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REGULARITY AND TRANSITIVITY OF LOCAL-AUTOMORPHISM SEMIGROUPS OF LOCALLY FINITE FORESTS

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Introduction

Investigations of regular semigroups (or regular elements of various semigroups) and other significant proper subclasses of the class of regular semigroups belong to interesting and useful directions of the algebraic theory of transformation and relation semigroups (cf. papers [1]–[3], [5], [8], [10], [12], [14]–[17], [19] and many others). The present paper is devoted to the description of locally finite trees and forests with regular, inverse, complete regular (and possessing another properties) semigroups (monoids in fact) of local automorphisms with respect to some modifications of the transitivity of the action of these semigroups on carrier sets of mentioned posets. Considerations are based on some results of L. A. Skornjakov [16] concerning endomorphism semigroups of monounary algebras which are close to mentioned questions in the connection with generalized transitive actions.

1. Preliminaries

We agree on the following notation: **Z** is the set of all integers, **N** is its subset of all positive integers and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. Further, ω is the first infinite ordinal, ω^d is the ordinal type dual to ω.

Let $(T, \leqq)$ be a (partially) ordered set. An ***up set*** of $(T, \leqq)$ (also called a dual semiideal of $(T, \leqq)$) generated by a subset $A \subset T$ is the set $[A)_{\leqq} = \{t\colon \exists\, a \in A, a \leqq t\}$. A *down subset* of $(T, \leqq)$ (also called a semiideal) is defined dually and denoted by $(A]_{\leqq}$. In case of A being a singleton, say $A = \{a_0\}$, we write as usually $[a_0)_{\leqq}$, $(a_0]_{\leqq}$ and these sets are called ***principal up set, principal down set***, respectively. An ordered set $(T, \leqq)$ is said to be an ***upper locally finite forest*** if every principal up subset of $(T, \leqq)$ is well ordered with the ordinal at most ω. An ordered set

$(T, \leqq)$ is said to be a *lower locally finite forest* if $(T, \leqq^d)$ (where $\leqq^d$ is the inverse ordering to $\leqq$) is an upper locally finite forest. A connected upper (lower) locally finite forest is called an *upper* (*lower*) *locally finite tree*. A root of an upper (lower) tree is the greatest (the least) element of this tree and a tree having the root is called *rooted*. A forest (especially a tree) is called *antirooted* if each of its maximal trees has no root. By Max $(T, \leqq)$, Min $(T, \leqq)$ we denote the set of all maximal, minimal elements of $(T, \leqq)$ respectively. An interval of $(T, \leqq)$ with the initial element s and the terminal element t, i.e. the set $\{x: s \leqq x \leqq t\}$ is denoted by $[s, t]$; $s \prec t$ means $[s, t] = \{s, t\}$. Since the successor of the element s is denoted by s^+ we have for the covering relationship $s \prec t$ the equivalent expression $s^+ = t$. A maximal chain of a tree is also called a branch.

An isotone selfmap f of a locally finite forest $(T, \leqq)$ is said to be a *local automorphism* of $(T, \leqq)$ (cf. 18) if for any pair of elements $s, t \in T$ such that $s < t$ the restriction $f \mid [s, t]$ is an order isomorphism of the interval $[s, t]$ onto the interval $[f(s), f(t)]$. The monoid of all local automorphisms of $(T, \leqq)$ will be denoted by LA$(T, \leqq)$. The full transformation monoid of a set X (i.e. X^X endowed with the binary operation of the composition of mappings) is denoted by M(X). Basic notions from the algebraic theory of semigroups can be found in [6], [9] or [11]. For a set T and a selfmap f of T we denote by $\mathrm{C}_T(f)$ the centralizer of the cyclic subsemigroup $\langle f \rangle$ of M(T), or which is the same the endomorphism monoid of a monounary algebra (T, f), i.e. $\mathrm{C}_T(f) = \{g: g \in \mathrm{T}^T, fg = gf\}$. (We put $f^0 = \mathrm{id}_T, f^n = ff^{n-1}$ for all $n \in \mathbf{N}$).

Following [16] we say that a monounary algebra (T, f) is a line with short tails or a cycle with short tails if $(f(T), f \mid f(T))$ is isomorphic to the algebra $(\mathbf{Z}, \sigma)$, where $\sigma(p) = p + 1$ for all $p \in \mathbf{Z}$, or there is $m \in \mathbf{N}$ such that $(f(T), f \mid f(T)) \cong$ $\cong (\{1, 2, \ldots, m\}, \sigma_m)$, where $\sigma_m(k) = 1$ if $k = m$ and $\sigma_m(k) = k + 1$ otherwise. A component (K, f_K) of a monounary algebra (T, f) is its maximal connected subalgebra.

In what follows the following results will be used. (It is to be noted that in the below stated two theorems the relationship card $A \mid$ card B means that either card B is infinite or both cardinals are finite and card A divides card B—cf. [16]).

1.1. Theorem ([16], Theorem 1). *The monoid $C_T(f)$ is regular iff each component of the monounary algebra (T, f) is either a cycle with short tails or a line with short tails and for any components K, L and M the following conditions are satisfied:*

(1) *if* card $f(L) \mid$ card $f(K)$, card $f(M) \mid$ card $f(L)$ *and* $L \neq M$ *then* card $f(K) =$ $=$ card $f(L)$.

(2) *if* card $f(L) \mid$ card $f(K)$, $K \neq f(K)$ *and* $L \neq f(L)$ *then* card $f(K) =$ card $f(L)$.

