Archivum Mathematicum

Miroslav Bartušek

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Archivum Mathematicum, Vol. 22 (1986), No. 3, 145--156

Persistent URL: http://dml.cz/dmlcz/107258

ARCHIVUM MATHEMATICUM (BRNO)
Vol. 22, No. 3 (1986) 145–156

ON OSCILLATORY SOLUTION OF THE DIFFERENTIAL EQUATION OF THE n-th ORDER

M. BARTUŠEK

(Received May 6, 1985)

Abstract. The properties of proper oscillatory solutions of the non-linear differential equation of the n-th order are studied. The sufficient conditions are given under which these solutions tend to zero or are unbounded.

Key words. Ordinary differential equations, oscillatory solutions, asymptotic behaviour.

MS Classification. 34 C 10.

1. Consider the differential equation

$$y^{(n)} = f(t, y, \dots, y^{(n-1)}) \tag{1}$$

where $n \geqq 2, f: D \to R$ is continuous, $D = R_+ \times R^n$, $R = (-\infty, \infty)$, $R_+ = [0, \infty)$ and there exists a number $\alpha \in \{0, 1\}$ such that

$$(-1)^\alpha f(t, x_1, \dots, x_n)\, x_1 \geqq 0 \qquad \text{in } D. \tag{2}$$

The solution y of (1), defined on R_+ is called proper if $\sup\limits_{\tau \leqq t < \infty} |y(t)| > 0$ for an arbitrary $\tau \in R_+$. The proper solution y is called oscillatory if there exists a sequence of its zeros tending to ∞.

A great number of papers is devoted to the existence of oscillatory solutions of (1) (see [5]). But the problem of asymptotic behaviour of such solutions for $n > 2$ is almost unsolved. The papers [6] and [7] are devoted to vanishing at infinity of solutions of (1) for linear case, the asymptotic behaviour for $n = 3, 4$; $\alpha = 1$ is studied in [1], [2]. Our aim is to study the behaviour of oscillatory solutions in the neighbourhood of the infinity, to give sufficient conditions under which solutions tend to zero or are unbounded.

Denote $N = \{1, 2, \dots\}$, n_0 the entire part of $n/2$, $C^{(0)}(I)$ the set of all continuous functions defined on I, $C^{(i)}(I)$, $i \in N$ the set of all continuous functions which have continuous derivatives to the order i, $L^{(i)}(I)$, $i \in N$ the set of all p-integrable functions on I, $L^{(\infty)}(I)$ the set of all bounded functions on I.

Further let $m \in N$ and $v \in C^0(R_+)$. Put

$$J_m(t; v) = \int_0^t \int_0^{\tau_m} \dots \int_0^{\tau_2} v(\tau_1)\, \mathrm{d}\tau_1 \dots \mathrm{d}\tau_m \qquad \text{and} \qquad J_0(t; v) = v(t), \qquad t \in R_+.$$

Let $y \in C^{(n-n_0-1)}(R_+)$. Put

$$Z(t; y) = \sum_{i=0}^{n-n_0-1} (-1)^{\alpha+i} \binom{n-i}{i} \frac{n}{2(n-i)} J_{2i}(t; [y^{(i)}]^2). \tag{3}$$

Let $O_{n\alpha}$ be the set of all oscillatory solutions of (1) and (2).

Lemma 1. *Let $y \in O_{n\alpha}$. Then*

$$Z^{(n)}(t; y) = (-1)^\alpha y^{(n)}(t)\, y(t) + [n - 2n_0 - 1]\, (-1)^{\alpha+n_0} y^{(n_0)2}(t), \qquad t \in R_+.$$

Moreover, if either

$$n = 2n_0, \qquad n_0 + \alpha \qquad \text{is odd} \tag{4}$$

or

$$n = 2n_0 + 1 \tag{5}$$

then $Z^{(n)}(t; y) \geqq 0$, $t \in R_+$.

Proof. Let $n = 2n_0$. For n odd the proof is similar.

Put $Z(t) = Z(t; y)$, $d_s^j = (-1)^{\alpha+s}\binom{j}{s}$,

$$Z_j(t) = \sum_{i=0}^{j-1} \mathrm{d}_i^{n-j} J_{j+i}(t; y^{(j)} y^{(i)}), \qquad j = 1, 2, \dots, n_0, Z_0(t) \equiv 0, \tag{6}$$

$$Z_{km}(t) = \sum_{i=0}^{k-m} \mathrm{d}_i^{n-k-1} J_{k+i+1}(t; y^{(k+1)} y^{(i)}) + \mathrm{d}_{k-m}^{n-k-1} J_{2k-m+1}(t; y^{(k)} y^{(k-m+1)}) +$$

$$+ \sum_{i=k-m+1}^{k} \mathrm{d}_i^{n-k} J_{k+i}(t; y^{(k)} y^{(i)}),$$

$$K_{jm} = -\mathrm{d}_{j-m}^{n-j-1} y^{(j)}(t_0)\, y_{(t_0)}^{(j-m)} \frac{(t-t_0)^{2j-m}}{(2j-m)!},$$

$$K_j = \sum_{r=0}^{j} K_{jr} + \frac{1}{2} \mathrm{d}_j^{n-j-1} y^{(j)2}(t_0) \frac{(t-t_0)^{2j}}{(2j)!},$$

