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ON DIFFEOMORPHIC SOLUTIONS OF SIMULTANEOUS ABEL'S EQUATIONS

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Abstract. In this paper there is considered the problem of the existence of C^r-diffeomorphic solution $\varphi : (a, b)$ **R** of the system of simultaneous Abel's equations

$$\varphi(f_i(x)) = \varphi(x) + c_i, \qquad i = 1, \ldots, N, \; x \in (a, b),$$

where $f_1, \ldots, f_N$ are given C^r-diffeomorphisms of an interval (a, b) onto itself and $c_i \neq 0$ are suitable constans.

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In the present paper we consider the problem of the existance of C^r solutions $\varphi : (a, b) \to \mathbf{R}$ of the system of simultaneous Abel's equations

$$(1) \qquad \varphi(f_i(x)) = \varphi(x) + c_i, \qquad i = 1, \ldots, N, \qquad x \in (a, b),$$

$c_1 = 1$ and $c_i \neq 0$, $i = 2, \ldots, N$, where $f_1, \ldots, f_N$ are given continuous bijections of an interval (a, b), $-\infty \leqq a < b \leqq \infty$ onto itself.

The problem arise from the theory of differential equations with deviating arguments. The existence of an invertible C^r transformation of the independent variable that converts a given differential equation with several deviating arguments f_i, $i = 1, \ldots, N$ into a differential equation with constant deviations is equivalent to the existance of a common C^r solution φ of system (1) such that $\varphi' \neq 0$. F. Neuman in [4], O. Borůvka in [1] and L. A. Beklarian in [9] gave some conditions ensuring the existence of such solution. In this paper we present an other approach to this problem and we give some immediate conditions on functions $f_1, \ldots, f_N$ which imply the existence of C^r solutions of (1). This is an answer of the problem set by F. Neuman in [5].

1. By f^n we denote here the n-th iterate of the function $f\colon I \to I$ that is

$$f^0 = id, \qquad f^{n+1} := f \circ f^n, \qquad f^{-n} := (f^n)^{-1}, \qquad n = 1, 2, \ldots$$

Let us note the following properties

Lemma 1. *If system* (1) *has an invertible solution, then*

$$f_i \circ f_j = f_j \circ f_i, \qquad i, j = 1, \ldots, N \tag{2}$$

and if $f_i^m(x_0) = f_j^k(x_0)$ *for some* $n, k \in \mathbf{Z}\setminus\{0\}$, $x_0 \in (a, b)$ *and* $i, j \in \{1, \ldots, N\}$, *then* $f_i^m = f_j^n$, *moreover*

$$f_i(x) \neq x \qquad \text{for } x \in (a, b), \qquad i = 1, \ldots, N. \tag{3}$$

Proof. Let φ be an invertible solution of (1). Then we have

$$\varphi(f_i(f_j(x))) = \varphi(f_j(x)) + c_i = \varphi(x) + c_j + c_i = \varphi(f_i(x)) + c_j = \varphi(f_j(f_i(x))).$$

Thus we get (2).

Let $f_i^m(x_0) = f_j^k(x_0)$ for an $x_0 \in (a, b)$. By (1) we have

$$\varphi(f_i^m(x)) = \varphi(x) + mc_i \qquad \text{and} \qquad \varphi(f_j^k(x)) = \varphi(x) + kc_j.$$

Hence $mc_i = kc_j$, and consequently $\varphi(f_i^m(x)) = \varphi(f_j^k(x))$. Thus $f_i^m = f_j^k$. Moreover, $f_i(x) \neq x$, since $c_i \neq 0$ for $i = 1, \ldots, N$.

Q.E.D.

Further in our investigations we consider the following two alternative hypotheses (C) and (H).

(C) $f_j : (a, b) \to (a, b), j = 1, \ldots, N$ are continuous bijections satisfying (2) and (3) and for every $i \in (2, \ldots, N\}$ there exist $m_i, k_i \in \mathbf{Z}\setminus\{0\}$ such that $f_1^{m_i} = f_i^{k_i}$,

however, let (H) be a hypothesis that one of the following assumptions holds:

(H$_1$) $f_j : (a, b) \to (a, b), j = 1, \ldots, N$ are continuous bijections satisfying (2) and (3) and $f_1^m(x) \neq f_i^k(x)$ for $x \in (a, b)$, $m, k \in \mathbf{Z}\setminus\{0\}$,

where $i = 2, \ldots, N$.

