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HAMILTONIAN LINES IN THE SQUARE OF GRAPHS. II.

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Abstract. A necessary and sufficient condition for the existence of a Hamiltonian circuit in the square of a graph whose every block is a complete graph is given.

Key words. Graph, K-graph, cactus, Hamiltonian circuit, square.

MS classification: 05 C 45.

Very well known Neuman's theorem says that the squaie of a tree with at least three vertices is Hamiltonian if and only if the given tree is caterpillar (i.e. such a tree which is either a path or after removing all end vertices we obtain a path). Its equivalent condition is that the given tree mustn't include the graph in the fig. 1a as its subgraph. Notice that both conditions consist in desciibing of the block-structure either of the permissible trees or the prohibited tree and we are not interested in the inner structure of particular blocks. On the other hand it is clear that the existence or non-existence of a Hamiltonian circuit in the square of any graph depends on the inner structure of the particular blocks (if G is a block the answer is known). We can constiuct easily the examples of the pairs of graphs whose block-structures are the same (i.e. their block-cut-vertex trees are isomorfic) and one has a Hamiltonian square and the other not (see fig. 1). Therefore the following questions: can there be found further types of graphs such that the

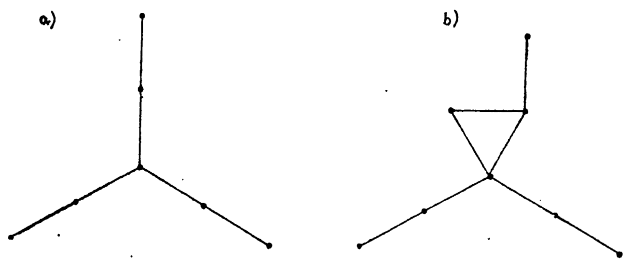

Fig. 1

existence or non-existence of a Hamiltonian circuit in the square of given graphs will depend only on the block-structure? We will deal with such question.

We use the common terminology from [1] and the following. A connected graph G is a K-graph if and only if every block of G is a complete graph. A connected graph is a cactus if and only if every block of G is an edge or a cycle (i.e. a regular graph of degree 2). Let G be a graph. A vertex v of G is free provided it is not a cut vertex. A block B is free provided at least $|V(B)| - 1$ its vertices are free in G. Otherwise B is an inner block. The set of all blocks and inner blocks is denoted by BL^G and $\overline{BL}^G$ respectively. The set of all blocks of G containing a common vertex w is denoted by $BL^G(w)$. For $BL \subseteqq BL^G$ we define $BL^G(BL, w) = BL^G(w) - BL$. If $I \subseteqq V(G)$ and B is a block of G, then for any positive integer k, $V^G(B, I, k)$ is the set of all vertices $x \in V(B) - I$ such that $|BL^G(B, x) \cap \overline{BL}^G| \geqq k$. For $k = 0$, $V^G(B, I, k)$ is the set of all vertices $x \in V(B) - I$ which are free in G. $\langle I \rangle_G$ is the subgraph of G induced by I. We say that a subgraph H of G is a BL-subgraph of G if and only if $BL^H \subseteqq BL^G$. The next used notions were defined in [2].

Definition 1. *Let G be a connected graph and x a vertex of G. A generating sequence of G from the vertex x is any sequence of graphs $G(1), \ldots, G(t) = G$ arising in the following manner.*

1. *$G(1) \bigcup_{C \in BL^G(x)} C$. The set $BL^G(x)$ is called the first growth and we say it is of the type $\{m\}$, where $m = |BL^G(x)|$.*

2. *Suppose we have constructed a graph $G(i - 1)$ and B is an arbitrary free block rom $G(i - 1)$ such that there is a vertex $b \in V(B)$ which is either a cut-vertex of $G(i - 1)$ or $b = x$ and at least one vertex of the set $V(B) - \{b\} = \{b_1, \ldots, b_n\}$ is a cut vertex of G. Then $G(i) = G(i - 1) \cup \bigcup_{j=1}^{n} \bigcup_{C \in BL^G(B, b_j)} C$ and the set $\{\bigcup_{j=1}^{n} BL^G(B, b_j)\}$ is called the i-th growth starting from the block B. It is of the type $\{m_1, \ldots, m_n\}$ if $m_j = |BL^G(B, b_j)|$ for every $j \in \{1, \ldots, n\}$.*

If there is no block B of the mentioned properties, then $G(i - 1) = G$ and the construction of a generating sequence stops.