(3) *if* card $f(L) \mid$ card $f(K)$, *and* card $(L \setminus f(L)) \geqq 2$ *then* $K = f(K)$ *or* $K = L$.

1.2. Theorem ([16], Theorem 2). *The monoid $C_T(f)$ is an inverse semigroup iff every element in the monounary algebra (T, f) has at most two predecessors, each* *of its components is either a cycle with short tails or a line with short tails and beyond conditions* (1) – (3), *the following are also fulfilled for any components* K, L *and* M*:*

(4) *if* $\operatorname{card} f(L) \mid \operatorname{card} f(K)$ *and* $\operatorname{card} f(M) \mid \operatorname{card} f(K)$ *then* $K = L$ *or* $L = M$,

(5) *if* $K \neq L$ *and* $\operatorname{card} f(L) \mid \operatorname{card} f(K)$ *then* $\operatorname{card} f(L) = 1$ *and* $\operatorname{card} f(K) > 1$, *and if, in addition,* $L \neq f(L)$ *then* $K = f(K)$.

1.3. Proposition ([4], Lemma 3.4). *Let* $(T, \leqq)$ *be a locally finite upper (lower) forest. If* $\operatorname{Max}(T, \leqq) = \emptyset$ $(\operatorname{Min}(T, \leqq) = \emptyset)$ *then there exists a transformation* $f \in T^T$ *with the property* $C_T(f) = \mathrm{LA}(T, \leqq)$.

Notice that in the proof of the above assertion the mapping f is defined in this way: For $t \in T$ we have $f(t) = t^+$ (or $f(t)$ is the predecessor of t in the case of a lower forest).

2. Regular and Transitive Semigroups

The following definitions are modifications of certain basic definitions from the topological transformation group theory; especially cf. Definition 9.02 [7], where we substitute a topology of the phase space by an ordering and the considered topological group by a transformation (discrete) semigroup. From the point of views of the well-known relationship between orderings and right respectively left order topologies the following notions seem to be useful.

2.1. Definition, *Let* $(X, \leqq)$ *be an ordered set,* F *be a transformation semigroup on* X, $x \in X$. *The semigroup* F *is said to be upper (lower) transitive at* x *and the element* x *is said to be upper (lower) transitive under* F *provided that if* U *is a nonvoid up (down) subset of the poset* $(X, \leqq)$, *then there exists* $f \in F$ *such that* $f(x) \in U$. *The transformation semigroup* F *is said to be upper (lower) transitive on* $(X, \leqq)$ *provided that* F *is upper (lower) transitive at every element of the set* X.

2.2. Definition. *A transformation semigroup* F *is said to be regionally upper (lower) transitive on a poset* $(X, \leqq)$ *provided that if* U, V *are nonvoid up (down) subsets of* $(X, \leqq)$, *then there exists* $f \in F$ *such that* $f(U) \cap V \neq \emptyset$.

It is to be noted that a semigroup $F \subset \mathrm{M}(X)$ is said to be universally transitive on X (also transitively acting on X or simply transitive on X) if for every pair of elements $x, y \in X$ there exists $f \in F$ with the property $f(x) = y$. Topological and algebraic characterizations of locally finite forests with universally acting monoids of local automorphisms are contained in paper [4] .Now, we shall consider the pointwise and regional transitivity of $\mathrm{LA}(T, \leqq)$ on locally finite trees and forests $(T, \leqq)$.

2.3. Lemma. *Let* $(T, \leqq)$ *be an upper locally finite tree. The following conditions are equivalent:*

1° $(T, \leqq)$ *does not possess any root.*

2° LA(T, $\leqq$) *is upper pointwise transitive on* T.

3° LA(T, $\leqq$) *is upper regionally transitive on* T.

Proof. 1° ⇒ 2°: Let $x \in T$ be an arbitrary element, $P \subset T$ be an arbitrary nonvoid up subset of the tree (T, $\leqq$). There exists $y \in P$ such that $x \leqq y$. Then there is $h \in$ LA(T, $\leqq$) with the property $h(x) = y$, hence 2° is satisfied.

The implication 2° ⇒ 1° is evident and the equivalence of conditions 1° and 3° follows immediately from the corresponding definitions as well. □

Remark. An analogical assertion to Lemma 2.3 holds also for lower locally finite trees, where it is necessary to change the notion of the upper pointwise (regional) transitivity by the notion of the lower pointwise (regional) transitivity. The condition of the antirootedness is not sufficient for the upper pointwise (regional) transitivity of LA(T, $\leqq$) in case of a locally finite forest (T, $\leqq$) which is not a tree (i.e. it is disconnected). This shows the following example: Denote by $\mathbf{N}'$ the set of all even positive integers and by $\mathbf{Z}''$ the set of all odd integers. Put $T = \mathbf{N}' \cup \mathbf{Z}''$ and for $s, t \in T$ put $s \leqq t$ whenever there is a nonnegative integer k such that $t - s = 2k$. The locally finite forest (T, $\leqq$) is a cardinal summ of a chain of the type $\omega^{\mathrm{d}} + \omega$ and a chain of the type ω. Evidently, for an arbitrary integer $t \in \mathbf{Z}''$ and $f \in$ LA(T, $\leqq$) we have $f(t) \notin \mathbf{N}'$.

2.4. Proposition. *For a locally finite upper tree* (T, $\leqq$) *the following conditions are equivalent:*

1° LA(T, $\leqq$) *is regular and upper pointwise transitive on* T.

2° LA(T, $\leqq$) *is regular and upper regionaly transitive on* T.

