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EDGE THEOREM FOR FINITE PARTIALLY ORDERED SETS

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Abstract. In this paper the fixed edge theorem is proved: Let P be a finite, connected partially ordered set and f an antitone map of P into itself. Then there exists a fixed edge of f or there exist connected subsets C, G of P of length one with $I(C) = I(G) = \emptyset$ such that $f(C) = G$ $f(G) = C$. Also, if P is dismantlable by irreducibles then P has the fixed edge property.

Key words. Partially ordered sets, crown, dismantlable, isotone and antitone mapping, fixed points, fixed edges.

MS Classification. Primary: 06 A 10, secondary: 05 A 05.

I

In a noted paper [6] Tarski has shown that every isotone map of a complete lattice into itself has a fixed point. Set P is said to have the *fixed point property* if every isotone map of P into itself has a fixed point. Antitone maps, on the other hand, may or may not have fixed points; however under certain conditions such maps must have a unique fixed point. The analogous problem for finite partially ordered sets has remained largely unexplored.

Rival [5] published a far-reaching extension: *Every isotone map of finite, dismantlable by irreducibles, partially ordered set P into itself has a fixed point.* In this paper there is introduce a concept of a *fixed edge* for the mapping of a (finite) partially ordered set into itself. The aim of this paper is to investigate conditions under which a mapping of a finite poset into itself has a fixed edge. The Fixed Edge Theorem is proved: *Every antitone mapping of a finite, dismantlable by irreducibles, partially ordered set P into itself has a fixed edge.*

Let P be a partially ordered set. Let f be a mapping of a poset P into itself and let $u \leqq v$ be elements of P. An ordered pair (u, v) is called a *fixed edge* of f if $f(u) = v$ and $f(v) = u$. Set P is said to have the *fixed edge property* if every antitone map of P into itself has a fixed edge.

In a noted paper [1] Baclawski and Björner (also see [4] Kurepa and [3] Klimeš) has shown that every *antitone map of a complete lattice into itself has a fixed edge.*

P is called *connected* if for all $a, b \in P$ there is a sequence $a = a_0, a_1, \ldots, a_n = b$ of elements of P such that a_i is comparable with a_{i+1} $(i = 0, 1, \ldots, n-1)$; otherwise, P is disconnected.

An element x of a finite poset P is said to *cover* y in P (or x is an upper cover of y or y is a lower cower of x) if $y < x$ and $y < z \leqq x$ implies $z = x$; x is *irreducible* in P if x has precisely one upper cover or precisely one lower cover in P.

A nonempty subset Q of P is obtained from P by *dismantling by irreducibles* if $P \backslash Q = \{a_1, a_2, \ldots, a_n\}$ and

$$a_i \in I(P \backslash \{a_1, a_2, \ldots, a_{i-1}\}), \qquad (i = 1, 2, \ldots, n),$$

where $I(P)$ denote the set of irreducible elements of P. We call P *dismantlable by irreducibles* if a singleton subset of P is obtained from P by dismantling by irreducibles. Note that a dismantlable partially ordered set is connected. For $n \geqq 4$ a subset $C = \{c_1, c_2, \ldots, c_n\}$ of P is a *crown* provided that $c_1 < c_n$ and $c_1 < c_2, c_2 > c_3, \ldots, c_{n-2} > c_{n-1}, c_{n-1} < c_n$ are the only comparability relations that hold in C and, in the case $n = 4$, there is no $a \in P$ such that $c_1 < a < c_2$, $c_3 < a < c_4$ (see Rival [5], Fig. 1).

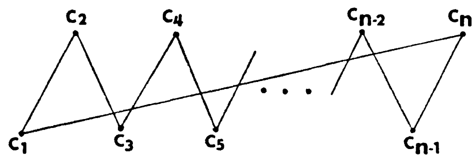

Fig. 1

The following result is proved in [5].

Theorem 1. (Rival [5]) *Let P be a finite, connected, partially ordered set of length one. The following conditions are equivalent:*

(FP) *P has the fixed point property,*

(DI) *P is dismantlable by irreducibles,*

(NC) *P does not contain a crown.*

II

Let $(P, \leqq)$ be a partially ordered set. For $x, y \in P$ and $x < y$, the set $]x, y[$ is defined by

$$]x, y[:= \{t : t \in P \text{ and } x \leqq t \leqq y\}.$$

We begin with a statements for conditionally complete sets (that is, every nonempty subset of P with upper bound has its supremum).