$j = 0, 1, 2, \dots, n_0$; $s = 0, 1, \dots, j$; $m = 0, 1, \dots, j$; $k = 0, 1, \dots, n_0 - 1$, $t \in R_+$.
It is easy to see that

$$Z_{jm}(t) = Z_{jm+1}(t) + K_{jm}, \qquad m = 0, 1, \dots, j-1, j = 0, 1, \dots, n_0 - 1$$

holds and thus

$$Z_{j+1}(t) = Z_{j0}(t) - \mathrm{d}_j^{n-j-1} J_{2j+1}(t; y^{(j)} y^{(j+1)}) = Z_{j,j}(t) -$$

$$- \frac{1}{2} \mathrm{d}_j^{n-j-1} J_{2j}(t; y^{(j)^2}) + K_j - K_{jj} =$$

$$= Z_j(t) + \left(\mathrm{d}_j^{n-j} - \frac{1}{2} \mathrm{d}_j^{n-j-1}\right) J_{2j}(t; y^{(j)^2}) + K_j, \qquad j = 0, 1, \dots, n_0 - 1.$$

From this and from (6) we have

$$Z(t) = Z_{n_0}(t) - \sum_{j=0}^{n_0-1} K_j,$$

(7) $$Z^{(n-1)}(t) = \sum_{i=0}^{n_0-1} d_i^{n_0}(y^{(n_0)}(t)\, y^{(i)}(t))^{(n_0-i-1)}.$$

Now, if we denote

(8) $$v_k(t) = \sum_{i=0}^{n_0-k-1} (-1)^{\alpha+i}\binom{i+k}{k} y_{(t)}^{(n-i-k-1)}\, y^{(i)}(t),$$
$$k = 0, 1, \ldots, n_0 - 1,\ v_{n_0}(t) = 0,$$

then

$$v_k'(t) = v_{k-1} - d_{n_0-k}^{n_0} y^{(n_0)}(t)\, y^{(n_0-k)}(t), \qquad k = 1, \ldots, n_0,$$
$$v_0(t) = \sum_{i=0}^{n_0-1} d_i^{n_0}[y^{(n_0)}(t)\, y^{(i)}(t)]^{(n_0-i-1)}.$$

Thus, according to (7), (8) and (2)

$$Z^{(n-1)}(t) = \sum_{i=0}^{n_0-1} (-1)^{\alpha+i}\, y^{(n-i-1)}(t)\, y^{(i)}(t),$$
$$Z^{(n)}(t) = (-1)^{\alpha} y^{(n)}(t)\, y(t) + (-1)^{\alpha+n_0-1}[y^{n_0}(t)]^2, \qquad t \in R_+$$

and lemma follows from (2). Lemma is proved.

Let (4) or (5) be valid. By virtue of Lemma 1 we can denote

(9) $$O_{n\alpha}^1 = \{v \mid v \in O_{n\alpha}, \lim_{t\to\infty} Z^{(n-1)}(t; v) = \infty\},$$
$$O_{n\alpha}^2 = \{v \mid v \in O_{n\alpha}, \lim_{t\to\infty} | Z^{(n-1)}(t; v) | < \infty\}.$$

2. This paragraph is devoted to the study of asymptotic behaviour of oscillatory solutions under the validity of the condition (4).

Lemma 2. *Let* $y \in O_{n\alpha}$ *and let* (4) *be valid. Then* $\int_0^\infty y^{(n_0)^2}(t)\, dt < \infty$ *if, and only if* $\lim_{t\to\infty} Z^{(n-1)}(t; y) = 0.$

Proof. Put $Z(t; y) = Z(t)$ for the simplicity. If $\lim_{t\to\infty} Z^{(n-1)}(t) = 0$, then according to Lemma 1 and (2)

$$-Z^{(n-1)}(o) = \int_0^\infty Z^{(n)}(t)\, dt \geqq \int_0^\infty [y^{(n_0)}(t)]^2\, dt$$

and the statement is valid. Let, on the contrary

(10) $$\int_0^\infty [y^{(n_0)}(t)]^2\, dt = M < \infty$$

hold. We prove the statement of lemma by the indirect proof. Thus, let $\lim_{t\to\infty} Z^{(n-1)}(t) = M_1$, $M_1 \in (-\infty, \infty]$, $M_1 \neq 0$ (the limit exists by virtue of Lemma 1). From this there exists a number $t_1 \in [0, \infty)$ such that

$$(11) \qquad |Z(t)| \geqq M_2 t^{n-1}, \qquad t \in [t_1, \infty),$$

where $M_2 = \frac{|M_1|}{2(n-1)!}$ for $M_1 < \infty$ and $M_2 = 1$ for $M_1 = \infty$. Further, according to Levin's lemma ([5], p. 50) and (10) there exist constants $M_3 > 0$ and $t_2 \in [0, \infty)$ with the properties

$$\int_\beta^t [y^{(i)}(t)]^2 \, dt \leqq M_3 t^{2(n_0 - i)} \int_\beta^t [y^{(n_0)}(t)]^2 \, dt, \qquad 0 \leqq \beta \leqq t < \infty, \; i \in \{0, 1, \ldots, n_0\},$$

$$\int_{t_2}^\infty [y^{(n_0)}(t)]^2 \, dt \leqq \varepsilon = \frac{1}{4} M_2 \left[\sum_{j=1}^{n_0 - 1} \binom{n-j}{j} \frac{n_0}{n-j} M_3 \right]^{-1}.$$