3° $T = K \cup$ Min (T, $\leqq$), *where* (K, $\leqq$) *is a chain of the type* $\omega^{\mathrm{d}} + \omega$.

Proof. The equivalence of conditions 1°, 2° follows from the above Lemma 2.3. Suppose 2°. Then (T, $\leqq$) has not any root, thus in virtue of Proposition 1.3 we have LA(T, $\leqq$) $= C_T(f)$, where $f(t) = t^+$ for every $t \in T$. From Theorem 1.1 there follows T has exactly one subset K which is a chain of the type $\omega^{\mathrm{d}} + \omega$ and $t^+ \in K$ for any $t \in T$. Hence the condition 3° is satisfied. The implication 3° ⇒ 1° follows also from Theorem 1.1 with respect to Lemma 2.3 and Proposition 1.3. □

It is evident that in case of an antirooted locally finite forest (T, $\leqq$) the monoid LA(T, $\leqq$) is a group if and only if (T, $\leqq$) is a chain of the type $\omega^{\mathrm{d}} + \omega$. Hence we shall suppose in the following considerations of this paragraph that LA(T, $\leqq$) is not a group.

2.5. Theorem. *Let* (T, $\leqq$) *be an upper locally finite antirooted forest such that* LA(T, $\leqq$) *is not a group. Suppose* $\{(T_i, \leqq): i \in I\}$ *is the system of all maximal trees of the forest* (T, $\leqq$). *The following conditions are equivalent:*

1° LA(T, $\leqq$) *is regular.*

2° *For every* $i \in I$ *we have* $T_i = K_i \cup$ Min (T_i, $\leqq$), *where* (K_i, $\leqq$) *is a chain of the type* $\omega^{\mathrm{d}} + \omega$ *and either* card Min (T_i, $\leqq$) $\leqq 1$ *and* card $I \geqq 2$ *or there* *exists exactly one index* $i_0 \in I$ *such that* card Min $(T_{i_0}, \leqq) \geqq 2$ *and at the same time* Min $(T_i, \leqq) = \emptyset$ *for any* $i \in I \setminus \{i_0\}$.

Proof. Again by Proposition 1.3 we have LA$(T, \leqq) = C_T(f)$, where $f(t) = t^+$ for any $t \in T$. In virtue of Theorem 1.1 we have every maximal tree $(T_i, \leqq)$ of the forest $(T, \leqq)$ has the form $T_i = K_i \cup$ Min $(T_i, \leqq)$, where $(K_i, \leqq)$ is a chain of the type $\omega^d + \omega$. Conditions (1), (2) from Theorem 1.1 are satisfied in the considered case. Condition (3) of the mentioned theorem says that for any two maximal trees $(K_i, \leqq)$, $(K_j, \leqq)$ of the forest $(T, \leqq)$ the inequality card $(K_i, \leqq) \geqq 2$ implies either Min $(K_j, \leqq) = \emptyset$ or $K_i = K_j$. Hence LA$(T, \leqq)$ is regular iff either every maximal tree $(K_i, \leqq)$ of the forest $(T, \leqq)$ has at most one minimal element or exactly one maximal tree has at least two minimal elements and every another tree is a chain of the type $\omega^d + \omega$ and simultaneously $t^+ \in K_i$ for any element $t \in T_i$ and any $i \in I$. □

2.6. Corollary 1. *Let* $(T, \leqq)$ *be an upper locally finite forest. The following conditions are equivalent:*

1° LA$(T, \leqq)$ *is regular and upper pointwise transitive on* T.

2° LA$(T, \leqq)$ *is regular and upper regionally transitive on* T.

3° *Condition* 2° *from Theorem* 2.5.

Proof. If some of conditions 1°, 2°, 3° is satisfied, then Max $(T, \leqq) = \emptyset$. The equivalence of conditions 1°, 2°, 3° then follows immediately from Theorem 2.5 with respect to corresponding definitions. □

2.7. Corollary 2. *For a locally finite forest* $(T, \leqq)$ *the following conditions are equivalent:*

1° LA$(T, \leqq)$ *is regular and transitive on* T.

2° $(T, \leqq)$ *is a cardinal summ of chains of the type* $\omega^d + \omega$ *over a nonempty antichain.*

Proof follows from the above Theorem 2.5 with respect to [18] Theorem 1 (part (a)) which says that LA$(T, \leqq)$ is (universally) transitive on T iff Min $(T, \leqq) \cup$ $\cup$ Max $(T, \leqq) = \emptyset$. □

It is easy to construct (similarly as in the remark following Lemma 2.3) an example of the least locally finite antirooted forest the local automorphism monoid of which is not a group but it is regular and universally transitive:

Example 1. Consider again the set **Z** of all integers and define an ordering $\lesssim$ on **Z** in this way: $m \lesssim n$ whenever there exists $k \in \mathbf{N}_0$ with the property $n - m = 2k$. Then $(\mathbf{Z}, \lesssim)$ is a cardinal summ (over a two-element antichain) of two chains of the type $\omega^d + \omega$.

3. Inverse, Complete Regular and Transitive Semigroups

Using Theorem 1.2 ([16] Theorem 2) we get a characterization of locally finite forest with the inverse semigroup of local automorphisms.

