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RETRACTS OF ABELIAN CYCLICALLY ORDERED GROUPS

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Dedicated to the memory of Professor Milan Sekanina

Abstract. In this paper it will be shown that a nonzero subgroup H of a cyclically ordered group G is a retract of G if and only if H is a large lexicographic factor of G.

Key words. Cyclically ordered group, retract, retract mapping, lexicographic product.
MS Classification. 06 F 20, 46 A 40.

Cyclically ordered groups were investigated in [1], [9], ..., [15]. The notion of cyclically ordered group is a generalization of the notion of linearly ordered group.

Retracts of partially ordered sets were studied in [2], ... ,[5].

Retracts of lattice ordered groups, and in particular, of linearly ordered groups, were investigated in [4]; cf. also [5].

All cyclically ordered groups dealt with in the present note are assumed to be abelian.

Let G be a cyclically ordered group. An endomorphism f of G will be said to be a retract mapping if $f(f(x) = f(x)$ for each $x \in G$. In such a case, the set $f(G)$ is called a retract of G.

It will be shown that to each retract of G there corresponds a two-factor lexicographic decomposition of G. More thoroughly, each retract mapping of G is a projection onto a large lexicographic factor of G, and conversely. This generalizes a result of [7] concerning retracts of linearly ordered groups.

1. PRELIMINARIES

For the sake of completeness we recall the definition of cyclically ordered group.

Let G be a group (the group operation will be denoted additively). Suppose that there is defined a ternary relation $[x, y, z]$ on G such that the following conditions are satisfied for each x, y, z, a, b a G:

I. If $[x, y, z]$ holds, then x, y and z are distinct; if x, y and z are distinct, then either $[x, y, z]$ or $[z, y, x]$.

II. $[x, y, z]$ implies $[y, z, x]$.
III. $[x, y, z]$ and $[y, u, z]$ imply $[x, u, z]$.
IV. $[x, y, z]$ implies $[a + x + b, a + y + b, a + z + b]$.

Under these assumptions G is said to be a cyclically ordered group; the ternary relation under consideration is said to be a cyclic order on G.

Each subgroup of G is considered as to be cyclically ordered under the induced cyclic order. The isomorphism of cyclically ordered groups is defined in the obvious way.

Let G and G' be cyclically ordered groups. A mapping $f : G \to G'$ is said to be a homomorphism if the following conditions are satisfied:

(i) f is a homomorphism with respect to the group operation;

(ii) whenever x, y and z are elements of G such that $[x, y, z]$ holds in G and the elements $f(x)$, $f(y)$, $f(z)$ are distinct, then the relation $[f(x), f(y), f(z)]$ is valid in G'.

Let L be a linearly ordered group. For distinct elements x, y and z of L we put $[x, y, z]$ if

$$x < y < z \quad \text{or} \quad y < z < x \quad \text{or} \quad z < y < x \tag{1}$$

is valid. Then G with the relation [ ] (which is said to be induced by the linear order) turns out to be a cyclically ordered group.

2. LEXICOGRAPHIC PRODUCTS

Let G_1 be a cyclically ordered group and let L be a linearly ordered group (each linearly ordered group is considered as to be cyclically ordered under the induced cyclic order).

Let $G_1 \times L$ be the (external) direct product of the groups G_1 and L. For distinct elements $u = (a, x)$, $v = (b, y)$ and $w = (c, z)$ of $G_1 \times L$ we put $[u, v, w]$ if some of the following conditions is satisfied:

(i) $[a, b, c]$;
(ii) $a = b \neq c$ and $x < y$;
(iii) $b = c \neq a$ and $y < z$;
(iv) $c = a \neq b$ and $z < x$;
(v) $a = b = c$ and $[x, y, z]$.

It is easy to verify that $G_1 \times L$ with this ternary relation is a cyclically ordered group; it will be denoted by $G_1 \oplus L$ and it is said to be a lexicographic product of G_1 and L. We call G_1 and L the large lexicographic factor or the small lexicographic factor of $G_1 \oplus L$, respectively.

If $G = G_1 \oplus L$, $g \in G$, $g = (u, x)$, then we denote $u = g(G_1)$ and $x = g(L)$.

Let us remark that if H_1 and H_2 are linearly ordered groups and if H is their lexicographic product $H_1 \circ H_2$ (cf., e.g., Fuchs [6]), then the cyclically ordered group H is a lexicographic product $H_1 \oplus H_2$ of the cyclically ordered groups H_1 and H_2, and conversely.