There exists a number $t_3 \in [t_2, \infty)$ such that

$$\int_0^t [y^{(i)}(t)]^2 \, dt \leqq \varepsilon M_3 t^{2(n_0 - i)} + \sum_{j=0}^{n_0 - 1} \int_0^{t_2} [y^{(j)}(t)]^2 \, dt \leqq 2\varepsilon M_3 t^{2(n_0 - i)},$$

$i \in \{0, 1, \ldots, n_0 - 1\}$, $t \in [t_3, \infty)$ holds. From this and from (3) there exists $t_4 \in [t_3, \infty)$ such that

$$|Z(t)| \leqq \frac{1}{2} y^2(t) + \left\{ 2 \sum_{j=1}^{n_0 - 1} \binom{n-j}{j} \frac{n_0}{n-j} M_3 \varepsilon \right\} t^{n-1} \leqq$$

$$\leqq \frac{1}{2} y^2(t) + \frac{M_2}{2} t^{n-1}, \qquad t \in [t_4, \infty).$$

This inequality is in contradiction to (11) for an arbitrary zero τ, $\tau \geqq t_1$, $\tau \geqq t_4$ of the function y. Lemma is proved. It is clear that the following theorem is valid.

Theorem 1. *Let* (4) *be valid. Then* $y \in O^1_{n\alpha}$ ($y \in O^2_{n\alpha}$) *if, and only if* $\int_0^\infty [y^{(n_0)}(t)]^2 \, dt = \infty$ ($< \infty$).

Theorem 2. *Let* (4) *be valid,* $y \in O^1_{n\alpha}$ *and* $M \in (0, \infty)$. *Then*

$$\limsup_{t\to\infty} (|y^{(n_0 - 1)}(t)| - M t^{1/2}) = \infty.$$

Proof. We prove the statement by the indirect proof. Thus suppose that there exist numbers $t_0 \in R_+$ and $M_1 \in (0, \infty)$ with the property

$$|y^{(n_0 - 1)}(t)| - M t^{1/2} \leqq M_1, \qquad t \in [t_0, \infty).$$

Then there exists $t_1 \geqq t_0$ such that

$$|y^{(i)}(t)| \leqq 2M t^{n_0 - i - 1/2}, \qquad t \in [t_1, \infty), \; 0 \leqq i < n_0$$

holds and according to (3)

$$(12) \qquad |Z(t;y)| \leqq M_2 t^{n-1} + \frac{1}{2} y^2(t), \qquad t \in [t_1, \infty),$$

where $M_2 < \infty$ is a suitable constant. On the other hand, as $y \in O^1_{n\alpha}$ there exists $t_2 \geqq t_1$ such that

$$Z^{(n-1)}(t;y) \geqq 3(n-1)!\, M_2,\ Z(t;y) \geqq 2M_2 t^{n-1},$$
$$t \in [t_2, \infty].$$

The last inequality contradicts the (12) for an arbitrary zero τ, $\tau \geqslant t_2$ of y. The theorem is proved.

Theorem 3. *Let* (4) *be valid and* $y \in O^1_{n\alpha}$. *Let there exist positive constant* M *and a nonnegative function* $g \in C^0(R_+)$ *such that*

$$(13) \qquad |f(t, x_1, \dots, x_n)| \leqq t^{\frac{n_0}{n_0-1}} g(|x_1|) \qquad \text{in } [M, \infty) \times R^n$$

holds. Then y *is unbounded.*

Proof. We prove the conclusion by the indirect proof. Thus suppose, that

$$(14) \qquad |y(t)| \leqq M_1 < \infty, \qquad t \in R_+.$$

According to Theorem 2 there exists a sequence $\{t_k\}_1^\infty$ such that

$$t_k \in [M, \infty), \qquad \lim_{k\to\infty} t_k = \mathrm{M},$$
$$(15) \qquad |y^{(n_0-1)}(t_k)| \geqq M_2 t_k^{1/2}, \qquad k \in N,$$
$$M_2 = 2^\sigma M_1^{\frac{n_0+1}{n}} [2 \max_{0 \leqq x \leqq M_1} g(x)]^{\frac{n_0-1}{n}},$$
$$\sigma = (3n_0 - 2)(n_0 + 1) + 1.$$

Denote

$$v_{jk} = \max_{M \leqq t \leqq t_k} |y^{(j)}(t)|, \qquad k \in N, j \in \{0, 1, \dots, n\}.$$

