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A LEIGHTON—BORŮVKA FORMULA FOR MORSE CONJUGATE POINTS

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Abstract. We find conditions for the conjugate point function of a system of linear differential equations depending on control variables to be differentiable and find the Leighton–Borůvka formula for its derivative. For nonlinear equations we determine conditions under which the control variable can be used to generate a preassigned conjugate point function locally.

Key words. Conjugate point, index, control, Leighton–Borůvka formula.

0.

W. Leighton [6] and O. Borůvka [1] have discovered a formula for the derivative of the first conjugate point of a second order linear differential equation $y'' + p(t)y = 0$. That formula has far-reaching consequences in the theory of these equations [4, 5]. The Leighton–Borůvka formula has been derived by Freedman [2] for 2×2 systems $x' = A(t)x$ under mild hypotheses on $A(t)$.

A number of authors [2, 3, 7] have studied the conjugate points of n-th order linear and nonlinear scalar equations; in certain cases, a determinant formulation of the Leighton–Borůvka formula holds for conjugate and focal points of solutions of these equations.

The present paper deals with systems of two n-dimensional linear equations that are either linear with real coefficient matrices

$$(1) \qquad \begin{aligned} u' &= A_{11}(t)u + A_{12}(t)v, \\ v' &= A_{21}(t)u + A_{22}(t)v \end{aligned}$$

or nonlinear equations

$$(2) \qquad \begin{aligned} u' &= F(u, v, t), \\ v' &= G(u, v, t). \end{aligned}$$

We shall prove a Leighton–Borůvka formula for (1) and show that, in a generic case of (2) initial values can be found for a prescribed conjugate point function.

In both cases, we shall assume that the solutions exist on an interval $a < t < b$ that contains the values t_0 and $s > t_0$ under consideration. The A_{ij} are supposed to be continuous; F and G are differentiable functions.

1.

The system (1) can be written as

(3) $$x' = A(t)\,x,$$

where

$$x = \begin{bmatrix} u \\ v \end{bmatrix}, \qquad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

The solution of the matrix equation

(4) $$X' = A(t)\,X, \qquad X(t_0) = I$$

is denoted by

$$X(t;\, t_0) = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}.$$

A value $s > t_0$ is a *conjugate point* of t_0 for (1) if there exists a solution $x = X(s;\, t_0)\, x_0$ of (3) for which

$$u(t_0) = u(s) = 0, \qquad u(t) \not\equiv 0.$$

This means that there exists an initial vector v_0 ($\neq 0$) which is an eigenvector of eigenvalue zero for $X_{12}(s;\, t_0)$:

(5) $$X_{12}(s;\, t_0)\, v_0 = 0.$$

The *index j* of the conjugate point s is the dimension of the space of eigenvectors v_0.

Definition: The conjugate point s is *regular* if all Jordan boxes belonging to eigenvalues 0 of $X_{12}(s;\, t_0)$ are of dimension one.

The conjugate point is regular if and only if rank $X_{12}(s;\, t_0) =$ rank $X_{12}^2(s;\, t_0) = n - j$.

If s is regular, $\mathbf{R}^n$ splits into the kernel $K(s)$ of $X_{12}(s;\, t_0)$ and a cokernel $C(s)$ on which $X_{12}(s;\, t_0)$ induces an automorphism. Let $v_1, \ldots, v_j$ be a basis of $K(s)$ and $w_1, \ldots, w_{n-j}$ one of $C(s)$. We choose the vectors so that

$$\det (v_1, \ldots, v_j, w_1, \ldots, w_{n-j}) = 1.$$

In order to use (5) to compute $s = s(t_0)$ by the implicit function theorem, we may restrict changes of v_0 $K(s)$ to vectors in $C(s)$ and put

$$\mathrm{d}v_0 = w_1\, \mathrm{d}\sigma_1 + \ldots + w_{n-j}\, \mathrm{d}\sigma_{n-j}.$$

From the condition

$$X_{12}(s + \mathrm{d}s;\, t_0 + \mathrm{d}t_0)(v_0 + \mathrm{d}v_0) = 0$$

we get

$$[(X_{12})_s\, \mathrm{d}s + (X_{12})_{t_0}\, \mathrm{d}t_0]\, v_0 + X_{12}\, \mathrm{d}v_0 = 0,$$

where all matrices are evaluated at $(s; t_0)$ and differentiation is indicated by a lower index. From (3), we have

$$(X_{12})_s = A_{11}(s)\, X_{12}(s; t_0) + A_{12}(s)\, X_{22}(s; t_0)$$

and

$$(X_{12})_s\, v_0 = A_{12}(s)\, X_{22}(s; t_0)\, v_0 .$$

From (4),

$$\begin{aligned} X(t; t_0 + \mathrm{d}t_0) &= X(t; t_0)\, X(t_0 + \mathrm{d}t_0; t_0)^{-1} \\ &= X(t; t_0)\, [I + A(t_0)\, \mathrm{d}t_0]^{-1} \qquad (\mathrm{mod}\ \mathrm{d}t_0^2) \\ &= X(t; t_0) - X(t; t_0)\, A(t_0)\, \mathrm{d}t_0 \qquad (\mathrm{mod}\ \mathrm{d}t_0^2). \end{aligned}$$