The following assertion is obvious.

2.1. Lemma. *Let G_1 be cyclically ordered groups and let L be a linearly ordered group. Put $G = G_1 \oplus L$ and for each $g \in G$ let $f(g) = g(G_1)$. Then f is a retract mapping of G.*

Let G_1 and L be as above and let φ be an isomorphism of a cyclically ordered group G onto $G_1 \oplus L$. Put

$$G_1^0 = \varphi^{-1}\{(a, 0) : a \in G_1\},$$
$$L^0 = \varphi^{-1}\{(0, x) : x \in L\}.$$

Then G_1^0 is isomorphic to G_1, and L^0 is isomorphic to L^0. The mapping

$$\varphi' : G \to G_1^0 \oplus L^0$$

defined by $\varphi'(g) = a^0 + x^0$, where $\varphi(g) = (a, x)$, $a^0 = \varphi^{-1}((a, 0))$ and $x^0 = \varphi^{-1}((0, x))$, is an isomorphism of G onto $G_1^0 \oplus L^0$. In such a case we write $G = G_1^0 \oplus_i L^0$ and G is said to be an internal lexicographic product of G_1^0 and L^0.

Analogously as above, G_1^0 and L^0 are called a large lexicographic factor and a small lexicographic factor of G, respectively.

In view of 2.1 we obtain:

2.2. Corollary. *Each large lexicographic factor of a cyclically ordered group G is a retract of G.*

Internal lexicographic product decompositions can be characterized intrinsically as follows.

2.3. Proposition. *Let G be a cyclically ordered group. Let G_1 and L be subgroups of G such that L is linearly ordered. Then the following conditions are equivalent:*

(a) $G = G_1 \oplus_i L$.

(b) The group G is an internal direct product of its subgroups G_1 and L. Whenever u, v and w are distinct elements of G with $u = a + x$, $v = b + y$, $w = x + z$ (where $a, b, c \in G_1$ and $x, y, z \in L$), then $[u, v, w]$ is valid if and only if some of the relations (i)–(v) above holds.

The proof can be performed by a routine verification. (Cf. also [10].)

Let us denote by K the set of all real numbers x with $0 \leq x < 1$; the operation $+$ on K is defined to be the addition mod 1. For distinct elements x, y and z of K we put $[x, y, z]$ if the relation (1) above is valid. Then K is a cyclically ordered group.

2.4. Theorem. (Cf. [12].) *Let G be a cyclically ordered group. Then there exist a subgroup K_1 of K and a linearly ordered group L such that G is isomorphic to $K_1 \oplus L$.*

A subgroup H of a cyclically ordered group G is said to be c-convex (cf. [9]) if some of the following conditions is fulfilled:

(i) $H = G$;

(ii) for each $h \in H$ with $h \neq 0$ we have $2h \neq 0$; if $h \in H$, $g \in G$, $[-h, 0, h]$ and $[-h, g, h]$, then $g \in H$.

The following lemma is an easy consequence of 2.4.

2.5. Lemma. *Let f be an endomorphism of a cyclically ordered group G. Then the kernel of f is a c-convex subgroup of G.*

3. LARGE LEXICOGRAPHIC FACTOR CORRESPONDING TO A GIVEN NONZERO RETRACT MAPPING

Let G be a cyclically ordered group. In view of the consideration performed in Section 2 and according to 2.4 there exist subgroups G_1 and L of G such that

(i) G_1 is isomorphic to a subgroup of K;

(ii) L is linearly ordered;

(iii) $G = G_1 \oplus_l L$.

3.1. Lemma. *Let f be an endomorphism of G. Then either $f(G) = \{0\}$ or $f^{-1}(0) \subseteq L_1$.*

Proof. This is a consequence of 2.5, and [9] (3.5 and 4.6).

An endomorphism f of G is said to be nonzero if $f(G) \neq \{0\}$. In what follows we assume that f is a nonzero endomorphism of G.