Then it follows from (13 – 15) and Kolmogorov – Horny Theorem ([4] p. 393) that there exists $s \in N$ with the property

$$M_2 t_s^{1/2} \leqq v_{n_0-1}, s \leqq 2^\sigma v_{os}^{\frac{n_0+1}{n}} v_{ns}^{\frac{n_0-1}{n}} \leqq 2^\sigma M_1^{\frac{n_0+1}{n}} v_{ns}^{\frac{n_0-1}{n}}.$$

If we define a number τ, such that $\tau \in [M, t_s]$, $|y^{(n)}(\tau)| = v_{ns}$ holds, then according to (13), (15) and (14) we have

$$2 \max_{0 \leqq x \leqq M_1} g(x)\, t_s^{\frac{n_0}{n_0-1}} \leqq v_{ns} \leqq \tau^{\frac{n_0}{n_0-1}} \max_{0 \leqq x \leqq M_1} g(x).$$

Then obtained contradiction proves the theorem.

Remark. For $y \in O^1_{4,1}$ the statement of Theorem 3 was proved without the validity of (13).

Lemma 3. *Let* (4) *be valid and* $y \in O^2_{n\alpha}$. *Let there exist continuous functions* $a : R_+ \to R_+$, $g : R_+ \to R_+$ *such that* g *is non-decreasing,*

$$H = \liminf_{t\to\infty} a(t)\, t^{\frac{n_0}{2}} g(t^{n_0/2}) > 0$$

and

$$|f(t, x_1, \ldots, x_n)| \geqq a(t)\, g(|x_1|) \qquad \text{in } D \tag{16}$$

holds. Then $\int_0^\infty t[y^{(n_0)}(t)]^2 \,\mathrm{d}t < \infty$, $\int_0^\infty t\,|y(t)\,y^{(n)}(t)|\,\mathrm{d}t < \infty$ *and* $\lim_{t\to\infty} Z^{(n-2)}(t; y) = C \neq \pm\infty$, $\lim_{t\to\infty} Z^{(n-1)}(t; y) = 0$.

Proof. The validity of $\lim_{t\to\infty} Z^{(n-1)}(t; y) = 0$ follows from Lemma 2. First we prove by the indirect proof that $\lim_{t\to\infty} Z^{(n-2)}(t; y) = C \neq \pm\infty$. As $y \in O^2_{n\alpha}$, then according to Lemma 1, Z^{n-2} is non-increasing on R_+. Thus suppose that

$$\lim_{t\to\infty} Z^{(n-2)}(t; y) = -\infty. \tag{17}$$

Now we prove the relation

$$\limsup_{t\to\infty} (|y^{(n_0-2)}(t)| - t) = \infty. \tag{18}$$

Thus suppose on the contrary that $|y^{(n_0-2)}(t)| \leqq t + M$, $t \in R_+$. From this there exist constants M_1 and $\tau \in R_+$ such that (see (3))

$$\left| Z(t) - \frac{n_0^2}{2} J_{n-2}(t; [y^{(n_0-1)}]^2) \right| \leqq M_1 t^{n-2}, \qquad t \in [\tau, \infty),$$

that contradicts to (17). Thus the relation (18) is valid. According to (18) there exists an increasing sequence $\{t_k\}_0^\infty$ such that

$$t_k - t_{k-1} \geqq 1, \qquad |y^{(n_0-2)}(t_k)| \geqq t_k, \qquad k \in N, \tag{19}$$

$y^{(i)}$, $i = 1, 2, \ldots, n_0 - 1$ has a zero in the interval

$$\Delta_k = [t_{k-1}, t_k], \max_{t\in\Delta_k} |y^{(n_0-2)}(t)| = |y^{(n_0-2)}(t_k)|, \qquad k \in N.$$

Put $v_{ik} = \max_{t\in\Delta_k} |y^{(i)}(t)|$, $i = 0, 1, \ldots, n_0 - 1$, $v_{n_0k} = t_{k-1}^{-1}$. Let $\Delta_{ik} \subset \Delta_k$ be an interval such that $\max_{t\in\Delta_{ik}} |y^{(i)}(t)| = v_{ik}$, $\min_{t\in\Delta_{ik}} |y^{(i)}(t)| = 0$ and $y^{(i)}$ does not change the sign on Δ_{ik}, $i = 0, 1, \ldots, n_0 - 1$, $k \in N$. Then

$$v_{ik}^2 \leqq 2 \int_{\Delta_{ik}} |y^{(i+1)}(t)\, y^{(i)}(t)|\,\mathrm{d}t \leqq 2v_{i+1,k} \int_{\Delta_{ik}} |y^{(i)}(t)|\,\mathrm{d}t \leqq 4v_{i+1,k}v_{i-1,k}, \tag{20}$$

$$i = 1, 2, \ldots, n_0 - 2,$$

$$v_{n_0-1,k}^2 \leqq 2 \int_{\Delta_{n_0-1,k}} [t^{-1} + t^{-1}(y^{(n_0)}(t))^2] \,|\, y^{(n_0-1)}(t) \,|\, dt \leqq$$
$$\leqq 4v_{n_0,k}v_{n_0-2,k} + 2v_{n_0-1,k}t_{k-1}^{-1} \int_{\Delta_k} [y^{(n_0)}(t)]^2 \, dt.$$

