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Abstract. Quantales can be viewed as a framework for noncommutative topology. The notion of a w-quantale is defined. Conjunctivity in quantales is developed. Weakly dually atomic w-quantales are characterized. Normal quantales are considered. Any normal quantale is a preliminary w-quantale. Any dually atomic normal quantale is a frame. An explicit description of the regular coreflection of a certain class of normal quantales is given.

Key words. Frame, quantale, w-quantale, q-quantale, (weakly) dually atomic quantale, congruence, nucleus, conjunctive frame, normal quantale, primitive element, preliminary quantale.

MS Classification. 06 F 99

The lattice-theoretical investigations of complete lattices equipped with an additional binary operation . which distributes over $\bigvee$ can be traced back to Ward and Dilworth [14]. Such a gadget is called a quantale (following C. J. Mulvey). Some topological properties of quantales were obtained by Borceux [4].

In this paper we introduce a certain class of quantales – the w-quantales. It is shown that a weakly dually atomic quantale is a w-quantale iff dual atoms are prime elements.

Following Banaschewski and Harting [2], we develop the notion of conjunctivity in quantales. Following [5], we discuss the properties of primitive elements and the notion of a preliminary quantale is introduced. Any preliminary weakly dually atomic quantale is a w-quantale.

In connection with [12] we consider normal quantales. Any normal quantale is a preliminary w-quantale. Any dually atomic normal quantale is a frame. An explicit description of the regular coreflection of a certain class of normal quantales is given.

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All terminology and notation on quantales which is not explained here is taken from [4] or [12].

§ 1. CONJUNCTIVITY IN QUANTALES

1.1. Remark. A quantale K is called a *w-quantale* (a *q-quantale*) if $a \vee b = 1$ implies (iff) $a \,.\, 1 \vee b = 1$ for all $a, b \in K$. Clearly, if a quantale K is idempotent or a q-quantale then K is a w-quantale. We can easily see that every 2-sided quantale is a q-quantale.

We say that an element $p \neq 1$ of a quantale K is *prime* if $a \,.\, b \leqq p$ implies $a \leqq p$ or $b \leqq p$ for all $a, b \in K$. The set of all prime elements will be denoted by $P(K)$.

Finally, let us recall that congruences on quantales are congruences with respect to . and $\bigvee$.

1.2. Lemma. *Let K be a quantale. Then the following holds*:

(*i*) *K is a w-quantale iff $a \vee c = 1 = b \vee c$ implies $a \,.\, b \vee c = 1$ for all $a, b, c \in K$.*

(*ii*) *K is a q-quantale iff $a \,.\, b = 1$ implies $a = 1 = b$ for all $a, b \in K$.*

Proof. (i) Let $a, b, c \in K$ such that $a \vee c = 1 = b \vee c$. Then $1 = a \,.\, 1 \vee c = a \,.\, b \vee a \,.\, c \vee c = a \,.\, b \vee c$. Conversely, let $a, b \in K$ so that $a \vee b = 1$. Clearly, $1 = 1 \vee b$ i.e. $1 = a \,.\, 1 \vee b$.

(ii) Let $a, b \in K$, $a \,.\, b = 1$. Clearly b $= 1$. Then $a \,.\, 1 \vee 0 = 1$ and this implies $a = 1$. Conversely, let $a, b \in K$, $a \vee b = 1$. Then $1 = (a \vee b) \,.\, 1 = (a \,.\, 1 \vee b) \,.\, 1$ i.e. $a \,.\, 1 \vee b = 1$. Now, let $a \,.\, 1 \vee b = 1$. Then $1 = (a \,.\, 1 \vee b) \,.\, 1 = (a \vee b) \,.\, 1$ i.e. $a \vee b = 1$.

1.3. Remark. Let us note that, for a quantale K, $D(K)$ is the set of all dual atoms of K. We say that K is weakly dually atomic if for any $a \neq 1$, $a \in K$ there is a dual atom $m \in D(K)$ such that $a \leqq m$. Following [12, 1.4. Lemma] it is easy to verify that if K is a w-quantale then every dual atom is prime.