Definition 2. *Let G be a graph, $G(1), \ldots, G(t) = G$ be any generating sequence of G from a vertex x. Suppose the i-th growth starts from a block B, $V(B) = \{b, b_1, \ldots, b_n\}$ where either $b = x$ or b is a cut vertex of $G(i - 1)$ and it is of the type $\{m_1, \ldots, m_n\}$. We say that the i-th growth is a right-growth if and only if*

1. *$m_j \leqq 2$ for each $j \in \{1, \ldots, n\}$,*
2. *$|M_0| = |M_2|$ where $M_l = \{j: j \in \{1, \ldots, n\}, m_j = l\}$, $l = 0, 2$,*
3. *all blocks of the i-th growth are free in G exepting the set of blocks $\bigcup_{j \in M_2} BL^G(B, b_j)$ which are the inner ones of G.*

Definition 3. *A graph G is a diad if there are a vertex x and a generating sequence $G(1), \dots, G(t) = G$ of G from the vertex x such that*

1. $t > 1$ *and the first growth is of the type* $\{1\}$,
2. *an i-th growth is a right-growth for each* $i \in \{2, \dots, t\}$.

The vertex x is called a root of G and the block $G(1)$ is called a root block of G. If $t = 2$ we say the diad G is prime.

Definition 4. *A graph G is called a 3-diad if there are the BL-subgraphs G_1, G_2, G_3 of G such that*

1. G_1, G_2, G_3 *are the mutually edge disjoint diads with a common root x,*
2. $\bigcup_{i=1}^{3} G_i = G$.

The vertex x is called a root of the 3-diad G.

Notes:

1. In the definitions 1. – 4. we defined notions which were resembling the notions used in [2]. Because their particular meanings are different we will have to distinguish carefully whenever we use them.
2. Any diad or 3-diad are always connected graphs and every vertex of theirs is adjacent to at most three different blocks.
3. If $G(1), \dots, G(t) = G$ is a generating sequence of G from a vertex x, then every $G(i)$ is a BL-subgraph of G. If B is any inner block of G then there is an index $j \in \{2, \dots, t\}$ such that the j-th growth starts from B. If x is a free vertex and $t \geqq 2$ then $G(1)$ is the single free block of G from which some growth starts (actually the second growth).
4. A 3-diad can be also defined as follows: there is a generating sequence $G(1), \dots, G(t) = G$ of G from a vertex x such that the first growth is of type $\{3\}$, every block of the first growth is the inner one of G and every further growth is a right-growth.
5. As there may be more tripples of the BL-subgraphs of a 3-diad fulfilling the conditions from definition 4, there are more roots in a 3-diad. The following assertion holds.

Lemma 1. *Every common vertex of any three different inner blocks of a 3-diad G is a root of G.*

Proof. Let $G(1), \dots, G(t) = G$ be a generating sequence of a 3-diad G from a root x and B_1, B_2, B_3 be any different inner blocks of G with common vertex y. If $x = y$ nothing is to be proved. Otherwise there is the smallest index s such that the s-th growth of G starts from some block of B_1, B_2, B_3, say B_1. Then both blocks B_2, B_3 belong to the s-th growth and both y-fragments including B_2 and B_3

are edge disjoint diads with a common root y. We will prove that the third y-fragment H containing B_1 is a diad with a root y, too.