3.1. Theorem. *Let $(T, \leqq)$ be an antirooted upper locally finite forest. The following conditions are equivalent:*

1° $\mathrm{LA}(T, \leqq)$ *is inverse.*

2° $(T, \leqq)$ *is a tree of the form* $T = K \cup \mathrm{Min}(T, \leqq)$, *where K is a chain of the type $\omega^d + \omega$ and for every element $t \in K$ there exists at most one element $a \in \mathrm{Min}(T, \leqq)$ with the property $a^+ = t$.*

Proof. According to Theorem 1.2 (condition (4)) we get the connectedness of $(T, \leqq)$ is a necessary condition for the regularity of $\mathrm{LA}(T, \leqq)$. Thus $(T, \leqq)$ is a locally finite tree. Further, by Theorem 2.5 we have $\mathrm{LA}(T, \leqq)$ is regular iff $T = K \cup \mathrm{Min}(T, \leqq)$, where K is a chain of the type $\omega^d + \omega$ and simultaneously every element $t \in T$ has at most two predecessors, i.e. for any $t \in K$ there exists at most one element $a \in \mathrm{Min}(T, \leqq)$ with the property $a^+ = t$. □

3.2. Corollary 1. *Let $(T, \leqq)$ be an upper locally finite forest with the inverse semigroup* $\mathrm{LA}(T, \leqq)$ *of local automorphisms. The following conditions are equivalent:*

1° $\mathrm{Max}(T, \leqq) = \emptyset$.

2° $\mathrm{LA}(T, \leqq)$ *is upper pointwise transitive on T.*

3° $\mathrm{LA}(T, \leqq)$ *is upper regionally transitive on T.*

3.3. Corollary 2. *Let $(T, \leqq)$ be a locally finite forest with the inverse monoid* $\mathrm{LA}(T, \leqq)$ *of all local automorphisms which is universally transitive on the set T. Then $(T, \leqq)$ is a chain of the type $\omega^d + \omega$, thus* $\mathrm{LA}(T, \leqq)$ *is the group of order automorphisms.*

Form the definition of a coregular semigroup ([2], [5]) it follows immediately the identity $x^2 = x^4$, i.e. x^2 is idempotent. In the class of antiinverse semigroups the identity $x^5 = x$ holds (cf. [3] Theorem 2.1 or [15] Lemma 1). From here there follows immediately:

3.4. Proposition. *If $(T, \leqq)$ is an upper or lower locally finite forest containing a chain of the type ω or ω^d then* $\mathrm{LA}(T, \leqq)$ *is neither coregular nor antiinverse.* □

Now we observe the question of the complete regularity of $\mathrm{LA}(T, \leqq)$. The following theorem shows that the requirement of the complete regularity of $\mathrm{LA}(T, \leqq)$ enforce a very simple structure of the poset $(T, \leqq)$. Suppose first the poset $(T, \leqq)$ is connected, i.e. $(T, \leqq)$ is a locally finite tree. Recall that an element a of a semigroup S is said to be completely regular if there exists an element $x \in S$ such that $a = axa$ and $ax = xa$. A semigroup each element of which is completely regular is said to be completely regular. Certain characterizations of completely regular elements in abstract semigroups are contained e.g. in [8].

3.5. Theorem. *Let $(T, \leqq)$ be an upper locally finite antirooted tree. The following conditions are equivalent:*

1° $\mathrm{LA}(T, \leqq)$ *is completely regular.*

2° $\mathrm{LA}(T, \leqq)$ *is regular and commutative.*

3° $\mathrm{LA}(T, \leqq)$ *is inverse and commutative.*

4° $T = K \cup \mathrm{Min}(T, \leqq)$, *where* $(K, \leqq)$ *is a chain of the type* $\omega^d + \omega$ *and* $\mathrm{card}\,\mathrm{Min}(T, \leqq) \leqq 1$.

Proof. The equivalence of conditions 3°, 4° follows immediately from Theorem 3.1 with respect to e.g. [13]. We shall verify these implications: 1° ⇒ 4° ⇒ ⇒ 2° ⇒ 1°.

1° ⇒ 4°: Since $\mathrm{LA}(T, \leqq)$ is completely regular, it is regular and the tree $(T, \leqq)$ has the form which is described in condition 2° of Theorem 2.5. Put $K = T \setminus \mathrm{Min}(T, \leqq)$ (which is a chain of the type $\omega^d + \omega$) and for any element $t \in T$ denote $P_t = \{x : x^+ = t\}$ (i.e. the set of all predecessors of the element t). Assume there exists an element $t \in T$ such that $\mathrm{card}\,P_t \geqq 3$. Consider a triad $x_1, x_2, x_3 \in P_t$ such that $x_1, x_2 \in \mathrm{Min}(T, \leqq)$, $x_3 \in K$ and a mapping $f: T \to T$ defined by $f(x_1) = x_2$, $f(x_2) = x_3$, $f(x) = x$ for every $x \in T \setminus \{x_1, x_2\}$. It is clear that $f \in \mathrm{LA}(T, \leqq)$ and for any local automorphism g of $(T, \leqq)$ satisfying the condition $f = fgf$ there holds $g(x_1) \in \{x_1, x_2, x_3\}$, $g(x_2) = x_1$, $g(x) = x$ for each $x \in T \setminus \{x_1, x_2\}$. Then we have $fg(x_1) \in \{x_2, x_3\}$, but $gf(x_1) = x_1$, i.e. $fg \neq gf$, which contradicts the assumption of the complete regularity of the monoid $\mathrm{LA}(T, \leqq)$. Hence $\mathrm{card}\,P_t \leqq 2$ for every element $t \in T$. Admit there exists a pair of different elements $t_1, t_2 \in T$ with the property $\mathrm{card}\,P_{t_1} = \mathrm{card}\,P_{t_2} = 2$. Without loss of generality we can suppose $t_1 < t_2$. Consider elements $x_i \in P_{t_1} \cap \mathrm{Min}(T, \leqq)$ for $i = 1, 2$ and $y \in P_{t_2} \setminus \mathrm{Min}(T, \leqq)$. Assume $f \in \mathrm{LA}(T, \leqq)$ is a local automorphism such that $f(x_1) = x_2$, $f(x_2) = f(y) \in K$ and $f(x) \in K$ for $x \in T$, $x \neq x_1$. For a local automorphism g of the tree $(T, \leqq)$ satisfying $f = fgf$ we have $g(x_2) = x_1$ since $f^{-1}(x_2) = \{x_1\}$ and $g(x) \in K$ for every element $x \in K$, thus $gf(y) = y$. Further, $fg(x_2) = x_2 \neq y = gf(y) = gf(x_2)$, which is a contradiction again. Therefore we have the tree $(T, \leqq)$ has at most one minimal element, i.e. condition 3° is satisfied.