1.4. Proposition. *Let K be a weakly dually atomic quantale. Then the following are equivalent*:

(i) *K is a w-quantale.*

(ii) *Every dual atom is prime i.e. $D(K) \subseteq P(K)$.*

Proof. (i) $\Rightarrow$ (ii). It follows from 1.3.

(ii) $\Rightarrow$ (i). Let $a \vee b = 1$, $a \,.\, 1 \vee b \neq 1$ for some $a, b \in K$. Then there exists a dual atom $m \in D(K)$ such that $a \,.\, 1 \vee b \leqq m$. Evidently, $a \leqq m$ because m is prime i.e. $1 = a \vee b \leqq m$, a contradiction.

1.5. Examples. (i) For any ring A with a unit the quantale $Lid(A)$ of all left ideals of A is a compact w-quantale.

(ii) For any C*-algebra A the quantale $L(A)$ of all closed left ideals of A is an idempotent weakly dually atomic quantale, which has enough points (see [4]). It is easy to prove that $L(A)$ is a q-quantale iff $L(A)$ is a frame i.e. A is a commutative C*-algebra.

1.6. Definition. Let K be a quantale, $j : K \to K$ an operator on K satisfying

(i) $a \leqq j(a)$

(ii) $j(a) = j(j(a))$

(iii) $j(a) . j(b) \leqq j(a . b)$

(iv) $a \leqq b$ implies $j(a) \leqq j(b)$

for all $a, b \in K$. We say that j is *a nucleus* on K. Let us put $K_j = \{a \in K; a = j(a)\}$. Recall that there is a natural $1 - 1$ correspondence between the congruences and the nuclei.

For an equivalence relation R on K we denote $\bar{R}$ the least congruence relation generated by R and j_R the nucleus which corresponds to $\bar{R}$.

We say that R is *multiplicative* if aRb, cRd implies $a . cRb . d$ for all $a, b, c, d \in K$.

1.7. Observation. *Let K be a quantale,* R a multiplicative equivalence on K. Then *for any $a \in K$ the following holds*:

$j_R(a) = a$ iff *(xRy implies $x \leqq a$ iff $y \leqq a$ for all $x, y \in K$).*

Proof. The proof immediately follows the idea of Kříž (see [11]).

1.8. Remark. Recall that a frame K is said to be *conjunctive* if for each two elements $a, b \in K$, $a \nleqq b$ there is an element $c \in K$ such that $a \vee c = 1$, $b \vee c \neq 1$. Such frames were studied by Isbell [7] who called them *subfit*. They were renamed by Simmons [13] because the defining property is dual to *Wallman's disjunctivity.* Later, they were studied by Banaschewski and Harting [2] in the context of compact frames.

1.9. Definition. Let K be a quantale. We put aSb iff ($a . 1 \vee c = 1$ iff $b . 1 \vee c = 1$ for any $c \in K$) for all $a, b \in K$. Obviously, S is a multiplicative equivalence relation on K. We define $s = j_s$. Recall that, for a frame K, K is conjunctive iff $K = K_s$ i.e. s is the identity mapping on K.

1.10. Proposition. *Let K be a quantale. Then K_s is a frame.*

Proof. Let $u \in K_s$. Then $uSu . 1$, $uSu . u$. In fact we obtain that $u = s(u) = s(u . 1) = u . 1 = s(u . u) = u . u$. Consequently, every element of K_s is *2-sided* and *idempotent* i.e. K_s is a frame.

1.11. Proposition. *Let K be a weakly dually atomic quantale. Then S is a congruence on K.*

Proof. We have to show that $x_i S y_i$, x_i, $y_i \in K$ implies xSy; here $x = \bigvee x_i$, $y = \bigvee y_i$. Let $x . 1 \vee c = 1$, $y . 1 \vee c \neq 1$ for some $c \in K$. Then there exists a dual atom $m \in D(K)$ such that $y . 1 \vee c \leqq m$, $x . 1 \nleqq m$. Evidently, there exists x_i such that $x_i . 1 \nleqq m$ i.e. $x_i . 1 \vee m = 1$. Now, we have $y_i . 1 \vee m = 1$ i.e. $y . 1 \vee m = 1$, a contradiction. The symmetiy argument concludes the proof.