If we denote $K_k = 2t_{k-1}^{-1} \int_{\Delta_k} [y^{(n_0)}(t)]^2 \, dt$, then by virtue of Theorem 1 $\lim_{k\to\infty} K_k = 0$ and thus

$$v_{n_0-1,k} \leqq \frac{1}{2}[K_k + \sqrt{K_k^2 + 16v_{n_0-2,k}v_{n_0,k}}] \leqq 4\sqrt{v_{n_0-2,k}v_{n_0,k}}, \qquad k \geqq k_0,$$

$k_0 \in N$ is a suitable number (see (19), too).
From this and from (20) we can easily get by means of the induction

$$v_{ik} \leqq 4^{(n_0-i)(n_0+i-1)} v_{ok}^{\frac{n_0-i}{n_0}} v_{n_0k}^{\frac{i}{n_0}}, \qquad k \geqq k_0, \; i \in \{0, 1, \dots, n_0\}. \tag{21}$$

Especially for $i = n_0 - 2$ and by virtue of (19) we have

$$t_k \leqq v_{n_0-2,k} \leqq 4^{2(n-3)} t_{k-1}^{-\frac{n_0-2}{n_0}} v_{ok}^{\frac{2}{n_0}} \leqq 2^{-\frac{2}{n_0}} v_{0k}^{\frac{2}{n_0}}, \qquad k \geqq k_1 \tag{22}$$

where $k_1 \geqq k_0$ is a suitable number.

Let $\{\bar{\Delta}_k\}$ be a sequence of intervals such that

$$\bar{\Delta}_k = [\sigma_k, \bar{\sigma}_k], \qquad \bar{\Delta}_k \subset \Delta_k, \qquad \bar{\sigma}_k - \sigma_k = 1, \qquad \max_{t\in\bar{\Delta}_k} (|\, y(t) \,|) = v_{0k},$$

$k \in N$. Then with respect to (21)

$$|\, y(t) \,| \geqq v_{0k} - \int_{\bar{\Delta}_k} |\, y'(t) \,|\, dt \geqq v_{0k} - v_{1k} \geqq v_{0k} - 4^{n_0(n_0-1)} t_{k-1}^{-\frac{1}{n_0}} v_{0k}^{\frac{n_0-1}{n_0}}, \qquad k \geqq k_0$$

and thus there exists $k_2 \geqq k_1$ such that by virtue of (22)

$$|\, y(t) \,| \geqq t_k^{\frac{n_0}{2}} \geqq \bar{\sigma}_k^{\frac{n_0}{2}}, \qquad t \in \bar{\Delta}_k, \; k \geqq k_2. \tag{23}$$

Let $\varepsilon > 0$, $\varepsilon \leqq \frac{H}{2}$ be an arbitrary number. As $y \in O_{n\alpha}^2$ it follows from Lemma 1 that $\lim_{k\to\infty} \int_{\bar{\Delta}_k} (-1)^\alpha y^{(n)}(t) \, y(t) \, dt = 0$ and therefore there exists a sequence $\{\varrho_i\}_1^\infty$ such that

$$\lim_{i\to\infty} \varrho_i = \infty, \qquad \varrho_i \in \bigcup_{k=1}^{\infty} \bar{\Delta}_k, \qquad |\, y^{(n)}(\varrho_i) \, y(\varrho_i) \,| \leqq \varepsilon, \; i \in N.$$

From this, and according to (1), (16) and (23) we have

$$\varepsilon \geqq \liminf_{i\to\infty} [a(\varrho_i) \, g(|\, y(\varrho_i) \,|) \,|\, y(\varrho_i) \,|] \geqq$$
$$\geqq \liminf_{i\to\infty} [a(\varrho_i) \, g(\varrho_i^{\frac{n_0}{2}}) \, \varrho_i^{\frac{n_0}{2}}] \geqq H \geqq 2\varepsilon.$$

This contradiction proves the validity of $\lim_{t\to\infty} Z^{(n_0-2)}(t; y) = C \neq \pm\infty$. From this, from Lemma 1 and by means of integration per partes we have for $v(t) = [y^{(n_0)}(t)]^2$, resp. $v(t) = (-1)^\alpha y^{(n)}(t)\, y(t)$:

$$\int_0^\infty tv(t)\,dt = \int_0^\infty \int_t^\infty v(t)\,dt\,dt \leqq \int_0^\infty \int_t^\infty Z^{(n)}(t; y)\,dt\,dt = Z^{(n-2)}(0; y) - C < \infty.$$

The lemma is proved.

Theorem 4. *Let* (4) *be valid and* $y \in O^2_{n\alpha}$. *Let positive constant* K *and the continuous, non-decreasing function* $g : R_+ \to R_+$ *exist such that* $\lim_{x\to\infty} g(x) > 0$ *and*

$$|f(t, x_1, \dots, x_n)| \geqq \frac{1}{t} g(|x_1|) \qquad \text{in } \; K, \infty) \times R^n$$

holds. Then $\lim_{t\to\infty} y^{(i)}(t) = 0$, $i = 0, 1, \dots, n_0 - 2$.