Together, we get

$$\text{(6)} \qquad \{A_{12}(s)\, X_{22}(s; t_0)\, \mathrm{d}s - [X_{11}(s; t_0)\, A_{12}(t_0) + X_{12}(s; t_0)\, A_{22}(t_0)]\, \mathrm{d}t_0\}\, v_0 = = -X_{12}(s; t_0)\, \mathrm{d}v_0 .$$

Since the right hand side is in $C(s)$, so is the left hand side of (6). This means that, for $v_0 \in K(s)$,

$$\det(v_1, \ldots, v_{i-1}, \{A_{12}(s) X_{22}(s; t_0) \mathrm{d}s - [X_{11}(s; t_0) A_{12}(t_0) + X_{12}(s; t_0) A_{22}(t_0)]\mathrm{d}t_0\} v_0, v_{i+1}, \ldots, v_j, w_1, \ldots, w_{n-j}) = 0.$$

We put

$$N_i = \det(v_1, \ldots, v_{i-1}, [X_{11}(s; t_0)\, A_{12}(t_0) + X_{12}(s; t_0)\, A_{22}(t_0)]\, v_0, v_{i+1}, \ldots, v_j, w_1, \ldots, w_{n-j}),$$

$$D_i = \det(v_1, \ldots, v_{i-1}, A_{12}(s)\, X_{22}(s; t_0)\, v_0, v_{i+1}, \ldots, v_j, w_1, \ldots, w_{n-j}).$$

Then

$$\text{(7)} \qquad \frac{\mathrm{d}s}{\mathrm{d}t_0} = \frac{N_i}{D_i}, \qquad i = 1, \ldots, j.$$

Let X_{12}^C be the nondegenerate matrix induced by $X_{12}(s; t_0)$ on $C(s)$ and p^C the projection of $\boldsymbol{R}^n$ onto $C(s)$ defined by $\boldsymbol{R}^n = K(s) + C(s)$. If (7) holds, $\mathrm{d}s$ can be computed from $\mathrm{d}t_0$ by (7) if the right-hand side is not of the form 0/0 and

$$\mathrm{d}v_0 = = -(X_{12}^C)^{-1} p^C \{A_{12}(s)\, X_{22}(s; t_0) \mathrm{d}s - [X_{11}(s; t_0) A_{12}(t_0) + X_{12}(s; t_0) A_{22}(t_0)]\, \mathrm{d}t_0\}\, v_0 .$$

The operator p^C changes n-columns into $(n - j)$-vectors. Since $C(s)$ is transversal to $K(s)$, the implicit function theorem can be used in a neighborhood of 0 in $C(s)$ to compute $s(\tau, v)$ and $v(\tau, v_0)$ where τ is the variable that was called t_0 up to now and v is the initial vector at $t = \tau$ which belongs to the conjugate point $s(\tau, v)$; $v(t_0, v_0) = v_0$. In particular, the index cannot change as long as (7) holds for definite values of the quotient and all $v_0 = v_i$ $(i = 1, \ldots, j)$. By the same reason, a regular conjugate point for which (7) holds is isolated. If (1) is self-adjoint, all conjugate points are isolated. In the general case, it may be that s is an isolated conjugate point for some values of v_0 but not for others. The same can happen if some eigenvectors of eigenvalue 0 of $X_{12}(s; t_0)$ are simple while others have companion vectors.

Theorem: *A regular conjugate point of* (1) *is isolated and differentiable if* $N_i \neq 0$, $D_i \neq 0$ $(i = 1, \ldots, j)$ *and* N_i/D_i *is independent of i. In that case,* $\partial s/\partial t_0 = N_i/D_i$.

2.

We define $s = s(t_0, v_0)$ to be a conjugate point of t_0 for (2) if

$$u(s) = 0 \qquad \text{for } u(t_0) = 0, v(t_0) = v_0 .$$

The Jacobian of u with respect to v_0 is denoted by u_{v_0}. Under our hypotheses, $u(t; t_0, v_0)$ is a differentiable function of all variables (8, Theorem 10.1). Hence, $u(s; t_0, v_0) = 0$ implies

$$u_s \, \mathrm{d}s + u_{t_0} \, \mathrm{d}t_0 + u_{v_0} \, \mathrm{d}v_0 = 0$$

or, from

$$u(s; t_0, v_0) = \int_{t_0}^{s} F(u(t), v(t), t) \, \mathrm{d}t,$$

(8) $$F(0, v(s), s) \, \mathrm{d}s - F(0, v_0, t_0) \, \mathrm{d}t_0 + u_{v_0} \, \mathrm{d}v_0 = 0.$$

Here the interesting case is

(9) $$\det u_{v_0} \neq 0,$$

which can never happen for linear F; (9) implies that v_0 is an isolated initial value that leads to a conjugate point. In the linear case, v_0 always defines a onedimensional subspace of initial vectors. In this case, an application of the implicit function theorem yields:

Theorem: *If* $\det u_{v_0} \neq 0$ *at a conjugate point* s_0, *the differentiable function* $s = s(\tau)$ *can be prescribed in the neighborhood of* $s_0 = s(t_0)$ *and uniquely defines an initial vector function* $v(s, \tau)$ *with* $v_0 = v(s_0, t_0)$.

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