4° ⇒ 2°: If condition 3° is satisfied then $\mathrm{LA}(T, \leqq)$ is regular according to Theorem 2.5. With respect to [13] we have $\mathrm{LA}(T, \leqq)$ is commutative, thus 2° holds.

The implication 2° ⇒ 1° is trivial, hence the proof is complete. □

Remark. Using the notion of a branch we can characterize (with respect to the above Theorem 3.5) a locally finite tree without root with the completely regular monoid of local automorphisms as follows:

Let $(T, \leqq)$ be a locally finite antirooted tree. Then $\mathrm{LA}(T, \leqq)$ is completely regular iff either $(T, \leqq)$ has the only one branch of the type $\omega^d + \omega$ (i.e. $\mathrm{LA}(T, \leqq)$ is an infinite cyclic group) or $(T, \leqq)$ is the union of exactly two branches, one of which has the type ω, the other has the type $\omega^d + \omega$ and their symmetrical difference is a singleton.

Removing the assumption of the connectedness of $(T, \leqq)$ we get this result:

3.6. Theorem. *Let* $(T, \leqq)$ *be an upper locally finite antirooted forest,* $\{(T_i, \leqq): i \in I\}$ *be the collection of all its maximal trees. The following conditions are equivalent:*

1° $\mathrm{LA}(T, \leqq)$ *is completely regular.*

2° $T = \bigcup_{i \in I} K_i \cup \text{Min}(T, \leqq)$, *where* $(K_i, \leqq)$ *is a chain of the type* $\omega^d + \omega$ *for any* $i \in I$, $1 \leqq \text{card}\, I \leqq 2$ *and* $\text{card Min}(T, \leqq) \leqq 1$.

Proof. 1° ⇒ 2°: Admit $\text{card}\, I \geqq 3$. Suppose $(T_i, \leqq)$, $i = 1, 2, 3$ are three different maximal trees (i.e. components) of the forest $(T, \leqq)$. Choose $t_i \in T_i$ for $i = 1, 2, 3$. Let $f \in \text{LA}(T, \leqq)$ be a local authomorphism such that $f(t_1) = t_2$, $f(t_2) = t_3$, $f(t) = t$ for each $t \in T_3$. If $g \in \text{LA}(T, \leqq)$ has the property $fgf = f$ then we have $g(t_2) = t_1$ and $gf(t_2) = g(t_2) = t_1$, $fg(t_1) \in T_2 \cup T_3$, thus $fg \neq gf$, which contradicts the assumption of the complete regularity of $\text{LA}(T, \leqq)$. Hence $1 \leqq \text{card}\, I \leqq 2$. Let $(T_i, \leqq)$ be an arbitrary component of the forest $(T, \leqq)$. Consider a subsemigroup S of $\text{LA}(T, \leqq)$ consisting of all local automorphisms $g \in \text{LA}(T, \leqq)$ such that $g(t) \in T_i$ for any $t \in T_i$ and $g(t) = t$ for $t \in T_j$, $j \in \{1, 2\}$, $i \neq j$. In virtue of Theorem 3.5 we have $T_i = K_i \cup \text{Min}(T_i, \leqq)$, where K_i is a chain of the type $\omega^d + \omega$ and $\text{card Min}(T_i, \leqq) \leqq 1$.

Now admit there exists at least one pair of elements $t_1, t_2 \in \text{Min}(T, \leqq)$ with $t_i \in T_i$, $i = 1, 2$, where $(T_1, \leqq)$, $(T_2, \leqq)$ are components of the forest $(T, \leqq)$. Consider a local automorphism f of the forest $(T, \leqq)$ such that $f(t_1) = t_2$, $f(t) = t$ for any $t \in T_2$. If $g \in \text{LA}(T, \leqq)$ is an automorphism with the property $fgf = f$, then $gf(t_1) = g(t_2) = t_1$ and $fg(t_1) \neq t_1$ for $t_1 \notin f(T)$. Thus $gf \neq fg$, which is a contradiction again. Hence $\text{card Min}(T, \leqq) \leqq 1$.