1.12. Remark. Let us note that the preceding proposition can be proved for compact frames without the Axiom of Choice (see [2]). Following [2], we can easily verify that the same holds for compact quantales.

1.13. Definition. An element a of a quantale K is said to be *primitive* if there is a dual atom $m \in D(K)$ such that $a = \tilde{m}$ is the greatest 2-sided element lying under m. The set of all primitive elements of K will be denoted by $Prim(K)$.

Recall that, for a quantale K, $\tilde{K}$ will denote the quantale of all 2-sided elements of K.

1.14. Proposition. *Let K be a quantale, $m \in D(K)$. Then the following are equivalent:*

(i) $m \in P(K)$.

(ii) $\tilde{m} \in K_s$.

(iii) $\tilde{m} \in P(K)$.

(iv) $\tilde{m} \in P(\tilde{K})$.

Proof. (i) $\Rightarrow$ (ii). Let $a = \tilde{m}$, xSy, $x \leqq a$. Then $y . 1 \leqq m$. Namely, if $y . 1 \vee m = 1$ then $x . 1 \vee m = 1$, which is a contradiction with $x . 1 \leqq a \leqq m$. Since m is prime we have $y \leqq m$ i.e. $y \vee y . 1 \leqq m$. Now, we have $y \leqq y \vee y . 1 \leqq a$ because $y \vee y . 1$ is 2-sided. The rest follows from 1.6.

(ii) $\Rightarrow$ (iii). Let $a = \tilde{m}$, $a \in K_s$, $x . y \leqq a$, $x \nleqq a$, $y \nleqq a$. Hence $x . 1 \nleqq a$, $y . 1 \nleqq a$ because $a \in K_s$. Since $a \in Prim(K)$ we have $x . 1 \vee m = 1 = y . 1 \vee m$ i.e. $1 = x . y . 1 \vee x . m \vee m \leqq a \vee m = m$, a contradiction.

(iii) $\Rightarrow$ (iv). It is evident.

(iv) $\Rightarrow$ (i). Let $x, y \in K$, $x . y \leqq m$, $x \nleqq m$, $y \nleqq m$. Then $x . y . 1 \leqq m$. Namely, if $x . y . 1 \vee m = 1$ then $1 = x . y . y \vee x . y . m \vee m = x . y \vee m$, a contradiction. Now, we have $(x \vee x . 1) . (y \vee y . 1) = x . y \vee x . y . 1 \leqq m$. Evidently, $(x \vee x . 1) . (y \vee y . 1) \leqq \tilde{m}$. Since $\tilde{m}$ is prime in $\tilde{K}$ and we have $x \vee x . 1$, $y \vee y . 1 \in \tilde{K}$ then $x \leqq x \vee x . 1 \leqq \tilde{m} \leqq m$ or $y \leqq y \vee y . 1 \leqq \tilde{m} \leqq m$, a contradiction.

1.15. Corollary. *Let K be a weakly dually atomic quantale. Then K is a w-quantale iff $Prim(K) \subseteq K_s \cap P(K)$.*

1.16. Corollary. *Let K be a quantale, $m \in D(K)$. Then m is 2-sided iff $m \in K_s$.*

Recall that, for a quantale K, K is said to be *simple* (see [4], [5]) if $\tilde{K} = \{0, 1\}$. Now, we adopt the following:

1.17. Definition. Let K be a quantale, $a \in \tilde{K}$. We say that the set $\uparrow(a) = \{x \in K; x \geqq a\}$ is *simple* if $\tilde{K} \cap \uparrow(a) = \{a, 1\}$.

The following generalizes the concept of a liminary quantale introduced in [5]. Let us recall that a quantale is said to be *liminary* if $\uparrow(a)$ is *atomic* (i.e. any element is a join of atoms) and *simple* for any $a \in K$ *primitive*.

1.18. Definition. Let K be a quantale. We say that K is *preliminary* if *primitive elements* are *dual atoms* in the quantale $\tilde{K}$. Equivalently, K is *preliminary* iff $\uparrow(a)$ is *simple* for any $a \in K$ *primitive*.

