\sigma_1\sigma'_\circ} + \varepsilon_{\sigma_2}$ is W_1-regular and $-\varepsilon_{\tau_1\sigma''_\circ} + \varepsilon_{\tau_2}$ is W_2-regular. This proves the claim. □

Remark. The above lemma can be easily generalized when $W = W_1 \times W_2 \times \cdots \times W_k$.

Lemma 3. *Suppose (σ_1, σ_2) is an edge in $\Gamma(W_1)$ with $\sigma_1 \rightarrow \sigma_2$ in W_1 and (τ_1, τ_2) is an edge in $\Gamma(W_2)$ with $\tau_1 \rightarrow \tau_2$ in W_2. Then the following pairs are edges in $\Gamma(W_1 \times W_2)$:*

$$\begin{aligned}
&(\sigma_1\tau_1, \sigma_2\tau_1), \quad (\sigma_1\tau_2, \sigma_2\tau_2), \quad (\sigma_1\tau_1, \sigma_1\tau_2),\\
&(\sigma_2\tau_1, \sigma_2\tau_2) \text{ and } (\sigma_1\tau_1, \sigma_2\tau_2).
\end{aligned}$$

In other words an edge in $\Gamma(W_1)$ and an edge in $\Gamma(W_2)$ gives a rectangle with a diagonal in $\Gamma(W_1 \times W_2)$.

Proof. From the definition it follows that $\sigma_1 \rightarrow \sigma_1$, $\sigma_2 \rightarrow \sigma_2$ in W_1 and $\tau_1 \rightarrow \tau_1$, $\tau_2 \rightarrow \tau_2$ in W_2. Therefore, the assumption $\sigma_1 \rightarrow \sigma_2$ in W_1 and $\tau_1 \rightarrow \tau_2$ in W_2 along with the lemma 2 gives the result. □

Corollary. *If each of* $\Gamma(W_1)$ *and* $\Gamma(W_2)$ *has at least one edge then* $g(\Gamma(W_1 \times W_2)) = 3$.

Lemma 4.

(i) *The graphs* $\Gamma(W_1)$ *and* $\Gamma(W_2)$ *are totally disconnected iff* $\Gamma(W_1 \times W_2)$ *is totally disconnected.*

(ii) *If* $\Gamma(W_1)$ *is totally disconnected and* $\Gamma(W_2)$ *is any graph then* $\Gamma(W_1 \times W_2)$ *consists of* $|W_1|$ *number of disjoint copies of* $\Gamma(W_2)$.

Proof.

(i) Let σ_1, $\sigma_2 \in W_1$ and τ_1, $\tau_2 \in W_2$. If either (σ_1, σ_2) or (τ_1, τ_2) is an edge then by lemma 2, $(\sigma_1\tau_1, \sigma_2\tau_2)$ is an edge. Conversely, suppose (ϱ_1, ϱ_2) is an edge in $\Gamma(W_1 \times W_2)$. Then $\varrho_1, \varrho_2 \in W_1 \times W_2$ implies that $\varrho_1 = \sigma_1\tau_1$, $\varrho_2 = \sigma_2\tau_2$ with unique $\sigma_1, \sigma_2 \in W_1$, τ_1, $\tau_2 \in W_2$. Then $(\sigma_1\tau_1, \sigma_2\tau_2)$ is an edge in $\Gamma(W_1 \times W_2)$ means that $\sigma_1\tau_2 \to \sigma_2\tau_2$ and $\sigma_1\tau_1 \neq \sigma_2\tau_2$. Therefore lemma 2 implies either $\sigma_1 \to \sigma_2$ or $\tau_1 \to \tau_2$. Also either $\sigma_1 \neq \sigma_2$ or $\tau_1 \neq \tau_2$ holds. Then either (σ_1, σ_2) is an edge in $\Gamma(W_1)$ or (τ_1, τ_2) is an edge in $\Gamma(W_2)$.

(ii) Suppose $\sigma \in W_1$. Then $\sigma \to \sigma$ in W_1. Let (τ_1, τ_2) be an edge in $\Gamma(W_2)$. Then, by lemma 2, $(\sigma\tau_1, \sigma\tau_2)$ is an edge in $\Gamma(W_1 \times W_2)$. We conclude that every edge in $\Gamma(w_2)$ is repeated at least $|W_1|$ times in $\Gamma(W_1 \times W_2)$. Suppose $(\sigma\tau_1, \varrho)$ is an edge in $\Gamma(W_1 \times W_2)$, where $\sigma \in W_1$, $\tau_1 \in W_2$, $\varrho \in W_1 \times W_2$ and $\sigma\tau_1 \to \varrho$. Then $\varrho = \sigma'\tau_1'$ with unique $\sigma' \in W_1$ and $\tau_1' \in W_2$ which in turn implies $\sigma \to \sigma'$ and $\tau_1 \to \tau_1'$ by lemma 2. If $\sigma \neq \sigma'$ then (σ, σ') is an edge in $\Gamma(W_1)$ which is a contradiction. Therefore $\sigma' = \sigma$ and we get an edge (τ_1, τ_1') in $\Gamma(W_2)$, as $\tau_1 \to \tau_1'$ and $\tau_1 \neq \tau_1'$ since $\sigma\tau_1 \neq \varrho = \sigma'\tau_1'$. This shows that every edgee in $\Gamma(W_1 \times W_2)$ at $\sigma\tau_1$ comes from an edge in $\Gamma(W_2)$ at τ_1. Therefore, the degree at $\sigma\tau_1$ in $\Gamma(W_1 \times W_2)$ is equal to the degree at τ_1 in $\Gamma(W_2)$. We conclude that each $\sigma \in W_1$ gives a copy of $\Gamma(W_2)$ and hence $\Gamma(W_1 \times W_2)$ has $|W_1|$ number of copies of $\Gamma(W_2)$. □