Lemma 1. *Let $(P, \leqq)$ be a partially ordered set and f an isotone mapping from P into P such that:*

(A) *f has a fork* i.e. *$a \leqq f(a) \leqq f(b) \leqq b$ for some $a, b \in P$,*
and

(B) *The set $]a, b[$ (or P) is a conditionally complete. Then the set $P(f) :=$ $:= \{x \in P: f(x) = x\}$ is nonempty.*

Proof. Since $a \leqq f(a)$ for some $a \in P$, then we have $a \leqq f(a) \leqq f(f(a)) \leqq$ $\leqq \ldots \leqq f(b) \leqq b$. Hence, the set S of elements $x = f^n(a) \in]a, b[$ for $n =$ $= 0, 1, 2, \ldots$, such that $x \leqq f(x)$ is nonempty and bounded from above, and $s = \vee S$ exists, by conditionally completeness of $]a, b[$. Since $f: P \to P$ is isotone and $x \leqq s$ for all $x \in S$, $x \leqq f(x) \leqq f(x)$ for all $x \in S$; hence $s = \vee S \leqq f(s)$. Since f is isotone, it follows, that $f(s) \leqq f(f(s))$ whence $f(s) \in S$. But this implies $f(s) \leqq s$, since $s = \vee S$. We conclude $s = f(s)$ i.e. $s \in P(f)$, therefore $P(f)$ is a nonempty. This completes the proof of Lemma 1.

Lemma 2. (Fixed Edge Lemma) *Let $P, \leqq)$ be a conditionally complete partially ordered set and f an antitone mapping from P into P such that f has a fork type*

(C) *$a \leqq f(b) \leqq f(a) \leqq b$ for some $a, b \in P$.*

Then there exists a fixed edge (u, v) of f and there exists an u with the least element in $]a, b[$ such that $(u, f(u))$ is the fixed edge of f.

Proof. Let $A = \{x \in]a, b[: x \leqq f^2(x)\}$ and $B = \{x \in]a, b[: f^2(x) \leqq x\}$. Hence, from (C),

$$A \supseteqq \{a, f(b), f^2(a), f^3(b), f^4(a), \ldots\},$$

and

$$B \supseteqq \{b, f(a), f^2(b), f^3(a), f^4(b), \ldots\}.$$

The sets A, B are bounded. Let $u = \wedge B$ and $v = \vee A$. According to Lemma 1 we can see that $u = f^2(u)$ and $v = f^2(v)$, since f^2 is an isotone mapping. Hence $u, v \in B$ and therefore $u \leqq v$. The preceding argument, which is due to Lemma 1, shows that $f(u) \in A$ and $f(v) \in B$. Hence, $u \leqq f(v) \leqq f^2(u) = u$, $f^2(v) = v \leqq$ $\leqq f(u) \leqq v$. It implies $u = f(v)$, $v = f(u)$, i.e. (u, v) is a fixed edge of f. If (x, y) is any edge of f with $x, y \in]a, b[$ then $x \in B$. Hence $u \leqq x$, which completes the proof.

Lemma 3. *Let P be a finite partially ordered set and f an antitone map of P into itself. If P is dismantlable by irreducibles then P is a conditionally complete set and f has a fork type (C).*

Proof. It is simple to prove that every finite dismantlable by irreducibles set is conditionally complete, so we omit the proof. Now let us prove that f has a fork. We proceed by induction on the number of elements of the set P, i.e. P_n $(n \in \mathbf{N})$. For $n = 2$: $P_2 = \{c_1, c_2\}$ is a connected set by the figure 2. The only three possible antitone mappings of the set P_2 into itself are $f(c_1) = c_2$, $f(c_2) = c_1$ or $f(c_1) = c_2$, $f(c_2) = c_2$ or $f(c_1) = c_1$, $f(c_2) = c_1$. All these mappings have forks.