1.19. Proposition. *Let K be a preliminary quantale. Then any primitive element is prime i.e. $Prim(K) \subseteq P(K)$. Moreover, if K is weakly dually atomic then K is a w-quantale.*

Proof. Let $x, y \in K$, $a = \tilde{m}$, $m \in D(K)$, $x . y \leqq a$, $x \nleqq a$, $y \nleqq a$. Then $a < a \vee x \vee x . 1 \in \tilde{K}$. Clearly, $x . 1 \vee x \vee a = 1$ i.e. $x . 1 \vee a = 1$. By the same argument we have $y . 1 \vee a = 1$. Hence $1 = x . y . 1 \vee x . a \vee a = x . y . 1 \vee a = a$, a contradiction. The rest follows immediately from 1.15.

1.20. Theorem. *Let K be a weakly dually atomic quantale. Then the following are equivalent*:

(i) K is a w-quantale.

(ii) $s(a) = \bigwedge\{u \geqq a; u \in Prim(K)\}$ *for any* $a \in K$.

Proof. (i) $\Rightarrow$ (ii). Let us put $p(a) = \bigwedge\{u \geqq a; u \in Prim(K)\}$ for any $a \in K$. We will show that $aSp(a)$ for all $a \in K$.

Suppose that non $aSp(a)$ for some $a \in K$. Then there exists an element $b \in K$ such that $p(a) . 1 \vee b = 1$, $a . 1 \vee b \neq 1$. Now, there exists a dual atom $m \in D(K)$, $a . 1 \vee b \leqq m, p(a) . 1 \nleqq m$. We put $u = \tilde{m}$. Then $a . 1 \leqq u, p(a) . 1 \nleqq u$ i.e. $a \leqq u$, $p(a) \nleqq u$, a contradiction. Consequently, $p(a) \leqq s(a)$. Conversely, $p(a) \geqq a$, $p(a) \in K_s$ i.e. $s(a) \leqq p(a)$. This proves that $s(a) = p(a)$ for all $a \in K$. (ii) $\Rightarrow$ (i). It is evident.

1.21. Proposition. *Let K be a compact quantale. Then K_s is a compact frame.*

Proof. Let $a_i \in K_s \subseteq \tilde{K}$, $\bigvee_{K_s} a_i = 1$. Since K is weakly dually atomic we have that a_iS1 i.e. $1 = 1 . 1 \vee 0 = (\bigvee a_i) . 1 \leqq \bigvee a_i$. Consequently, we have $\bigvee a_i = 1$ and by compactness in K there exists a finite set $a_{i_1}, \ldots, a_{i_n}$ such that $1 = \bigvee_{j=1}^{n} a_{i_j}$.

Recall that a quantale K is said to be *normal* (see [12]) if, given $a, b \in K$ with $a \vee b = 1$, we can find $d, c \in K$ with $d . c = 0$, $d \vee a = 1 = b \vee c$. It is trivial to check that a quantale K is normal iff given $a, b \in K$ with $a \vee b = 1$, we have $d, c \in \tilde{K}$ such that $d . c = 0$, $d \vee a = 1 = b \vee c$.

1.22. Proposition. *Let K be a normal quantale. Then*

(i) *K is a w-quantale.*

(ii) *K is preliminary.*

(iii) *Given two elements $a, b \in K$, then $a \vee b = 1$ implies $h(a) \vee b = 1$; here $h(a)$ is the greatest regular element below a (see* [12]).

Proof. (i) Let $a, b \in K$, $a \vee b = 1$. Then there exist $c, d \in K$ such that $a \vee d = 1 = b \vee c, d \,.\, c = 0$. Evidently, $1 = 1 \,.\, 1 = (a \vee d) \,.\, (b \vee c) = a \,.\, c \vee d \,.\, c \vee b = a \,.\, 1 \vee b$.

(ii) Let $m \in D(K)$, $\tilde{m} < a \neq 1$, $a \in \tilde{K}$. Then $a \vee m = 1$ i.e. there exist $c, d \in \tilde{K}$ such that $d \,.\, c = 0$, $a \vee d = 1 = c \vee m$. Obviously, $d \leqq m$ i.e. $d \leqq \tilde{m}$. Now, we have $1 = d \vee a = \tilde{m} \vee a = a$, a contradiction.