Let $H(1), \ldots, H(r) = H$ be a generating sequence of H from the vertex y. The first growth is of type $\{1\}$ and $r > 1$ because B is an inner block of G. Let us consider that a q-th growth, $q \geqq 2$, starts from a block $D \in BL^{H(q-1)}$ and $d \in V(D)$ is a cut-vertex of $H(q - 1)$ or $d = y$. As D is an inner block of G, there is an index $p \in \{2, \ldots, t\}$ such that the p-th growth of G starts from D and it is a right-growth. Let $\bar{d} \in V(D)$ be a cut-vertex of $G(p - 1)$.

If $d = \bar{d}$ the q-th growth of H and the p-th growth of G are the same. Thus the q-th growth of H is the right-growth.

If $d \neq \bar{d}$, then $| BL^G(D, \bar{d}) \cap \overline{BL}^G | = | BL^G(D, d) \cap \overline{BL}^G | = 2$ and the q-th growth of H is the right-growth, too.

Hence H is a diad with a root y and y is a root of 3-diad G.

Corollary 1. *At least one vertex of every inner block of a 3-diad G is a root of G.*

Note. If G is a 3-C-diad then every common vertex of any three different inner blocks of G is a root of G. The Corollary holds, too. The proof is almost the copy of the previous one. Only in case $d \neq \bar{d}$ we must still realize that there are just two $(d, \bar{d})$-pathes in D. All vertices of the first $(d, \bar{d})$-path are free in G (exepting d and $\bar{d}$) and all vertices of the second $(d, \bar{d})$-path are the cut-vertices in G.

Definition 5. *Let G be a connected graph. We say a cactus $\bar{G}$ is a C-relative to G if*

1. $V(G) = V(\bar{G})$,
2. *$\bar{G}$ is a subgraph of G,*
3. *for any $M \subseteq V(G)$, $\langle M \rangle_G$ is a block of G if and only if $\langle M \rangle_{\bar{G}}$ is a block of $\bar{G}$.*

Notes.

1. If G is any connected graph then there is a C-relative cactus to G if and only if there is a Hamiltonian circuit in every block of G with at least three vertices. Then a C-relative cactus arises by replacing every block of G by a cycle (i.e. a regular connected graph of degree 2) defined by a Hamiltonian circuit in it. Because there may be many different Hamiltonian circuits in the singular blocks, a C-relative cactus is not defined uniquely. A graph G and C-relative cactus have always the same cut-vertices.

2. If a diad or a 3-diad is a K-graph we will use the term a K-diad or a 3-K-diad respectively. It is clear every K-graph has a C-relative cactus.

Lemma 2. *Let G be a cactus with a free block B such that $| BL^G(B, b) | \geqq 2$ for a cut-vertex $b \in V(B)$ and let G contain a 3-C-diad as its subgraph. Then G-B contain a 3-C-diad as its BL-subgraph.*

Proof. Let H be a 3-C-diad which is a BL-subgraph of G. If $B \notin BL^H$ nothing is to be proved. Suppose $B \in BL^H$. Because every free block touches just a single block in every 3-C-diad there is a block $C \in BL^G(B, b)$ such that $C \notin BL^H$. Then $(H - B) \cup C$ is a 3-C-diad which is a BL-subgraph of G-B.

Note. The same assertion can be proved for K-graphs. The proof is the same, too.