Proof. Let $M > 0$ be a constant such that $g(M) > 0$ and let $D_1 = \{t : t \in R_+, |y(t)| \leqq M\}$, $D_2 = R_+ - D_1$, $y_i(t) = |y(t)|$ for $t \in D_i$, $y_i(t) = 0$ for $t \in R_+ - D_i$, $i = 1, 2$. Then, according to Theorem 1 $y_i^{(n_0)} \in L^2(R_+)$, $i = 1, 2$, $y_1 \in L^{(\infty)}(R_+)$. As the assumptions of Lemma 3 are fulfiled, then

$$\infty > \int_0^\infty t\,|y^{(n)}(t)\, y(t)|\,dt \geqq \int_K^\infty g(|y(t)|)\,|y(t)|\,dt \geqq g(M) \int_K^\infty |y_2(t)|\,dt. \tag{24}$$

Thus $y_2 \in L^1(R_+)$ and according to [3] p. 236

$$|y^{(i)}(t)| \leqq K_1 < \infty, \qquad t \in R_+, \qquad i = 0, 1, \dots, n_0 - 1 \tag{25}$$

for a suitable constant K_1. We prove by the indirect proof that $\lim_{t\to\infty} y(t) = 0$. Thus suppose on the contrary that there exist a sequence $\{t_k\}_1^\infty$ and a constant $K_2 > 0$ such that

$$|y(t_k)| \geqq K_2, \qquad k \in N, \qquad \lim_{k\to\infty} t_k = \infty, \qquad t_k \geqq K. \tag{26}$$

Let $\tau_k \in R_+$ be the first zero of y lying on the left of t_k, $\Delta_k = [\tau_k, t_k]$. Then it follows from (24), (25) and (26)

$$\infty > \int_K^\infty g(|y(t)|)\,|y(t)|\,dt \geqq \sum_{i=2}^{\infty} \int_{\Delta_i} g(|y(t)|)\,|y(t)|\,dt \geqq$$
$$\geqq \sum_{i=2}^{\infty} [\max_{t\in\Delta_i} |y'(t)|]^{-1} \int_0^{K_2} g(s)\,s\,ds = \infty.$$

This contradiction shows that $\lim_{t\to\infty} y(t) = 0$ and the statement follows from (25) a Kolmogorov–Horny Theorem ([4]).

Remark. The statement of Theorem 4 was proved for the linear equation under weaker assumptions in [6].

Theorem 5. *Let* $y \in O^2_{4,1}$. *Then* $\lim_{t\to\infty} y'(t) = 0$. *Moreover, if there exist positive constant* K *and continuous functions* $g : R_+ \to R_+$, $g_1 : R^3 \to (0, \infty)$ *such that* $g > 0$ on $(0, \infty)$,

$$|f(t, x_1, x_2, x_3, x_4)| \geqq \frac{1}{t} g(|x_1|)\, g_1(x_2, x_3, x_4) \tag{27}$$

on

$$[K, \infty) \times R^4, \qquad \text{then} \qquad \lim_{t\to\infty} y^{(i)}(t) = 0, \qquad i = 0, 1.$$

Proof. Put for the simplicity $Z(t; y) = Z(t)$. It is clear according to (3) that

$$\begin{aligned} Z''(t) &= -y''(t)\, y(t) + y'^2(t); \\ Z'''(t) &= -y'''(t)\, y(t) + y'(t)\, y''(t). \end{aligned} \tag{28}$$

It was proved in [2] that there exist sequences $\{t^i_k\}^\infty_{k=1}$, $i = 0, 1, 2, 3$ such that it holds $t^i_k \in [K, \infty)$, $y^{(i)}(t^i_k) = 0$, $y^{(i)}(t) \neq 0$ for $t \in [t^0_1, \infty)$, $t \neq t^i_k$ and $t^0_k < t^1_k < t^2_k < t^3_k < t^0_{k+1}$, $k \in N$, $i \in \{0, 1, 2, 3\}$. From this

$$(-1)^{i+1} y^{(i)}(t)\, y(t) > 0 \ (< 0) \qquad \text{for } t \in (t^0_k, t^i_k) \tag{29}$$

(for $t \in (t^i_k, t^0_{k+1})$), $k \in N$.
It follows from (28), (29) that $z'''(t) \leqq y'(t)\, y''(t)$, $t \in [t^0_k, t^1_k]$ and thus

$$Z''(t^1_k) - Z''(t^0_k) \leqq -2y'^2(t^0_k) = -2Z''(t^0_k). \tag{30}$$

As Z'' is according to (10), (28) non-decreasing and non-negative, we can conclude from (28), (30)

$$\lim_{t\to\infty} Z''(t) = 0, \qquad \lim_{k\to\infty} y'(t^2_k) = 0, \qquad \lim_{t\to\infty} y'(t) = 0. \tag{31}$$