(iii) Let $a, b \in K$, $a \vee b = 1$. Then there exist $c, d \in K$ such that $d \,.\, c = 0$, $d \vee a = 1 = c \vee b$. Obviously, since K is a w-quantale we have $1 = c \,.\, 1 \vee b = c \,.\, d \vee c \,.\, a \vee b = c \,.\, d \,.\, c \vee c \,.\, d \,.\, b \vee c \,.\, a \vee b = c \,.\, a \vee b$. Evidently, $c \,.\, a \leqq a$, $c \,.\, a \lhd a$ i.e. $h(a) \vee b = 1$.

As a consequence of the above Proposition 1.22 we obtain the following:

1.23. Corollary. *Let K be a quantale. Then the following are equivalent:*

(i) *K is normal.*

(ii) *Given two elements $a, b \in K$, then $a \vee b = 1$ implies that there exist elements $c, d \in RK$ such that $d \,.\, c = 0$, $d \vee a = 1 = c \vee b$; here RK is the quantale of all regular elements of K (see* [12]).

1.24. Lemma. *Let K be a normal quantale, $m \in D(K)$ such that $\tilde{m} = \bigwedge\{n; n \in D(K)\}$. Then $m = \tilde{m}$ i.e. m is 2-sided.*

Proof. Since $\tilde{m} \in P(K)$ we have from [12] 1.6 that $\tilde{m}$ is contained in exactly one dual atom i.e. $m = \tilde{m}$.

1.25. Proposition. *Let K be a dually atomic (i.e. any element is a meet of dual atoms) normal quantale. Then K is a frame.*

Proof. Clearly, $D(K) \subseteq \tilde{K}$ by Lemma 1.24. The rest is evident.

1.26. Corollary. *Let A be a unital C^*-algebra. Then the following are equivalent:*

(i) *$L(A)$ is a frame i.e. A is commutative.*

(ii) *$L(A)$ is normal.*

1.27. Proposition. *Let K be a compact quantale. Then the following are equivalent:*

(i) *K is normal.*

(ii) *$h(a) \vee h(b) = h(a \vee b)$ for all $a, b \in K$.*

Proof. Analogous to [12], Theorem 3.4.

1.28. Lemma. *Let K be a normal quantale such that S is a congruence on K. Then*

(i) *$s(a) = s(h(s(a)))$ for any $a \in K$.*

(ii) *$h(a) = h(s(h(a)))$ for any $a \in K$.*

Proof. (i) Evidently, $s(h(s(a))) \leqq s(a)$. Now, we have to show that $s(a) \,.\, 1 \vee c = 1$ implies $s(h(s(a))) \,.\, 1 \vee c = 1$ for all $c \in K$. Clearly, $s(a) \vee c = 1$ i.e. $1 = h(s(a)) \vee c = h(s(a)) \,.\, 1 \vee c = s(h(s(a))) \,.\, 1 \vee c$.

(ii) We have $h(a) \leqq h(s(h(a)))$. Let $x \leqq h(s(h(a)))$, $x \in K$, $x \lhd h(s(h(a)))$. Then $x^* \vee h(s(h(a))) = 1$ i.e. $1 = x^* \vee s(h(a)) = x^* . 1 \vee s(h(a)) . 1 = x^* \vee s(h(a) . 1 = = x^* \vee h(a) . 1$. Now, we have $x = h(a) . x \leqq h(a)$ i.e. $h(s(h(a))) \leqq h(a)$.

1.29. Theorem. *Let K be a normal quantale such that S is a congruence on K. Then the regular coreflection RK is, up to isomorphism, exactly the frame K_s.*

Proof. We denote $\bar{h} = h/K_s$, $\bar{s} = s/RK$. From 1.28 we have $\bar{h} . \bar{s} = id_{RK}$, $\bar{s} . \bar{h} = id_{K_s}$ i.e. $\bar{h}, \bar{s}$ are bijections. It is easy to check that $\bar{h}, \bar{s}$ are morphisms of quantales.

1.30. Corollary. *Let K be a normal compact quantale. Then K_s is the compact regular coreflection of K.*

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