Theorem 1. *Let G be a K-graph not containing any 3-K-diad as its BL-subgraph. Let $B_1, \ldots, B_m$ be all blocks of G such that $| V^G(B_i, \emptyset, 2) | < | V^G(B_i, \emptyset, 0) |$ for each $i \in \{1, \ldots, m\}$. Let, for each $i \in \{1, \ldots, m\}$, $b_1^i, \ldots, b_{m_i}^i$ be any sequence formed from the set $V^G(B_i, \emptyset, 0)$, where $1 \leqq m_i \leqq | V^G(B_i, \emptyset, 0) | - | V^G(B_i, \emptyset, 2) | + 1$ if $V^G(B, \emptyset, 2) \neq \emptyset$ and $1 \leqq m_i \leqq | V^G(B_i, \emptyset, 0) |$ if $V^G(B_i, \emptyset, 2) = \emptyset$. Then there is a C-relative cactus $\bar{G}$ to G not containing any 3-C-diad as its BL-subgraph and having the vertices b_j^i, b_{j+1}^i adjacent for each $i \in \{1, \ldots, m\}, j \in \{1, \ldots, m_i - 1\}$.*

Proof. By induction on $| BL^G |$. If $| BL^G | = 1$, the theorem holds. Suppose G is a K-graph such that $| BL^G | = n > 1$.

I. Let $V^G(B, \emptyset, 0) = \emptyset$ for every inner block B of G. Because G is a K-graph there is a C-relative cactus $\bar{G}$ to G such that in G given vertices are adjacent in given order. If $\bar{G}$ contained a 3-C-diad like its BL-subgraph, then there would be three different inner blocks with common vertex in $\bar{G}$ and in G, too. These blocks would be root blocks of three edge-disjoint prime K-diads. It is not possible, hence $\bar{G}$ does not contain any 3-C-diad.

II. Suppose there is an inner block B such that $| V^G(B, \emptyset, 0) | = l \neq 0$ and $| V^G(B, \emptyset, 2) | = k \leqq l$.

1. Let $k \neq 0$. Let b be any vertex from $V^G(B, \emptyset, 2)$, G_1 be a component of $G - B$ containing the vertex b and $G_2 = G - G_1$. The K-graphs G_1, G_2 fulfil the conditions of the theorem, $| BL^{G_1} | < n$, $| BL^{G_2} | < n$ and for each $i \in \{1, \ldots, m\}$ and suitable $r \in \{1, 2\}$ it is $| V^{G_r}(B_i, \emptyset, 2) | \leqq | V^G(B_i, \emptyset, 2) | < | V^G(B_i, \emptyset, 0) | \leqq \leqq | V^{G_r}(B_i, \emptyset, 0) |$.

a) $k = 1$. Then $V^{G_2}(B, \emptyset, 2) = \emptyset$ and $| V^{G_2}(B, \emptyset, 0) | = l + 1$. According to the induction there are C-relative cacti $\bar{G}_1, \bar{G}_2$ to G_1, G_2 respectively not containing any 3-C-diad as their BL-subgraphs and such that the vertices b_j^i, b_{j+1}^i are adjacent for each $i \in \{1, \ldots, m\}, j \in \{1, \ldots, m_i - 1\}$ and the vertex b is adjacent either to $b_{m_s}^s$ if $B \in \{B_1, \ldots, B_m\}$, say $B = B_s$, or b is adjacent to a vertex $\bar{b} \in V^{G_2}(B, \{b\}, 0)$ if $B \notin \{B_1, \ldots, B_m\}$ (then $V^{G_2}(B, \emptyset, 0) = \{b, \bar{b}\}$).

Suppose a cactus $\bar{G} = \bar{G}_1 \cup \bar{G}_2$ contains a 3-C-diad H as its BL-subgraph. Then $\bar{B} \in BL^H$, where $\bar{B} = \langle V(B) \rangle_{\bar{G}_2}$. If $\bar{B}$ were a free block of H then either $\bar{G}_2$ or $\bar{G}_1 \cup \bar{B}$ would contain H as its BL-subgraph. It is not possible due to induction and Lemma 2. Hence $\bar{B}$ is an innei block and at least one vertex of $\bar{B}$ must be a root of H. Because b is the only vertex of $\bar{G}$ in which three different inner blocks touch each other, b is a root and $\bar{B}$ is a root block of the C-diad which is a BL-subgraph of $\bar{G}_2$. But it is not possible because $V^{\bar{G}_2}(\bar{B}, \{b\}, 0) \neq \emptyset$ and $V^{\bar{G}_2}(B, \emptyset, 2) = \emptyset$.