2° ⇒ 1°: Let $f \in \text{LA}(T, \leqq)$ be an arbitrary local automorphism. We are going to show first that the restriction $f_1 = f \mid f(T)$ is an order automorphism of the forest $(T, \leqq)$ onto itself. Since $f \in \text{LA}(T, \leqq)$ we have either $f(T) = K_1 \cup K_2$ or $f(T) = K_i$, where $i \in \{1, 2\}$. Suppose $x \in f(T)$ and $t \in T$ with the property $x = f(t)$. If $t = x$ we have $x = f^2(t)$ thus $x \in f^2(T)$. If $t \neq x$ and $f(x) \neq x$ then either x is comparable with $f(x)$ or the element t is comparable with $f(x)$. In the first case there exists t_1 (belonging to the component containing x and $f(x)$) with the property $x = f(t_1)$, in the second case there exists an element t_1 comparable with x such that $f(t_1) = t$. Since in the first case t_1 is comparable with x, there exists an element t_2 comparable with x with the property $f(t_2) = t_1$, hence $f^2(t_2) = f(t_1) = x$. In the second case $f^2(t_1) = f(t) = x$, thus $x \in f^2(T)$. Therefore we get the inclusion $f(T) \subset f^2(T)$, implies (with respect to the evident opposite inclusion) the equality $f(T) = f^2(T)$, consequently the mapping $f_1 = f \mid f(T)$ is surjective. Assume $t_1, t_2 \in f(T)$, $t_1 \neq t_2$. If elements t_1, t_2 are comparable then elements $f(t_1)$, $f(t_2)$ are different and comparable as well. Assume $t_1 \parallel t_2$ and admit $f(t_1) = f(t_2)$. Since $\text{Min}(T, \leqq) \subset T \setminus f(T)$, we can suppose under a suitable notation $t_i \in K_i$ for $i = 1, 2$. Suppose $f(t_1) = f(t_2) \in K_1$. There is $t \in T$ with $f(t) = t_2$. In case $t \in K_2$ elements t, t_2 are endpoints of an interval, which contradicts the fact $f \in \text{LA}(T, \leqq)$. In case $t \in K_1$ the elements t, t_1 determine an interval in K_1, which is a contradiction again. In the same way we get a contradiction under the assumption $f(t_1) = f(t_2) \in K_2$, hence $f(t_1) \neq f(t_2)$. We have got the mapping f_1

is a bijection of $f(T)$ onto itself. Since it is a restriction of a local automorphism, f_1 is an order automorphism of the forest $(f(T), \leqq)$ onto itself. Now define a mapping $g \in T^T$ by the rules: $g(t) = f_1^{-1}(t)$ for $t \in f(T)$, $g(t) = f_1^{-2}(f(t))$ for $t \in T \setminus f(T)$. For any element $t \in T$ we have $fgf(t) = f_1(f_1^{-1}(f(t))) = f(t)$ and $gf(t) = f_1^{-1}(f(t))$. If $t \in f(T)$, we have $f_1^{-1}(f(t)) = f_1^{-1}(f_1(t)) = t = f_1(f_1^{-1}(t)) = fg(t)$ for $f_1^{-1}(t) \in f(T)$. If $t \in T \setminus f(T)$ then $fg(t) = f(f_1^{-2}(f(t))) = f_1 f_1^{-2}(f(t)) = f_1^{-1}(f(t)) = gf(t)$, hence $fg = gf$.

It remains to verify that g is a local automorphism of the forest $(T, \leqq)$. Suppose $s, t \in T$, $s < t$. If $s, t \in f(T)$, then $[s, t] \subset f(T)$ and since f_1 is an automorphism of the forest $(f(T), \leqq)$ we have $f_1^{-1} \in \mathrm{LA}(f(T), \leqq)$ hence the interval $[g(s), g(t)]$ is order-isomorphic to the interval $[s, t]$. Suppose $[s, t] \cap (T \setminus f(T)) \neq \emptyset$. Then either

a) $[s, t] \subset T \setminus f(T)$ or

b) $s \in T \setminus f(T)$ and $x \in f(T)$ for every $x \in (s, t]$.

In case a) $[f(s), f(t)] \cong [s, t]$ and since f_1^{-2} is an automorphism of the forest $(f(T), \leqq)$ we have $[g(s), g(t)] = [f_1^{-2}(f(s)), f_1^{-2}(f(t))] \cong [f(s), f(t)]$, therefore $[g(s), g(t)] \cong [s, t]$.

Consider case b). Since $f(s) \prec f(s^+)$ and $f(s), f(s^+) \in f(T)$ we have $f_1^{-2}(f(s)) \prec f_1^{-2}(f(s^+))$. There exists an element $x_s \in f(T)$ such that $f(x_s) = f(s)$ and $x_s \prec s^+$. Then $x_s = f_1^{-1}(f(s))$ and further $g(s) = f_1^{-2}(f(s)) = f_1^{-1}(x_s) \prec f_1^{-1}(s^+) = g(s^+)$ which implies $[g(s), g(s^+)] \cong [s, s^+]$. Further we have $[g(s^+), g(t)] \cong [s^+, t]$ (case (a)), hence $[g(s), g(t)] = [g(s), g(s^+)] \oplus [(g(s^+))^+, g(t)] \cong [s, s^+] \oplus [s^{++}, t] = [s, t]$. Consequently, the mapping g is a local automorphism of the forest $(T, \leqq)$, therefore $\mathrm{LA}(T, \leqq)$ is completely regular. □

Remark. In case of a disconnected forest the complete regularity of $\mathrm{LA}(T, \leqq)$ implies neither commutativity nor the property to be inverse (on the contrary to the case ot a tree) which shows Example 1. If $(\mathbf{Z}, \lesssim)$ is a locally finite forest considered in the mentioned Example 1, then $\mathrm{LA}(\mathbf{Z}, \lesssim)$ is not inverse in virtue of Theorem 3.1. According to Theorem 3.6 $\mathrm{LA}(\mathbf{Z}, \lesssim)$ is completely regular. However it is not commutative, since for mappings $f, g \in \mathrm{LA}(\mathbf{Z}, \lesssim)$ defined by $f(1) = 2$, $g(2) = 1$ and $f(2) = 2$, $g(1) = 1$ we have $fg \neq gf$.