Let us assume that $f_1, \ldots, f_N$ satisfy hypothesis (H) or (C). Let us fix an $x_0 \in \in (a, b)$. The sequence $\{f_1^n(x_0), n \in \mathbf{Z}\}$ is strictly monotonic and $\lim_{n\to\infty} f_1^n(x_0) = a$ (or b) and $\lim_{n\to-\infty} f_1^n(x_0) = b$ (or a), since $f_1(x) \neq x$ in (a, b) (see [3] p. 21). Hence for every $n \in \mathbf{N}$ and every $k \in \{2, \ldots, N\}$ there exists exactly one $m_n^k \in \mathbf{Z}$ such that

$$f_1^{m_n^k - 1}(x_0) \underset{(\geqq)}{\leqq} f_k^n(x_0) \underset{(>)}{<} f_1^{m_n^k}(x_0).$$

It has been proved in paper [8] (Th. 3) that there exist limits

$$\lim_{n\to\infty} \frac{m_n^k}{n} =: s_k, \qquad k = 2, \ldots, N \tag{4}$$

and s_k do not depend of x_0. Moreover, $s_k \in \mathbf{Q}$ if there exist $u_k, v_k \in \mathbf{Z}\setminus\{0\}$ such that $f_1^{u_k} = f_k^{v_k}$. Then $s_k = \dfrac{u_k}{v_k}$. Also an immediate conclusion of Th. 3, Th. 2 and Remark 2 in [8] is the following

Lemma 2. *If hypothesis (C) or (H) is fulfilled and system* (1) *has a continuous solution, then $c_k = s_k$ for $k = 2, \dots, N$.*

Further we consider the system

$$(5) \qquad \varphi(f_i(x)) = \varphi(x) + s_i, \qquad i = 1, \dots, N, \qquad x \in (a, b),$$

where $s_1 = 1$ and $s_2, \dots, s_N$ are given by (4).

2. Let us start with case (C).

Lemma 3. (see [7]). *If f and g are strictly increasing selfmappings of (a, b), $f \circ g = g \circ f$ and $f^n = g^n$ for an $n \in \mathbf{N}$, then $f = g$.*

Lemma 4. *If f and g are strictly increasing selfmappings of (a, b), $f \circ g = g \circ f$, $f^v = f^u$ for some $u, v \in Z\setminus\{0\}$ and $mv = nu$, then $f^n = g^m$.*

Proof. Let $f^v = g^u$. Then $f^{mnv} = g^{mnu} = g^{m^2v}$. Hence by Lemma 2 $f^n = g^m$.

Q.E.D.

Theorem 1. *Let hypothesis (C) be fulfilled and let $f_1^{u_i} = f_i^{v_i}$ for some $u_i, v_i \in \mathbf{Z}\setminus\{0\}$, $i = 1, \dots, N$. Let $s \in Q$, $m_1, \dots, m_N, k_1, \dots, k_N \in \mathbf{Z}\setminus\{0\}$ be such that*

$$\frac{u_i}{v_i} = m_i s =: s_i, \qquad for\ i = 1, \dots, N$$

and let $m_1, \dots, m_N$ be relatively prime and

$$m_1 k_1 + \dots + m_N k_N = 1. \text{ (}^1\text{)}$$

Then system (5) is equivalent to the equation

$$(6) \qquad \varphi(f(x)) = \varphi(x) + s,$$

where $f := f_1^{k_1} \circ \dots \circ f_N^{k_N}$.

Proof. Let φ satisfy system (5), then

$$\varphi(f_i^{k_i}(x)) = \varphi(x) + k_i s_i, \qquad i = 1, \dots, N.$$

(1) From the elementary number theory it is known that such integers always exist.

Hence

$$\varphi(f(x)) = \varphi \circ f_1^{k_1} \circ \dots \circ f_N^{k_N}(x) = \varphi(x) + k_1 s_1 + \dots + k_N s_N =$$
$$= \varphi(x) + (k_1 m_1 + \dots + k_N m_N)\, s = \varphi(x) + s.$$

Thus φ satisfies (6).