b) $k > 1$. Then $V^{G_2}(B, \emptyset, 2) \neq \emptyset$ and $| V^{G_2}(B, \emptyset, 0) | - | V^{G_2}(B, \emptyset, 2) | + 1 = = (l + 1) - (k - 1) + 1 = l - k + 3$. According to the induction there are C-relative cacti $\bar{G}_1, \bar{G}_2$ to G_1, G_2 respectively not containing any 3-C-diad as its BL-subgraph such that the vertices b^i_j, b^i_{j+1} are adjacent for each $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, m_i - 1\}$ and the vertex b is adjacent either to the vertices $b^s_{n_s}, \bar{b}$, where $\bar{b} \in V^{G_2}(B, \emptyset, 0) - \{b^s_1, \ldots, b^s_{m_s}, b\}$ if $B \in \{B_1, \ldots, B_m\}$, say $B = B_s$, or to any vertices $b_1, b_2 \in V^{G_2}(B, \emptyset, 0) - \{b\}$, if $B \notin \{B_1, \ldots, B_m\}$. If a cactus $\bar{G} = \bar{G}_1 \cup \bar{G}_2$ contained a 3-C-diad H as its BL-subgraph, then by the same way as in the case a) we prove that $\bar{B}, \bar{B} = \langle V(B) \rangle_{\bar{G}_2}$, is an inner block and one of its vertices is a root of H. If b is a root of H then there is a C-diad with a root b and a root block $\bar{B}$ which is a BL-subgraph of $\bar{G}_2$. But it is not possible due to definition of a C-diad, because both vertices which are adjacent to b in $\bar{B}$ are free. If some other vertex from $V^{\bar{G}}(\bar{B}, \{b\}, 2)$ is a root of H, then by the same reason $BL^H \cap BL^{\bar{G}_1} = \emptyset$ and H is a BL-subgraph of $\bar{G}_2$ which is a contradiction to the induction assumption.

The cactus $\bar{G}$ is, in both cases a) and b), a C-relative to G not containing any 3-C-diad like its BL-subgraph. The remaining part of the theorem follows immediately from the definition of $\bar{G}$ and from the induction.

2. Let $k = 0$. Then $B \in \{B_1, \ldots, B_m\}$, say $B = B_s$. Let $b \in V(B)$ be any cut-vertex, G_* be a component of $G - B$ containing b and $G_1 = G_* \cup B$, $G_2 = G - G_*$. For each $i \in \{1, \ldots, m\} - \{s\}$ and convenient $r \in \{1, 2\}$ it is $V^G(B_i, \emptyset, 0) = = V^{G_r}(B_i, \emptyset, 0)$, $V^G(B_i, \emptyset, 2) = V^{G_r}(B_i, \emptyset, 2)$ and $V^{G_2}(B_s, \emptyset, 0) = V^G(B_s, \emptyset, 0) \cup \cup \{b\}$. As $| BL^{G_1} | < n$, $| BL^{G_2} | < n$ and both G_1 and G_2 fulfil the conditions of the theorem, there are C-relative cacti $\bar{G}_1, \bar{G}_2$ to G_1, G_2 respectively not containing any 3-C-diad as its BL-subgraph such that the vertices b^i_j, b^i_{j+1} are adjacent for each $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, m_i - 1\}$ and b is adjacent to $b^s_{n_s}$ in $\bar{G}_2$. We can suppose $\langle V(B) \rangle_{\bar{G}_1} = \langle V(B) \rangle_{\bar{G}_2} = \bar{B}$. If $\bar{G} = \bar{G}_1 \cup \bar{G}_2$ contains a 3-C-diad H as its BL-subgraph, then $\bar{B} \in BL^H$. $\bar{B}$ is not an inner block of H because $V^G(B, \emptyset, 2) = \emptyset$. Hence $\bar{B}$ is a free block and H is a BL-subgraph either of $\bar{G}_1$ or $\bar{G}_2$. It is a contradiction to the induction. Therefore $\bar{G}$ is a C-relative cactus to G not containing any 3-C-diad like its BL-subgraph and the whole assertion follows from the definition of $\bar{G}$ and from the induction.