3. Girth of $\Gamma(W)$

Now we prove the following lemma which improves the result on $g(\Gamma(A_n))$ for $n \geq 7$.

Lemma 5. $g(\Gamma(A_7)) = 3$. *Hence* $g(\Gamma(A_n)) = 3$ *for* $n \geq 7$.

Proof. The Dynkin diagram of A_7 is a chain of length 6 with vertices labelled by $1, 2, \ldots, 7$ from left to right, i.e. in this case $I = \{1, 2, \ldots, 7\}$. Take $J = \{1, 2, 3, 5, 6, 7, \}$. Then $\Phi_J = \Phi_1 \times \Phi_2$ where each of Φ_1 and Φ_2 is a root system of type A_3. Now $\Gamma(A_3)$ has edges and therefore $g(\Gamma(\Phi_J)) = 3$ by corollary to lemma 3. By lemma 1, $g(\Gamma(A_7)) = 3$. Since $\Gamma(A_7)$ is an induced subgraph of $\Gamma(A_n)$ for $n \geq 7$, again by lemma 1, we have $g(\Gamma(A_n)) = 3$ for $n \geq 7$. □

In view of lemma 5, we can rewrite the theorem on $g(\Gamma(W))$ mentioned earlier as follows:

Theorem 1. *Let* Φ *be an irreducible root system. Then*

(i) $\Gamma(A_1)$, $\Gamma(A_2)$, $\Gamma(A_3)$ *and* $\Gamma(B_2)$ *have no cycles.*

(ii) $g(\Gamma(A_4)) = 4$, $3 \leq g(\Gamma(A_5))$, $g(\Gamma(A_6)) \leq 4$ *and* $g(\Gamma(G_2)) = 4$.
(iii) $g(\Gamma(\Phi)) = 3$ *when* Φ *is not of type mentioned in (i) and (ii).*

Now we state our main result in the following theorem which generalize theorem 1 for arbitrary root system.

Theorem 2. *Let* Φ *be an arbitrary root system. Suppose* $\Phi = \Phi_1$ *or* $\Phi = \Phi_1 \times \Phi_2$ *where* Φ_1 *is a root system of type* $A_1^{k_1} \times A_2^{k_2}$ *where* k_1, k_2 *are non-negative integers and* Φ_2 *does not contain a root system of type* A_1 *or* A_2 *as a factor. Then following holds:*
(a) *Suppose* $\Phi = \Phi_1$. *Then* $\Gamma(\Phi)$ *does not contain a cycle.*
(b) *Suppose* $\Phi = \Phi_1 \times \Phi_2$ *and* Φ_2 *is irreducible.*

(i) *If* Φ_2 *is of type* B_2 *or* A_3, *then* $\Gamma(\Phi)$ *does not contain a cycle.*
(ii) *If* Φ_2 *is of type* G_2 *or* A_4 *then* $g(\Gamma(\Phi)) = 4$.
(iii) *If* Φ_2 *is of type* A_5 *or* A_6 *then* $3 \leq g(\Gamma(\Phi)) \leq 4$.
(iv) *In all other cases* $g(\Gamma(\Phi)) = 3$.

(c) *Suppose* $\Phi = \Phi_1 \times \Phi_2$ *and* Φ_2 *is reducible. Then* $g(\Gamma(\Phi)) = 3$.

Proof. Recall that $\Gamma(A_1)$ and $\Gamma(A_2)$ are totally disconnected graphs. Therefore $\Gamma(\Phi_1)$ is a totally disconnected graph by repeated application of lemma 4 (i). This proves (a). Part (b) follows from lemma 4 (ii) and theorem 1. In case of (c) let $\Phi_2 = \Phi_2' \times \Phi_3'$. Then each of $\Gamma(\Phi_2')$ and $\Gamma(\Phi_3')$ contains at least one edge as Φ_2 does not contain a root system of types A_1 or A_2 as a factor. By corollary to lemma 3, $g(\Gamma(\Phi_2)) = 3$. Then by lemma 4 (ii), $g(\Gamma(\Phi)) = 3$. □

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