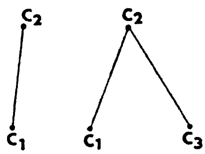

Fig. 2

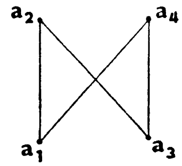

Fig. 3

The inductive hypothesis is that the statement is true in the case of $n-1$ elements set P i.e. P_{n-1}. If P has **n** elements, then every antitone mapping $f: P_n \to P_n$ has a fork since the restriction (of f on P_{n-1}) $f \mid P_{n-1}$ has a fork (by the inductive hypothesis). Therefore the extension of that mapping f to P_n has a fork. This proves the statement.

Remark. The following will show that P *contains no crown* i.e. dismantlable by irreducibles of Lemma 3 may be dropped.

Example. Define posets P_n for $n = 4$ by the diagram of Fig. 3. Then, P_4 is finite, connected partially ordered sets of length one with a crown and P_4 is a not conditionally complete set. Let f: $P_4 \to P_4$ defined by $f(a_1) = a_2$, $f(a_2) = a_3$, $f(a_3) = a_4$, and $f(a_4) = a_1$. Then f is an antitone mapping without forks (C). Also f has not fixed edge.

An immediate corollary of the preceding statements i.e. Lemma 2 and Lemma 3 is the following result.

Theorem 2. (Fixed Edge Theorem) *Let P be a finite partially ordered set and f an antitone map of P into itself. If P is dismantlable by irreducibles then there exists a fixed edge of f.*

Also, an immediate corollary of the preceding Lemmas, Theorems 1, 2 and some results of [5] is the following statement.

Theorem 3. (Fixed Edge Alternative) *Let P be a finite, connected partially ordered set and f an antitone map of P into itself. Then there exists a fixed edge of f or there exist connected subsets C, G of P of length one with $I(C) = I(G) = \Phi$ such that $f(C) = G$, $f(G) = C$.*

Proof *of Theorem* 3. Let $g := f^2$ and $Q := g^n(P)$ be the subset of P guaranted by Lemma 8 of [5] and let G be the set of all elements maximal or minimal in Q. Also, from [5], Lemma 7 ensures that G is connected. Moreover, since $g \mid Q$ is an isomorphism, $f^2(G) = G$. Let $G = G_0 \supset G_1 \supset \ldots \supset G_n = C$ be the maximal descending chain satisfying $G_{i-1} \backslash G_i = I(G_{i-1})$. Since $g \mid G$ is an isomorphism we have that $f^2(I(G)) = I(G)$ and iterating $f^2(I(G_{i-1})) = I(G_{i-1})$ for each $i = 1, 2, \ldots, n$; hence $f^2(C) = C$. Hence, $f^2(C) = C \subset G = f^2(G)$. Analogous the proof of Lemma 2, also, we have $C \subset f(G) \subset f^2(C) = C$ and $f^2(G) \subset f(C) \subset G$. It implies $C = f(G)$ and $G = f(C)$. Also, by [5], C and G are nonempty and connected. Finally, the maximality of the chain implies that $I(C) = I(G) = \Phi$. This proves the theorem.

This proof is analogous of the proof of Proposition 9 of [5].

We want to remark that the corresponding assertion by Rival (Proposition 9 of [5], p. 317) can be proved in another way. Namely we can apply the Lemma 1 and the same method as in the proof of the Theorem 2.

REFERENCES

[1] K. Baclawski and A. Björner, *Fixed points in partially ordered sets*, Advances in Math. 31 (1979), 263–287.

[2] C. Blair and A. Roth, *An extension and simple proof of a constrained lattice fixed point theorem*, Algebra Universalis 9 (1979), 131–132.

[3] J. Klimeš, *Fixed edge theorems for complete lattices*, Arch. Math. 4. scripta, 17 (1981), 227–234.

[4] D. Kurepa, *Fixpoints of decreasing mappings of ordered sets*, Publ. Inst. Math., 32 (1975), 111–116.

[5] I. Rival, *A fixed point theorem for finite partially ordered sets*, J. Combin. Theory, 21 (A), 1976, 309–318.

[6] A. Tarski, *A lattice theoretical fixpoint theorem and its applications*, Pacific J. Math., 5 (1955), 283–309.

[7] A. Björner, *Order-reversing maps and unique fixed points in complete lattices*, Algebra Universalis 12 (1981), 402–403.

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