Thus the first part of the statement is valid.
By virtue of (31)

$$\int_0^\infty t\, |y^{(4)}(t)\, y(t)|\, dt \leqq \int_0^\infty t Z^{(4)}(t)\, dt \leqq \int_0^\infty \int_t^\infty Z^{(4)}(t)\, dt\, dt = Z''(0) < \infty. \tag{32}$$

$$\lim_{t\to\infty} y''(t)\, y(t) = 0. \tag{33}$$

We prove by the indirect proof that $\lim_{t\to\infty} y(t) = 0$. Thus suppose without loss of generality that there exists a constant $M > 0$ with the property

$$|y(t^1_k)| \geqq M, \qquad k \in N. \tag{34}$$

Denote $\{\tau_k\}$, $k \in N$ the sequence such that $\tau_k \in (t^0_k, t^1_k)$, $|y(\tau_k)| = \dfrac{M}{2}$, $k \in N$. Then it follows from (33), (34), (28), (31) that for a suitable $M_1 < \infty$ we have

$$|y^{(i)}(t)| \leqq M_1, \qquad t \in \Delta_k = [\tau_k, t^1_k], \qquad k \in N,\ i = 1, 2, 3.$$

From this and from (27), (32) and (31)

$$0 \underset{k\to\infty}{\leftarrow} \int_{\Delta_k} t\,|\,y^{(4)}(t)\,y(t)\,|\,dt \geqq M_2 \int_{\Delta_k} g(|\,y(t)\,|)\,|\,y(t)\,|\,dt \geqq$$

$$\geqq \frac{M_2}{\max\limits_{t\in\Delta_k} |\,y_k'(t)\,|} \int_{M/2}^{M} g(s)\,s\,ds \underset{k\to\infty}{\to} \infty.$$

$M_2 = \min\limits_{|x_i|\leq M_1,\, i=2,3,4} g_1(x_2, x_3, x_4) > 0$. The gained contradiction proves the theorem.

3. This paragraph deals with the case when (5) is valid.

Theorem 6. *Let $y \in O^1_{n\alpha}$ and (5) be valid. Then the following statements hold:*
a) $y^{(n_0)}$ is unbounded on R_+.
b) If $\alpha + n_0$ is odd and $M \in (0, \infty)$, then

$$\limsup_{t\to\infty} (|\,y^{(n_0-1)}(t)\,| - Mt) = \infty.$$

c) Let there exist a non-negative function $g \in C^0(R_+)$ such that

$$|\,f(t, x_1, x_2, \dots, x_n)\,| \leqq t^{\frac{n}{n_0-1}\sigma} g(|\,x_1\,|) \tag{35}$$

holds in D, where $\sigma = \frac{1}{2}[1 - (-1)^{\alpha+n_0}]$. Then y is unbounded on R_+.

Proof. The statement a) can be proved similarly to the Theorem 2. Now, we prove the case b). Put

$$Z_1(t) = Z(t; y) + \frac{n}{2} J_{n-1}(t, [y^{(n_0)}(t)]^2), \qquad t \in R_+$$

and suppose, on the contrary, that

$$|\,y^{(n_0-1)}(t)\,| - Mt \leqq M_1 < \infty, \qquad t \in R_+.$$

Then according to (3)

$$|\,Z_1(t)\,| \leqq M_2 t^{n-1}, \qquad t \in R_+, \tag{36}$$

where $M_2 < \infty$ is a suitable constant. As $y \in O^1_{n\alpha}$, then

$$\lim_{t\to\infty} Z_1^{(n-1)}(t) = \lim_{t\to\infty}\left[Z(t; y) + \frac{n}{2}[y^{(n_0)}(t)]^2\right] = \infty.$$

This relation contradicts to (36) and b) is valid. The case c): If $\alpha + n_0$ is odd, the proof is similar to that of Theorem 3. If $\alpha + n_0$ is even, then the statement follows from Kolmogorov–Horny Theorem, (35) and a). The theorem is proved.

Theorem 7. *Let $y \in O_{30}$. Then $y \in O^2_{30}$. Moreover, if there exist continuous functions $g : R_+ \to R_+$, $h : R_+ \to (0, \infty)$ such that $g(0) = 0$, $g(x_1) > 0$ for $x_1 > 0$ and*

(37) $$|f(t, x_1, x_2, x_3)| \geqq g(|x_1|)\, h(|x_2|) \quad \textit{in } R_+ \times R^2 \textit{ holds.}$$

Then $\lim_{t\to\infty} y(t) = 0$ *and* y' *is bounded on* R_+.