Note that $f_i^{v_i u_k} = f_k^{v_k u_i}$ for $i, k = 1, \dots, N$, since $f_1^{u_i} = f_i^{v_i}$ for $i = 1, \dots, N$. On the other hand

$$\frac{v_i u_k}{v_k u_i} = \frac{s_k}{s_i} = \frac{m_k}{m_i},$$

whence, by Lemma 4 it follows that

$$f_i^{m_k} = f_k^{m_i}, \qquad i, k = 1, \dots, N.$$

Therefore

$$f^{m_i} = (f_1^{k_1} \circ \dots \circ f_N^{k_N})^{m_i} = (f_1^{m_i})^{k_1} \circ \dots \circ (f_N^{m_i})^{k_N} =$$
$$= (f_i^{m_1})^{k_1} \circ \dots \circ (f_i^{m_N})^{k_N} = f_i^{m_1 k_1 + \dots m_N k_N} = f_i^1 = f_i,$$

for $i = 1, \dots, N$.

Now, let φ satisfy (6), then

$$\varphi(f_i(x)) = \varphi(f^{m_i}(x)) = \varphi(x) + m_i s = \varphi(x) + s_i, \qquad i = 1, \dots, N,$$

so φ satisfies system (5)

Q.E.D.

Let J be an interval and $r \geqq 1$ be an integer. Denote by $\mathrm{Diff}^r J$ the set of all functions $\varphi\colon J \to \mathbf{R}$ continuously differentiable up to r-order such that $\varphi'(x) \neq 0$ for $x \in J$.

If $f \in \mathrm{Diff}^r (a, b)$ and $a < f(x) < x$ or $b < f(x) < x$ in (a, b), then equation (6) has a solution $\varphi \in \mathrm{Diff}^r (a, b)$ depending on an arbitrary function (see [3] Th. 4.1, Th. 3.1 and L. 5.1). Hence in view of Theorem 1 we get the following

Corollary 1. *If $f_1, \dots, f_N \in \mathrm{Diff}^r (a, b)$ satisfy hypothesis (C), then system (5) has a solution $\varphi \in \mathrm{Diff}^r (a, b)$ depending on an arbitrary function.*

3. Now we shall consider case (H)

Theorem 2. *Let $f_1, \dots, f_N$ satisfy hypothesis (H). Then system (5) has a continuous solution unique up to an additive constant. This solution is monotonic. Moreover, if $s_k \notin Q$, then every continuous solution of the system*

$$(7) \qquad \begin{aligned} \varphi(f_i(x)) &= \varphi(x) + 1 \\ \varphi(f_k(x)) &= \varphi(x) + s_k \end{aligned} \qquad x \in (a, b)$$

fulfils system (5).

Proof. It follows by hypothesis (H), that at least ony coefficient $s_2, \dots, s_N$, which are defined by (4), is irrational. Let $s_k \in Q$. It has been proved in [8], that under our assumptions system (7) has a continuous solution φ unique up to an additive constant and this solution is monotonic. Let $2 \leqq i \leqq N$, $i \neq k$ and φ be a continuous solution of (7), then by (2) we have

$$\varphi \circ f_i(f_1(x)) = \varphi \circ f_1(f_i(x)) = \varphi \circ f_i(x) + 1$$

and

$$\varphi \circ f_i(f_k(x)) = \varphi \circ f_k(f_i(x)) = \varphi \circ f_i(x) + s_k.$$

Therefore the function $\varphi \circ f_i$ satisfies (7), and consequently from the uniqueness of continuous solutions of system (7) it follows that there exists $c_i \in \mathbf{R}$ such that

$$\varphi \circ f_i(x) = \varphi(x) + c_i.$$

Thus φ satisfies system (1) with $c_k = s_k$. Moreover, according to Lemma 2, $c_i = s_i$ for $i = 1, \dots, N$. Therefore, φ is a solution of system (5).

Let us note that by the way we have proved also the second thesis of the theorem.

Q.E.D.

Now we give some conditions which imply that solution φ of system (5) described in Theorem 2 is a C^r diffeomorphism on (a, b) i.e. $\varphi \in \mathrm{Diff}^r(a, b)$.

It is well-known that one can write each irrational number α uniqely in the form of infinite continuous fraction

$$\alpha = [\alpha] + \cfrac{1}{a_1(\alpha) + \cfrac{1}{a_2(\alpha) + \cfrac{1}{a_3(\alpha) + \ddots}}},$$

where $a_i(\alpha)$, $i = 1, 2, \dots$ are positive integers. Denote by A the set of all $\alpha \in \mathbf{R}\setminus\mathrm{Q}$ such that

$$\lim_{B\to\infty} \limsup_{n\to\infty} \sum_{\substack{1 \leqq i \leqq n \\ a_i(\alpha) \geqq B}} \log(a_i(\alpha) + 1) / \sum_{1 \leqq i \leqq n} \log(a_i(\alpha) + 1) = 0.$$

In paper [2] Ch. V. 9 it has been shown that the set A is of full Lebesgue measure in $\mathbf{R}$, i.e. $|\mathbf{R}\setminus A| = 0$.