4. Height Preserving Transitivity and Regularity of Local-automorphism Semigroups of Rooted Trees

In this paragraph we shall consider locally finite trees with roots especially for the case of upper trees (the dual case of lower trees is similar), i.e. we suppose that every tree $(T, \leqq)$ possesses the greatest element.

4.1. Definition. The *height function in a rooted upper locally finite tree* $(T, \leqq)$ *is a function* $\lambda\colon T \to \mathbf{N}_0$ *such that* $\lambda(t) = 0$ *iff* t *is the root of* $(T, \leqq)$ *and* $\lambda(t) = \lambda(t^+) + 1$ *for all* $t \in T$ *different from the root of* $(T, \leqq)$. *The integer* $\lambda(t)$ *is called the height of the element* t *in the rooted tree* $(T, \leqq)$.

Convention. For an integer $n \in \mathrm{N}_0$ we put $\mathrm{L}(n) = \{t: t \in T, \lambda(t) = n\}$ and the set $\mathrm{L}(n)$ is called the n-th layer of $(T, \leqq)$.

Recall that the ordinal which is the supremum of the set of ordinals of all maximal chains of $(T, \leqq^d)$ (for an upper tree $(T, \leqq)$) is said to be the height of the tree $(T, \leqq)$ and it is denoted by $\mathrm{h}(T, \leqq)$. In case of a lower tree we consider $(T, \leqq)$ instead of $(T, \leqq^d)$.

4.2. Definition. *Let $(T, \leqq)$ be a rooted tree. The monoid* $\mathrm{LA}(T, \leqq)$ *is said to be height preserving transitive (shortly HP-transitive) if for any pair of elements $a, b \in T$ such that $\lambda(a) = \lambda(b)$ there exists $f \in \mathrm{LA}(T, \leqq)$ with the property $f(a) = f(b)$.*

4.3. Proposition. *Let $(T, \leqq)$ be an upper locally finite rooted tree. The following conditions are equivalent:*

1° $\mathrm{LA}(T, \leqq)$ *is HP-transitive on T.*

2° *Either every maximal chain in $(T, \leqq)$ is finite and any two maximal chains in $(T, \leqq)$ have the same length or* $\mathrm{Min}\,(T, \leqq) = \emptyset$.

The proof is straightforward and hence omitted.

Remark. A similar proposition to the above one holds also for lower trees ($\mathrm{Min}\,(T, \leqq)$ should be changed by $\mathrm{Max}\,(T, \leqq)$). The same remark concerns to the below Theorem 4.6.

Put $U_T = \{t: t \in T, \exists\,(x_1, x_2) \in (T \times T) \setminus \triangle_T;\ x_1 \prec t,\ x_2 \prec t\}$.

4.4. Theorem. *Let $(T, \leqq)$ be a locally finite rooted tree with the HP-transitive monoid of all local automorphisms. Then $\mathrm{LA}(T, \leqq)$ is regular iff $\mathrm{h}(T, \leqq) < \omega$ and* card $U_T \leqq 1$.

Proof. Consider the case of an upper tree. Suppose $\mathrm{LA}(T, \leqq)$ is regular. Admit the tree $(T, \leqq)$ contains a chain $(C, \leqq)$ of the type ω^d, thus (by the assumption and Proposition 4.3) every maximal chain of $(T, \leqq)$ has the type ω^d. Define a mapping $g: C \to C$ by the rule: $[g(c)]^+ = c$ for any $c \in C$. Since $(C, \leqq)$ is an LA-retract of the tree $(T, \leqq)$ (cf. [4]), there exists $h \in \mathrm{LA}(T, \leqq)$ such that $h(T) = C$ and $h \mid C = \mathrm{id}_C$. Then the mapping $f = gh$ belongs to $\mathrm{LA}(T, \leqq)$ and for any $\varphi \in T^T$ with the property $f = f\varphi f$ we have $\varphi \notin \mathrm{LA}(T, \leqq)$, which contradicts the assumption. Every chain of $(T, \leqq)$ is finite and with respect to Proposition 4.3 all maximal chains of $(T, \leqq)$ have the same finite length, thus $h(T, \leqq) < \omega$. Now admit card $U_T \geqq 2$. Let $a, b \in U_T$ be a pair of elements such that $a < b$. Consider elements $b_1, b_2 \in T$ with the property $b_1^+ = b_2^+ = b$, and simultaneously $a \leqq b_2$. Further suppose $\{t_i: i = 1, 2, 3\} \subset T$ is a set satisfying conditions: $t_1 < b_1$, $t_2^+ = t_3^+ = a$, $\lambda(t_1) = \lambda(t_2) = \lambda(t_3)$. Define a selfmap f of T in this way: $f(t_1) = t_2$, $f(t_2) = f(t_3) = t_3$, the restriction $f \mid (T \setminus ((b_1]_{\leqq} \cup (a]_{\leqq}))$ is an identity mapping, and further we extend f onto the set $(b_1]_{\leqq} \cup (a]_{\leqq}$ in such a way that $f \in \mathrm{LA}(T, \leqq)$. The down set $(b_1]_{\leqq}$ is then mapped onto a chain containing elements t_2, a. The mapping f satisfying the mentioned conditions exists in virtue of Proposition 4.3.