If neither I nor II occurs then for every inner block B of G containing free vertices $| V^G(B, \emptyset, 0) | < | V^G(B, \emptyset, 2) |$ holds and at least one such block must exist. Then there are three different inner blocks C_1, C_2, C_3 in G having a common vertex c. If every vertex of C_1 is a cut-vertex, C_1 is a root block of a prime K-diad with a root c. If at least one vertex of C_1 is free, then $0 < | V^G(C_1, \{c\}, 0) | < < | V^G(C_1, \{c\}, 2) |$. We can discuss the blocks C_2, C_3 and all inner blocks from $\bigcup_{x \in V^G(C_1, \{c\}, 2)} BL^G(C_1, x)$ by the same way as C_1. From the definition of a K-diad it is clear that C_1, C_2, C_3 are the root blocks of the three edge-disjoint K-diads with a common root c which are the BL-subgraphs of G. It is not possible and so either I or II must occur.

Corollary 2. *Let G be a K-graph not containing any 3-K-diad as its BL-subgraph, A be the set of all vertices of type X in G, for each $a \in A$, B_a be an arbitrary block of $BL^G(\overline{BL}^G, a)$ and $b_1^a, \dots, b_{n_a}^a$ be any sequence formed from all free vertices of B_a. Next, let $B_1, \dots, B_m$ be all blocks of G different from B_a such that $| V^G(B_i, \emptyset, 2) | < | V^G(B_i, \emptyset, 0) |$ for each $i \in \{1, \dots, m\}$ and $b_1^i, \dots, b_{m_i}^i$ be any sequence formed from the set $V^G(B_i, \emptyset, 0)$ where $m_i = | V^G(B_i, \emptyset, 0) | - | V^G(B_i, \emptyset, 2) | + 1$ if $V^G(B, \emptyset, 2) \neq \emptyset$ and $m_i = | V^G(B_i, \emptyset, 0) |$ if $V^G(B_i, \emptyset, 2) = \emptyset$. Then there is a Hamiltonian circuit h in G^2 having the following properties.*

1. *There is a transform of h of the form*

$$(x_a), a, b_1^a, \dots, b_{m_a}^a, (y_a)$$

for each $a \in A$.

2. *There is a transform of h of the form*

$$(x_i), b_1^i, \dots, b_{m_i}^i, (y_i)$$

for each $i \in \{1, \dots, m\}$.

Proof. It follows immediately from the previous theorem and from Theorem 2 from [2].

Theorem 2. *Let G be a K-graph. Then there is a C-relative cactus $\bar{G}$ to G such that $\bar{G}^2$ is Hamiltonian if and only if G^2 is Hamiltonian.*

Proof. Let h be a Hamiltonian circuit in the square of a graph G (it need not be necessary a K-graph) and B, $V(B) = \{b_1, \dots, b_n\}$, be a block of G which is a complete graph different from K_2, K_3. For every vertex $b_i \in V(B)$, let $G_{b_i}^B$ be either the union of all B, b_i-fragments, if b_i is a cut-vertex, or $G_{b_i}^B$ is a graph with the single vertex b_i, if b_i is a free vertex.