We shall prove the following

Theorem 3. *Let* $3 \leqq r \leqq \omega$ *and* $f_1, \dots, f_N \in \mathrm{Diff}^r(a, b)$ *satisfy hypothesis* (H). *If there is* $k \in \langle 2, \dots, N\}$ such that $s_k \in A$, *then system* (5) *has a solution* $\varphi \in \mathrm{Diff}^{r-2}(a, b)$ *unique up to an additive constant.*

Proof. Let $\gamma \in \mathrm{Diff}^r(a, b)$ be a solution of the equation

(8) $$\gamma(f_1(x)) = \gamma(x) + 1, \qquad x \in (a, b).$$

Such solution exists and depends on an arbitrary function (see [3], Th. 4.1, Th. 3.1 and L. 5.1). We may write the last relation as follows

$$\gamma^{-1}(t+1) = f_1(\gamma^{-}(t)), \qquad t \in \mathbf{R}. \tag{9}$$

Let $s_k \in A$. Put

$$g := \gamma \circ f_k \circ \gamma^{-1}. \tag{10}$$

Note that $g \in \mathrm{Diff}^r\,\mathbf{R}$ and

$$g(t+1) = g(t) + 1, \qquad t \in \mathbf{R}. \tag{11}$$

In fact, according to (9) and (2) we have

$$\begin{aligned} g(t+1) &= \gamma \circ f_k \circ \gamma^{-1}(t+1) = \gamma \circ f_k \circ f_1 \circ \gamma^{-1}(t) = \gamma \circ f_1 \circ f_k \circ \gamma^{-1}(t) = \\ &= \gamma \circ f_1 \circ \gamma^{-1} \circ \gamma \circ f_k \circ \gamma^{-1}(t) = \gamma \circ f_1 \circ \gamma^{-1} \circ g(t) = g(t) + 1. \end{aligned}$$

In view of Theorem 2 system (7) has a continuous and monotonic solution $\varphi : (a, b) \to \mathbf{R}$. Note that the function $\Psi := \varphi \circ \gamma^{-1}$ is a continuous and increasing solution of the system

$$\begin{aligned} \Psi(t+1) &= \Psi(t) + 1 \\ \Psi(g(t)) &= \Psi(t) + s_k \end{aligned} \qquad t \in \mathbf{R}. \tag{12}$$

Indeed, by (9), (10) and (7) we have

$$\Psi(t+1) = \varphi \circ \gamma^{-1}(t+1) = \varphi \circ f_1 \circ \gamma^{-1}(t) = \varphi \circ \gamma^{-1}(t) + 1 = \Psi(t+1)$$

and

$$\Psi(g(t)) = \varphi \circ \gamma^{-1} \circ \gamma \circ f_k \circ \gamma^{-1}(t) = \varphi \circ f_k \circ \gamma^{-1}(t) + s_k.$$

Moreover,

$$f_1^m(x) = \gamma^{-1}(m + \gamma(x)) \qquad \text{and} \qquad g^m(t) = \gamma \circ f_k^m \circ \gamma^{-1}(t), \tag{13}$$

for $m \in \mathbf{Z}$, $x \in (a, b)$ and $t \in \mathbf{R}$.

It follows from hypothesis (H_k) that for every $n \in \mathbf{N}$ there is $m_n^k \in \mathbf{Z}$ such that

$$f_1^{m_n^k - 1}(x) \underset{(\geqq)}{\leqq} f_k^n(x) \underset{(>)}{<} f_1^{m_n^k}(x).$$

Hence by (13) we get

$$m_n^k - 1 + t \leqq g^n(t) \leqq m_n^k + t, \qquad n \in \mathbf{N},\ t \in \mathbf{R}$$

and consequently in view of (4)

$$\lim_{n \to \infty} \frac{g^n(t)}{n} = s_k, \qquad t \in R, \tag{14}$$

Further we apply the following result of M. Herman (see [2], Ch. IX, Th. 5.1). If $g \in \mathrm{Diff}^r \mathbf{R}$ $(3 \leqq r \leqq \omega)$ satisfies (11) and (14) and $s_k \in A$, then system (12) has a solution $\bar{\psi} \in \mathrm{Diff}^{r-2} \mathbf{R}$.

