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A CATEGORICAL CHARACTERIZATION OF SETS AMONG CLASSES

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Abstract. There is given a categorical characterization of sets among classes. The characterization is connected with the coding of subclasses of a class.

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We will consider two naturally connected questions concerning categorical properties of classes: the characterization of classes X for which 2^X exists (we will call them small) and the categorical characterization of sets among classes. Evidently, any set is small. Without the axiom of regularity, there are small proper classes (e.g. A is small in a permutation model with a proper class A of atoms). In the presence of regularity, we do not know whether small proper classes can exist. We will prove that X is a set if and only if any X – indexed union of small classes is small. It is a categorical characterization of sets among classes, which could be applied to the context of [2]. The first version of this paper was presented at the 8th Winter School on Abstract Analysis (see [3]).

We will work in the Gödel–Bernays set theory. A class X is *small* if there is a class 2^X and a map $E : 2^X \times X \to 2$ such that for any class Z and any map $F : Z \times X \to 2$ there is a unique map $H : Z \to 2^X$ such that $E . (H \times 1) = F$. The definition specifies the categorical scheme of an object of subobjects (see, e.g. [1]) but it can be rewritten in a more set-theoretical spirit. Having a relation R, we put $Ext_R(x) = \{y \setminus [x, y] \in R\}$ and $D(R) = \{x \setminus Ext_R(x) \neq \emptyset\}$. R is called nowhere constant if $Ext_R(x) \neq Ext_R(y)$ for any $x, y \in D(R)$, $x \neq y$ (see [4]). It is easy to see that X is small iff there exists a nowhere constant relation E such that for any subclass $\emptyset \neq Y \subseteq X$ there is $x \in D(E)$ such that $Y = Ext_E(x)$. This scheme was used in [4] (see the axiom (Pot)). It is evident that any subclass of a small class is small and if X is small and $H : X \to Y$ surjective then Y is small.

Proposition 1. *The universal class V is not small.*

Proof. Assume that V is small. Put $Y = \{y \in D(E) \setminus y \notin Ext_S(y)\}$. If $Y \neq \emptyset$ then $Y = Ext_E(x)$ for some $x \in D(E)$ and neither $x \in Y$ nor $x \notin Y$ is possible. Hence $y \in Ext_E(y)$ for any $y \in D(E)$. For any $y \in D(E)$ there is $z \in D(E)$ such that $\{y\} = Ext_E(z)$. Since $z \in Ext_E(z)$, it holds $Ext_E(y) = \{y\}$, which is a contradiction.

Proposition 2. *The following two conditions are equivalent for any class X:*

(i) *X is a set*

(ii) *If $F: Y \to X$ is a map and $F^{-1}(x)$ are small for any $x \in X$ then Y is small.*

Proof. Take F from (ii) such that X is a set. Let E_x code the subclasses of $F^{-1}(x)$ and A be the set of all maps $f: Z \to \bigcup_{x \in X} D(E_x)$ such that $Z \subseteq X$ and $f(x) \in D(E_x)$. Then $E = \{[a, b] \setminus a \in A, b \in \bigcup_{x \in X} Ext_{E_x}(x)\}$ codes Y. Hence (i) $\Rightarrow$ (ii).

Let X satisfy (ii) and $G: X \to X'$ be surjective. Assume that $F': Y' \to X'$ has small fibres and form the pullback

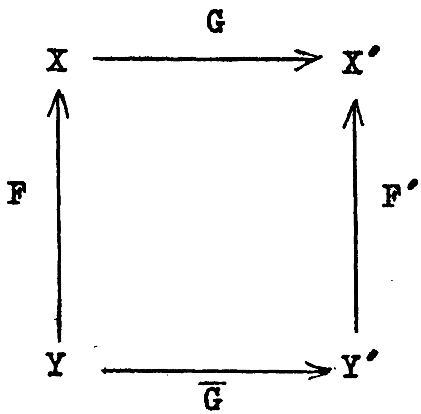

Fig. 1

Then F has small fibres and hence Y is small because X satisfies (ii). Since $\bar{G}$ is surjective, Y' is small. We have proved that X' satisfies (ii).

Now let X satisfy (ii) and denote by $r: V \to Ord$ the rank function. The image $A = r(X)$ satisfies (ii), as well. If X is not a set then neither is A and we can define a map $H: Ord \to A$ such that $H(\alpha)$ is the smallest ordinal $\beta \in A$, $\beta \geq \alpha$. The composition $V \xrightarrow{r} Ord \xrightarrow{h} A$ has set fibres and therefore V is small because A satisfies (ii). It contradicts Proposition 1.

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