1. Suppose, for each $i \in \{1, \dots, n\}$ there is a section $w(b_i)$ of h such that $V(w(b_i)) = V(G_{b_i}^B)$. Then h is of the form $(w(b_{i_1})), \dots, (w(b_{i_n})), F(w(b_{i_1}))$, where $(i_1, \dots, i_n)$ is a suitable permutation of $\{1, \dots, n\}$. Let us define a graph G_* like that: $V(G_*) = V(G), E(G_*) = (E(G) - E(B)) \cup \bigcup_{j=1}^{n-1} \{b_{i_j}, b_{i_{j+1}}\} \cup \{b_{i_1}, b_{i_n}\}$ (we replace a block B by a cycle induced by the sequence of the vertices $b_{i_1}, \dots, b_{i_n}$). Then h is a Hamiltonian circuit in G_*^2.

2. Suppose there is a vertex $b_j \in V(B)$ such that no section in h is formed by just all vertices from $V(G_{b_j}^B)$ (then b_j is a cutvertex). Let h_{b_j} be a simplification of h at b_j. Then there is an ordering $F_1, \dots, F_p$ of all b_j-fragments such that a transform of h_{b_j} is of the form $b_j, (d_1), \dots, (d_p), (d_{p+1}), b_j$, where $V(d_i) = V(F_i)$ for

each $i \in \{1, \ldots, p\}$ and $V(d_{p+1}) \subseteq V(F_1)$, if $V(d_{p+1}) \neq \emptyset$. Let $B \in BL^{F_r}$, then $G^B_{b_j} = \bigcup_{\substack{i=1 \\ i \neq r}}^{p} F_i$. Suppose $r = 1$ and $V(d_{p+1}) \neq \emptyset$. As $L(d_1)$, $F(d_{p+1}) \in V(F_1)$ are adjacent to b_j, $L(d_{p+1}) \in V(B)$ or $L(d_{p+1})$ is adjacent to a vertex from $V(B) - \{b_j\}$ and $F(d_2)$ is adjacent to b_j in $G^B_{b_j}$. So $d(L(d_1), L(d_{p+1})) \leq 2$ and $d(F(d_{p+1}), F(d_2)) = 2$ (see fig. 2). Then $\bar{h}_{b_j} = b_j, (d_1), (d_{p+1}^{-1}), (d_2), \ldots, (d_p), b_j$ is a Hamiltonian circuit in G^2. If $r \neq 1$ or $V(d_{p+1}) = \emptyset$, then $\bar{h}_{b_j} = h_{b_j}$. Now, there is a section in $\bar{h}_{b_j}$ formed by just all vertices of $V(G^B_{b_j})$.

If there is a section of h formed by just all vertices of $G^B_{b_l}$ for any $b_l \neq b_j$, then

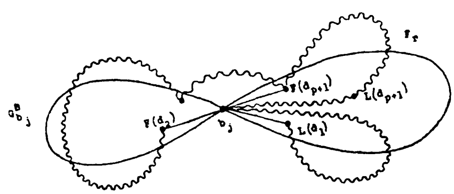

Fig. 2

from the definition of simplification and the fact that this section does not include the vertex b_j it follows that there is a section in $\bar{h}_{b_j}$ formed by just all vertices of $V(G^B_{b_j})$, too. In this way we can construct a Hamiltonian circuit in G^2 so that the case 1 occurs.

If we apply this procedure to every block of a K-graph we will obtain a C-relative cactus whose square is Hamiltonian.

The converse assertion is obvious.

Notice a graph G on the fig. 3a. $h = x, 1, 2, \ldots, 14, x$ is a Hamiltonian circuit in G^2 and $G(3, x)^2$ is Hamiltonian. The only C-relative cactus to G is a C-diad with a root x, therefore the only C-relative cactus to $G(3, x)$ is a 3-C-diad. Hence Theorem 2 does not hold for any graph containing a C-relative cactus. h is a Hamiltonian circuit in the square of the graph H on the fig. 3b and $H(3, x)^2$ is Ha-