Let $\bar{\psi} \in \mathrm{Diff}^{r-2} \mathbf{R}$ be a solution of (12). According to Theorem 2 system (12) has a continuous solution unique up to an additive constant, so $\bar{\psi} = \Psi + c$ for a $c \in \mathbf{R}$, whence we infer that $\Psi \in \mathrm{Diff}^{r-2} \mathbf{R}$ and consequently $\varphi = \Psi \circ \gamma \in \mathrm{Diff}^{r-2} \mathbf{R}$. Moreover, in view of Theorem 2 φ satisfies (5). Q.E.D.

If in Theorem 3 we drop the assumption that there exists $k \in \{2, \ldots, N\}$ such that $s_k \in A$, then system (5) may have a continuous solution which is not of class C^1 (see [2] Ch. XII.1). Moreover M. Herman has shown (see [2] Ch. XI that the set of functions g for which system (12) has a C^r solution $(1 \leqq r < \infty)$ is of the first Baire cathegory in the space of all functions $g \in \mathrm{Diff}^r \mathbf{R}$ fulfilling conditions (11) and (14) with C^r norm in $[0, 1]$.

4. In this section we give some conditions which ensure the existence of C^r diffeomorphic solutions of system (5).

Theorem 4. *Let $a > -\infty$ and let $f_1, \ldots, f_N\colon (a, b) \to (a, b)$ be continuous bijections fulfilling relation* (2) *and such that $f_i(x) < x$ for $x \in (a, b)$, $i = 1, \ldots, N$. If $f_1, f_2 \in \mathrm{Diff}^r (a, b)$, $r \geqq 2$ and*

$$(15) \qquad 0 < f_i'(a) < 1, \qquad i = 1, 2 \qquad \text{and} \qquad \log f_1(a)/\log f_2'(a) \notin \mathbf{Q},$$

then there exist $c_2, \ldots, c_N$ such that system (1) *has a solution $\varphi \in \mathrm{Diff}^1(a, b)$.* This solution is unique up to an additive constant.

Theorem 5. Let $a > -\infty$ *and let $f_1, \ldots, f_N\colon (a, b) \to (a, b)$ be continuous bijections fulfilling relation* (2) *such that $f_i(x) < x$* for $x \in (a, b)$, $i = 1, \ldots, N$. If $f_1, f_2 \in \mathrm{Diff}^1 (a, b)$ *are convex or concave and satisfy* (15), *then there exist $c_2, \ldots, c_N$ such that system* (1) *has a solution $\varphi \in$* $\mathrm{Diff}^1 (a, b)$. This solution is unique up to an additive constant.

We shall prove Theorems 4 and 5 simultaneously.

Proof. Put $p_1 := f_1'(a)$. If $f_1 \in \mathrm{Diff}^r [a, b)$ $(r \geqq 2)$, then it follows from Th. 6.1 in [3] that there exists a unique C^r diffeomorphism Ψ on $[a, b)$ such that

$$(16) \qquad \Psi(f_1(x)) = p_1 \Psi(x) \qquad \text{for } x \in [a, b)$$

and $\Psi'(a) = 1$. However, if $f_1 \in \mathrm{Diff}^1 [a, b)$ is convex (concave), then in view of Th. 4 in [6] equation (16) has a convex (concave) and strictly increasing solution $\Psi \in \mathrm{Diff}^1 (a, b)$ unique up to a multiplicative constant.

Using (2) and (16) we obtain

$$(\Psi \circ f_2) \circ f_1(x) = (\Psi \circ f_1) \circ f_2(x) = p_1 \Psi \circ f_2(x), \qquad x \in [a, b).$$

Thus the function $\alpha := \Psi \circ f_2$ satisfies equation (16) and respectively $\alpha \in \mathrm{Diff}^r [a, b)$ or $\alpha \in \mathrm{Diff}^1 (a, b)$ and is convex (concave).