3.2. Lemma. *Assume that f is a retract mapping of G. Then $f(x) \in L$ for each $x \in L$.*

Proof. By way of contradiction, assume that there exists an element $x \in L$ such that $f(x) \notin L$. Thus there are $a \in G_1$ and $y \in L$ with $f(x) = a + y$, $a \neq 0$. This yields that $f(a + y) = a + y$, hence $f(a + y - x) = 0$. The element $a + y - x$ does not belong to L, therefore the kernel of f fails to be a subset of L. In view of 3.1, $f(G) = \{0\}$, which is a contradiction.

Denote $f_2 = f \mid L$. According to 3.2 we have

3.3. Corollary. *Let f be as in 3.2. Then f_2 is a retract mapping of L.*

3.4. Lemma. *Let f be as in 3.2. Next let $f_1 = f \mid G_1$. Then f_1 is an isomorphism of G_1 onto $f(G_1)$.*

Proof. According to the definition, f_1 is a homomorphism of G_1 onto $f(G_1)$. Let $a \in G_1$, $a \neq 0$, $f(a) = a_1 + x$, $a_1 \in G_1$, $x \in L$. Hence $f(a_1 + x) = a_1 + x$, thus $f(-a + a_1 + x) = 0$. In view of 3.1, $-a + a_1 + x \in L$ and therefore $a = a_1$. Hence $f(a) \neq 0$. Thus f_1 is a monomorphism. By summarizing, f_1 is an isomorphism.

We have clearly $f(G_1) \cap L = \{0\}$. If $g \in G$ and $g = a + x$, $a \in G_1$, $x \in L$, $f(a) = a + x_1$, then $g = (a + x_1) + (-x_1 + x)$ with $a + x_1 \in f(G_1)$ and $-x_1 + x \in L$. Hence we infer:

3.5. Lemma. *The group G is a direct product of the groups $f(G_1)$ and L.*

3.6. Lemma. *Let f be as in* 3.2. *Then $G = f(G_1) \oplus_i L$.*

The proof consists in a routine verification by applying 3.5 and 2.3.

3.7. Lemma. *Let f_2 be as above. There are subgroups L_1 and L_2 of L such that $f_2(L) = L_1$ and $L = L_1 \oplus_i L_2$.*

Proof. Since L is linearly ordered and since in view of 3.2, f_2 is a retract mapping of L as cyclically ordered group, it is also a retract mapping of L as linearly ordered group. Thus, according to [7], Theorem 3.4, there are l-subgroups L_1 and L_2 of L such that

$$L = (i)\, L_1 \circ L_2, \tag{2}$$

(an internal lexicographic product of linearly ordered groups L_1 and L_2, cf. [7]). From (2) we obtain that the relation

$$L = L_1 \oplus_i L_2.$$

holds.

Put $L_3 = f(G_1) + L_1$. The relation $L_1 \subseteq L$ and Lemma 3.6 yield

$$L_3 = f(G_1) \oplus_i L_1. \tag{3}$$

Next, from $f(L) = L_1$ we obtain

$$f(G) = L_3. \tag{4}$$

Also, from 3.6 and 3.7 we infer that

$$G = f(G_1) \oplus_i (L_1 \oplus_i L_2). \tag{5}$$

Clearly

$$f(G_1) \oplus_i (L_1 \oplus_i L_2) = (f(G_1) \oplus_i L_1) \oplus_i L_2 = f(G) \oplus_i L_2.$$

Thus in view of (5) we obtain

$$G = f(G) \oplus_i L_2. \tag{6}$$

Let $g \in G$. In view of (6) there are uniquely determined elements $a \in g(G)$ and $x \in L_2$ such that $g = a + x$. Then $f(a) = a$. Next we have $f(x) \in f(G)$ and in view of 3.2, $f(x) \in L_2$. Hence $f(x) \in f(G) \cap L_2 = \{0\}$ and so $f(x) = 0$. We obtain

$$f(g) = f(a) + f(x) = a.$$

By summarizing, we have the following result:

3.8. Theorem. *Let f be a nonzero retract mapping of an abelian cyclically ordered group G. Then the retract f(G) is a large lexicographic factor of G and for each $g \in G$, f(g) is the component of the element g in the factor f(G).*

Theorem 3.8 and Lemma 3.1 yield:

3.9. Corollary. *Let G be an abelian cyclically ordered group and let $H \neq \{0\}$ be an l-subgroup of G. Then the following conditions are equivalent:*

(i) *H is a retract of G.*

(ii) *H is a large lexicographic factor of G.*

This generalizes Theorem 3.4, [7] concerning retracts of linearly ordered groups.

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