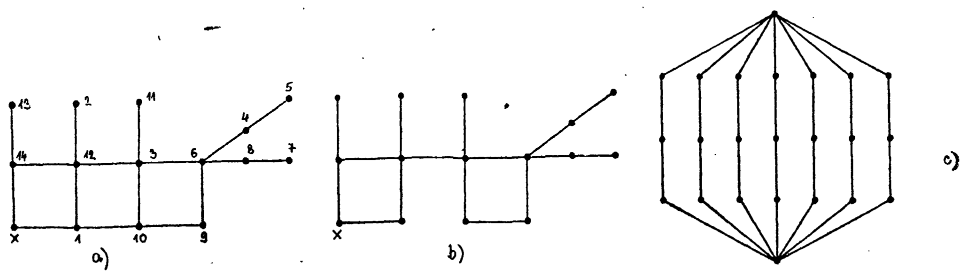

Fig. 3

miltonian, too. The only C-relative graph $H(3, x)$ does not contain any 3-C-diad as its BL-subgraph. So we could put a question wheather a graph G, whose square is Hamiltonian, must contain a subgraph $\bar{G}$ which is a cactus such that $V(G) = V(\bar{G})$ (not longer necessarily C-relative to G). Unfortunately, it is far from valid (see fig. 3c).

Theorem 3. *Let G be a 3-K-diad. Then any C-relative cactus to G contains a 3-C-diad as its BL-subgraph.*

Proof. Let b be a root of a 3-K-diad G and $\bar{G}$ be a cactus C-relative to G. If b is a root of a 3-C-diad which is a BL-subgıaph of $\bar{G}$, the theorem holds. Otherwise, there is a b-fragment of $\bar{G}$ which does not contain any C-diad with a root b like its BL-subgraph. Let B be the only block of its containing the vertex b. Evidently $V^{\bar{G}}(B, \{b\}, 2) \neq \emptyset$.

1. Suppose that it is $V(p) \cap V^{\bar{G}}(B, \emptyset, 0) \neq \emptyset$ for every vertex $x \in V^{\bar{G}}(B, \{b\}, 2)$ and for every (b, x)-path p in B. Because then $|V^{\bar{G}}(B, \emptyset, 0)| \geqq 2$, $|V^{\bar{G}}(B, \{b\}, 2)| = |V^{\bar{G}}(B, \{b\}, 0)|$, there are the vertices $u, v \in V^{\bar{G}}(B, \{b\}, 2)$ and a (u, v)-path q in B such that $V(q) \cap V^{\bar{G}}(B, \emptyset, 0) = \emptyset$ and $V(q) \cap V^{\bar{G}}(B, \emptyset, 2) = \{u, v\}$. Let $\bar{G}_1 = \bigcup_{x \in V(q)} G_x \cup B$, where G_x is a component of $\bar{G} - B$ containing the vertex x. If either u or v is a root of a 3-C-diad which is a BL-subgraph of G_1 the theorem holds. Otherwise at least one of B, u-fragments or B, v-fragments does not contain any C-diad with a root u or v as its BL-subgraph.

2. Suppose there is a vertex $w \in V^{\bar{G}}(B, \{b\}, 2)$ and a (w, b)-path t such that $V(t) \cap V^{\bar{G}}(B, \emptyset, 0) = \emptyset$. Then at least one B, v-fragment does not contain a C-diad with a root w as its BL-subgraph.

If we continue the procedure we must obtain a 3-C-diad which is a BL-subgraph of G.

Theorem 4. *Let G be a K-graph with at least three vertices. Then G^2 is Hamiltonian if and only if G does not contain any 3-K-diad as its BL-subgraph.*

Proof. Suppose G^2 is Hamiltonian and G contains a 3-K-diad as its BL-subgraph. According to Theorems 2 and 3 there is a C-relative cactus $\bar{G}$ to G which contains a 3-C-diad as its BL-subgraph and has the Hamiltonian square. But it is not possible by Theorem 4 in [2].

The converse implication was proved by Corollary 2. ■

Corollary 3. *If a graph G contains a 3-diad as its BL-subgraph then G^2 is not Hamiltonian.*

REFERENCES

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