Proof. It follows from [1] and (37) that there exist sequences $\{t_k^i\}_{k=1}^{\infty}$, $i = 0, 1, 2$ such that $t_k^0 < t_k^1 < t_k^2 < t_{k+1}^0$, $\lim_{k\to\infty} t_k^0 = \infty$,

(38) $$y^{(i)}(t_k^i) = 0, \qquad (-1)^{i+1} y^{(i)}(t)\, y(t) > 0 \qquad \text{for } t \in (t_k^0, t_k^i),$$
$$(-1)^i y^{(i)}(t)\, y(t) > 0 \qquad \text{for } t \in (t_k^i, t_{k+1}^0),\ k = 1, 2, \ldots, i = 1, 2.$$

According to (3)

$$Z''(t; y) = -\frac{1}{2} y'^2(t) + y(t)\, y''(t); \qquad Z'''(t, y) = y(t)\, y'''(t) \geqq 0$$

holds. From this (for $t = t_k^0$) we can see that $\lim_{t\to\infty} Z''(t; y) = M$, $M \in (-\infty, 0]$ and thus $y \in O_{30}^2$ and

(39) $$\int_{t_1^0}^{\infty} y(t)\, y'''(t)\, \mathrm{d}t < \infty, \qquad \lim_{t\to\infty} |y'(t_k^2)| = \sqrt{-M}.$$

We can conclude that y' is bounded, $|y'(t)| \leqq M_1$. Further, it follows from (39) and (2) that

$$0 \underset{k\to\infty}{\leftarrow} \int_{t_k^0}^{t_k^1} y(t)\, y'''(t)\, \mathrm{d}t \geqq \frac{M_2}{M_1} \int_{t_k^0}^{t_k^1} y(t)\, g(|y(t)|)\, y'(t)\, \mathrm{d}t \geqq \frac{M_2}{M_1} \int_0^{|y(t_k^1)|} s g(|s|)\, \mathrm{d}s,$$
$$M_2 = \min_{0 \leqq x \leqq M_1} h(x) > 0.$$

Thus $\lim_{k\to\infty} y(t_k^1) = 0$ and $\lim_{t\to\infty} y(t) = 0$. The theorem is proved.

Theorem 8. *Let* $y \in O_{31}$ *and let a constant* $M > 0$ *and continuous functions* $g_1 : R_+^3 \to R_+$, $g_2 : R_+^3 \to R_+$ *exist such that* $g_1(x_1, x_2, x_3) > 0$ *for* $x_1 > 0$,

(40) $$g_1(|x_1|, |x_2|, |x_3|) \leqq |f(t, x_1, x_2, x_3)|,$$
$$(t, x_1, x_2, x_3) \in R_+ \times R^3$$

and

$$|f(t, x_1, x_2, x_3)| \leqq g_2(|x_1|, |x_2|, |x_3|),$$
$$(t, x_1, x_2, x_3) \in R_+ \times R^3, \qquad |x_3| \leqq M \textit{ holds. Then } y \in O_{31}^1.$$

Proof. According to [1] and (40) there exist sequences $\{t_k^i\}_{k=1}^{\infty}$, $i = 0, 1, 2$ such that $t_k^0 < t_k^2 < t_k^1 < t_{k+1}^0$, $\lim_{k\to\infty} t_k^0 = \infty$, $y^{(i)}(t_k^i) = 0$, $y^{(i)}(t)\, y(t) > 0$ for $t \in (t_k^0, t_k^i)$ $y^{(i)}(t)\, y(t) < 0$ for $t \in (t_k^i, t_{k+1}^0)$, $k = 1, 2, \ldots, i = 1, 2$. By virtue of (3) $Z''(t; y) = \frac{1}{2} y'^2(t) - y(t)\, y''(t)$, $Z'''(t, y) = -y'''(t)\, y(t) \geqq 0$ holds. If $y \in O_{31}^2$, then

$\lim_{k\to\infty} Z''(t; y) = M_1 < \infty$ and $\frac{1}{2} y'^2(t_k^2) = Z''(t_k^2; y) \underset{k\to\infty}{\to} M_1$. Thus y' is bounded on R_+ that contradicts to Theorem 5 of [1]. Theorem is proved.

REFERENCES

[1] M. Bartušek: *The asymptotic behaviour of solutions of the differential equation of the third order*. Arch. Math. (Brno) 20, 3, 1984, 101–112.

[2] M. Bartušek: *The asymptotic behaviour of oscillatory solutions of the equation of the fourth order*. Arch. Math. (Brno) 21, 2, 1985, 93–104.

[3] E. F. Beckenbach, R. Bellman: *Inequalities*. Springer-Verlag, Berlin, 1961.

[4] Г. Г. Харди, Д. Е. Диттльвуд, Г. Полиа, *Неравенства,* Изд. иност. литературы, Москва, 1948.

[5] И. Т. Кигирадзе, *Некотόрые сингулярные краевые задачи для обыкновенных дифференциалных уравиений*. Из. Тбилисс. унив., Тбилиси 1975.

[6] I. T. Kiguradze: *On vanishing at infinity of solutions of ordinary differential equations*. Czech. Math. J. 33 (108), 1983, 613–646.

[7] M. Švec: *Sur le component asymptotic des intégrales de l'equation differentielle* $Y^{(4)} + Q(x) Y = 0$, Czech. Math. J. 8 (83), 1958, 230–245.

M. Bartušek
Department of Mathematics
Faculty of Science, J. E. Purkyně University
Janáčkovo nám. 2a
662 95 Brno
Czechoslovakia