By the uniqueness of the solutions of equation (16) it results that there exists a $p > 0$ such that $\alpha = p\Psi$. On the other hand $\alpha = \Psi \circ f_2$, so

$$\Psi(f_2(x)) = p\Psi(x), \qquad x \in [a, b). \tag{17}$$

We have that $\Psi(x) > 0$ for $x \in (a, b)$, since $\Psi(0) = 0$ and Ψ is strictly increasing. Put

$$\varphi := \log \Psi / \log p_1 .$$

From (16) and (17) we get

$$\begin{aligned} \varphi(f_1(x)) &= \varphi(x) + 1, \\ \varphi(f_2(x)) &= \varphi(x) + c, \end{aligned} \tag{18}$$

where $c = \log p / \log f_1'(a)$. Moreover $\varphi \in \mathrm{Diff}^r (a, b)$ or $\varphi \in \mathrm{Diff}^1 (a, b)$. Further, taking into account Lemma 1 we see that f_1 and f_2 satisfy hypothesis (C) or (H_2) for $N = 2$. Note, that finally f_1 and f_2 satisfy only hypothesis (H_2), since otherwise $f_1'(a)^n = f_2'(a)^m$ for some $m, n \in \mathbf{Z}\setminus\{0\}$, which is contradiction to (15). Now put $\varphi_i := \varphi \circ f_i$ for $i = 3, \ldots, N$. By (2) and (18) we get

$$\varphi_i(f_1(x)) = \varphi \circ f_i(f_1(x)) = \varphi \circ f_1(f_i(x)) = \varphi \circ f_i(x) + 1 = \varphi_i(x) + 1$$

and

$$\varphi_i(f_2(x)) = \varphi \circ f_i(f_2(x)) = \varphi \circ f_2(f_i(x)) = \varphi \circ f_i(x) + c_2 = \varphi_i(x) + c_2 .$$

Therefore φ_i for $i = 3, \ldots, N$ are continuous solutions of system (18). On the other hand, according to Theorem 3 system (18) has a unique continuous solution up to an additive constant, so for every $i \in \{3, \ldots, N\}$ there exists c_i such that

$$\varphi(f_i(x)) = \varphi_i(x) = \varphi(x) + c_i, \qquad \text{for } x \in (a, b).$$

Thus φ satisfies (1). Q.E.D.

Proving Theorems (4) and (5) we have proved also the following

Corollary 2. *Let $a > -\infty$ and $f_1, \ldots, f_N$ satisfy hypothesis (H_2). If $f_1 \in \mathrm{Diff}^r [a, b)$, $f_2 \in \mathrm{Diff}^r (a, b)$, $r \geqq 2$ ($f_1 \in \mathrm{Diff}^1 [a, b)$, $f_2 \in \mathrm{Diff}^1 (a, b)$ and f_1, f_2 are convex or concave), $0 < f_1'(a) < 1$, then system* (5) *has a solution $\varphi \in \mathrm{Diff}^r (a, b)$ ($\varphi \in \mathrm{Diff}^1 (a, b)$).*

Theorem 6. *If $f_1, \ldots, f_N$ satisfy hypothesis (H_2) for $b = \infty$, $f_1, f_2 \in \mathrm{Diff}^1 (a, \infty)$ are convex or concave and $\lim\limits_{x \to a^+} f_1'(x) = 1$, then* system (5) *has a solution $\varphi \in \mathrm{Diff}^1$ (a, ∞).*

Proof. Let us note that equation

$$\varphi(f_i(x)) = \varphi(x) + c_i$$

is equivalent to the equation

$$\varphi(f_i^{-1}(x)) = \varphi(x) - c_i.$$

Hence we may assume that f_1 and f_2 are convex functions.

It follows from Th. 7.4 in [3] that the equation

$$\varphi(f_1(x)) = \varphi(x) + 1 \tag{19}$$

has a convex solution $\varphi \in \mathrm{Diff}^1\, \mathbf{R}$ unique up to an additive constant.

Put $\varphi_i := \varphi \circ f_i$, $i = 2, \ldots, N$. By (2) and (19) we have

$$\varphi_2(f_1(x)) = \varphi \circ f_2(f_1(x)) = \varphi \circ f_1(f_2(x)) = \varphi(f_2(x)) + 1 = \varphi_2(x) + 1.$$

Therefore φ_2 satisfies (19) and φ_2 is a convex C^1 diffeomorphism. Hence by the uniqueness of convex solutions of equation (19) we infer that there exists c_2 such that $\varphi_2 = \varphi + c_2$, whence we get

$$\varphi(f_2(x)) = \varphi(x) + c_2, \qquad x \in (a, \infty). \tag{20}$$

Further by the same arguments as in the last lines of the proof of Theorem 4 we deduce that there exist $c_3, \ldots, c_N$ such that φ satisfies system (1). Moreover, from Lemma 2 we obtain that $c_i = s_i$, for $i = 2, \ldots, N$. Thus φ satisfies (5).

Q.E.D.

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