If $g \in T^T$ is an arbitrary mapping satisfying the equality $f = fgf$, then $g(t_2) \in (b_1]_{\leq}$, $g(t_3) \in (a]_{\leq}$ and either $g(a) \in (b_1]_{\leq}$ or $g(a) = a$. Then either $g(t_3) \parallel g(a)$ or $g(t_2) \parallel g(a)$, thus $g \notin \mathrm{LA}(T, \leqq)$, which is a contradiction. Hence card $U_T \leqq 1$.

Now suppose $\mathrm{h}(T, \leqq) < \omega$, card $U_T \leqq 1$. If $U_T = \emptyset$, then $\mathrm{LA}(T, \leqq) = \{\mathrm{id}_T\}$, hence the implication is satisfied. Assume $U_T = \{a\}$, where $a \in T$ is an element of a height $\lambda(a) = m \in \mathrm{N}_0$ and $f \in \mathrm{LA}(T, \leqq)$. If $\mathrm{h}(T, \leqq) = n \in \mathrm{N}$ then the monounary algebra (T, f) has at least $n + 1$ components, where $(\mathrm{L}(k), f_k)$ for $k \leqq m(f_k = f \mid \mathrm{L}(k))$ are one-element cycles. Consider a monounary algebra $(\mathrm{L}(m + 1), f_{m+1})$, where $f_{m+1} = f \mid \mathrm{L}(m + 1)$. Define a mapping $g_{m+1}\colon \mathrm{L}(m + 1) \to \mathrm{L}(m + 1)$ in this way: For $t \in \mathrm{L}(m + 1)$ with the property $f_{m+1}^{-1}(t) = \emptyset$ we put $g_{m+1}(t) = t$. For $t \in \mathrm{L}(m + 1)$ such that $f_{m+1}^{-1}(t) \neq \emptyset$ we choose one element $s \in f_{m+1}^{-1}(t)$ arbitrarily and put $g_{m+1}(t) = s$. Suppose the mapping $g_{m+k}\colon L(m + k) \to \mathrm{L}(m + k)$, where $1 \leqq k \leqq n - m$ and $m + 1 < n$ is defined. For $t \in \mathrm{L}(m + k + 1)$ we put $g_{m+k+1}(t) = s$, where $s^+ = g_{m+k}(t)$. Further we put $g(t) = t$ for any element $t = \bigcup_{k=0}^{m} \mathrm{L}(k)$ and $g(t) = g_k(t)$ for each $t \in \mathrm{L}(k)$, $k = m + 1, \ldots, n$. From the construction of the mapping g it follows $g \in \mathrm{LA}(T, \leqq)$ and $f = fgf$, hence f is a regular element of the monoid $\mathrm{LA}(T, \leqq)$. Consequently $\mathrm{LA}(T, \leqq)$ is regular. □

From the just proved theorem there follows immediately with respect to [6] Theorem 1.17:

4.5. Corollary. *Let $(T, \leqq)$ be a locally finite rooted tree such that* $\mathrm{LA}(T, \leqq)$ *is* HP-*transitive on the set T. Then* $\mathrm{LA}(T, \leqq)$ *is inverse iff it is trivial* (i.e. $\mathrm{LA}(T, \leqq) = \{\mathrm{id}_T\}$). □

4.6. Theorem. *Let $(T, \leqq)$ be an upper locally finite rooted tree. The following conditions are equivalent:*

1° *T is finite and* card Min $(T, \leqq) \leqq 2$.

2° $\mathrm{LA}(T, \leqq)$ *is coregular.*

3° $\mathrm{LA}(T, \leqq)$ *is anti-inverse.*

Proof. Suppose 1° holds. If $f \in \mathrm{LA}(T, \leqq)$ then either $f = \mathrm{id}_T$ or $f^2 = f$ or $f^2 = \mathrm{id}_T, f \neq \mathrm{id}_T$ (an involutory nonidentical local automorphism exists in $\mathrm{LA}(T, \leqq)$, if the tree $(T, \leqq)$ has two branches of the same length). In the all possible cases we have $f^3 = f$, thus $\mathrm{LA}(T, \leqq)$ is coregular. Suppose 2° holds. Then $f^3 = f$ for any $f \in \mathrm{LA}(T, \leqq)$ (cf. [2] Theorem 3), hence any automorphism $f \in \mathrm{LA}(T, \leqq)$ is selfanti-inverse. Therefore Condition 3° is satisfied. Now assume 3°. Admit the tree $(T, \leqq)$ possesses a chain $(C, \leqq)$ of the type ω^d. A mapping $g\colon C \to C$ defined (as in the proof of Theorem 4.4) by $[g(c)]^+ = c$ for any $c \in C$, belongs to $\mathrm{LA}(C, \leqq)$. For $h \in \mathrm{LA}(T, \leqq)$ such that $h(T) = C$, $h \mid C = \mathrm{id}_C$, we have $f = gh \in \mathrm{LA}(T, \leqq)$ and $f^5 \neq f$. This is a contradiction e.g. in virtue of [3] Theorem 2.1. Therefore any chain of the tree $(T, \leqq)$ is finite. Assume card Min $(T, \leqq) \geqq 3$. Then there is at least one triad $a, b, c \in \mathrm{Min}\,(T, \leqq)$ of different elements which satisfies (without loss of generality) this condition: $\lambda(a) \leqq \lambda(b) \leqq \lambda(c)$. Then there exists $f \in LA(T, \leqq)$ such that $f(a) = b$, $f(b) = c$, $f(c) = c$, consequently $f^5 \neq f$ again. This contradiction shows that 1° holds. □

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