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GLOBAL TRANSFORMATIONS OF LINEAR DIFFERENTIAL EQUATIONS AND QUADRATIC FUNCTIONALS, II.

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Dedicated to my father to his 70th birthday.

This paper is a continuation of the paper [1] in which there were studied extremal properties of quadratic functionals of the type

$$J(y) = \int_0^B [y'^2(t) + q(t)\, y^2(t)]\, dt,$$

where $q(t) \in C^0(0, B]$ and $y(t)$ are A-admissible functions on $[0, B]$:

i) $y(t) \in C^0[0, B]$, $y(0) = y(B) = 0$

ii) $y(t)$ is absolutely continuous and $y'^2(t)$ is Lebesque integrable on each closed subinterval of the interval $(0, B]$.

Throughout the paper only Lebesque integral is used (briefly L-integral).

The associated Euler equation (E-equation) to the functional $J(y)$ is denoted by (q), i.e.

$$(q) \qquad y'' = q(t) \,.\, y$$

In the paper [1] there was introduced the new unified approach to study of the quadratic functionals of type $J(y)$. This approach is based on the basic result of Borůvka's theory consisting in the fact that each linear differential equation of the second order on its whole definition interval can be globally transformed to the equation $y'' + y = 0$ on a suitable interval. Firstly by classical means of direct evaluation there are derived the properties of one special functional corresponding to E-equation $y'' + y = 0$ and then by means of Borůvka's global transformation the result are extended to quadratic functionals of type $J(y)$.

In this paper we are going to demonstrate that all results of quadratic functionals study of type $J(y)$ can be transmised to associated Euler equation of the quadratic functionals of type

$$(\mathrm{p, q}) \qquad (p(t) \,.\, y'(t))' = q(t) \,.\, y(t)$$

by special global transformations. Further we find a certain invariant of all associated E-equations (p, q) and the new class of these functionals that achieve absolute minimum equal to 0 on a set of A-admissible functions $y(t)$ explicitly given

The quadratic functionals were systematically studied by means of classical methods mainly by W. Leightou [6]. Further it was J. Krbiľa [7] who dealt with the quadratic functionals. Using Borůvka's theory of central dispersion, he studied special properties of a selfadjoint linear diferential equation of second order

$$(p(t) \,.\, y'(t))' + q(t) \,.\, y(t) = 0$$

whose coefficiens $p(t)$, $q(t)$ are the complex functions of a real argument.

I. THE MAIN RESULTS OF PAPER [1]

Theorem 1. ([1], Theorem 1)

$$\liminf_{e \to 0+} \int_e^b [y'^2(t) - y^2(t)]\,dt \geqq 0 \tag{I}$$

for each A-admissible function $y(t)$ if only if $0 < b \leqq \pi$. If $b < \pi$, then the equality in (I) *occurs exactly for $y(t) \equiv 0$.*

If $b = \pi$, then the equality in (I) *occurs exactly for the system of functions $y(t) = k \,.\, \sin t$, $k \in R$. If $b > \pi$, then the relation* (I) *is not satisfied for all A-admissible functions on* $[0, b]$.

Theorem 2. ([1] Corollary 3)

Let $Y(T)$ be A-admissible function on $[0, B]$ and $Q(T) \in C^0_{(0, B]}$ be a fixed chosen function.

$$1\,\text{m}\inf_{E \to 0+} \int_E^B [Y'^2(T) + Q(T)\, Y^2(T)]\,dT \geqq 0 \tag{II}$$

for each A-admissible function $Y(T)$ if and only if there exists an increasing phase $A(T)$ of E-equation (Q) *such that*

$$\lim_{T \to 0+} A(T) = 0, \qquad \lim_{T \to B-} A(T) \leqq \pi.$$

If $\lim_{T \to B-} A(T) < \pi$, then the equality in (II) *occurs just for $Y(T) \equiv 0$.*

If $\lim_{T \to B-} A(T) = \pi$, then the equality in (II) *occurs exactly for the functions*

$$Y(T) = \frac{k \sin (A(T))}{\sqrt{A'(T)}}, \qquad k \in R.$$

If $\lim_{T \to B-} A(T) > \pi$, then the relation (II) *is not satisfied for all A-admissible functions on* $[0, B]$.

Theorem 3. ([1] Corollary 5)

All differential equations (Q1) *with the basic central dispersion* φ *of the first kind satisfying* $\varphi(T + B) = \varphi(T)$ *are given*

$$Y''(T) = \left\{-\frac{\pi^2}{B^2} + \left[f''\left(\frac{\pi}{B}T\right) + f'^2\left(\frac{\pi}{B}T\right) + 2f'\left(\frac{\pi}{B}T\right)\operatorname{cotg}\frac{\pi}{B}T\right]\frac{\pi^2}{B^2}\right\} Y(T).$$

(Q1)

where the function f *is an arbitrary* π*-periodic function of the class* $C^2_{(-\infty,\infty)}$ *with the following property*:

$$f(0) = f'(0) = 0; \qquad \int_0^\pi \frac{\exp(-2f(T)) - 1}{\sin^2 T}\,\mathrm{d}T = 0.$$

Then

$$\liminf_{E\to 0+} \int_E^B [Y'^2(T) + Q1(T)\,Y^2(T)]\,\mathrm{d}T = 0$$

or each A-admissible function $Y(T)$ *on* $[0, B]$. *This functional achieves its absolute minimum equal to zero exactly on the functions*

$$Y(T) = k\,e^{f\left[\frac{\pi}{B}T\right]}\sin\left(\frac{\pi}{B}T\right),$$

where $k \in R$.

II. THE STUDY OF THE QUADRATIC FUNCTIONALS WITH THE ASSOCIATED E-EQUATIONS OF THE TYPE $(py')' = qy$

Notation. Let us denote by (p^-, q) the functional of the type

$$\int_0^b [p(t)\,y'^2(t) + q(t)\,y^2(t)]\,\mathrm{d}t,$$

where $(p(t) > 0,\ q(t)) \in C^0_{(0,b]}$, $y(t)$ are A-admissible functions on $[0, b]$ and corresponding L-integral is taken on open interval $(0, b)$. The associated E-equation to the functional (p^-, q) is denoted by (p, q) i.e.

(p, q) $$(p(t)\,.\,y'(t))' = q(t)\,.\,y(t)$$

Since E-equation is defined on an open interval $(0, b)$, we define all terms of Borůvka's theory [2] for the E-equations (p, q) analogously as for the equations $y''(t) = q(t)\,.\,y(t)$, for example phase, dispersion, type, kind, character (see [2] or [1]). The functional of the type $(1^-, q)$ is denoted by $(\bar{q})$ and the associated E-equation $(1, q)$ is denoted by (q).

Assumption. Let us assume that there exists L-integrals

$$\int_0^b \frac{dt}{p(t)} \quad \text{and} \quad \int_0^b q(t)\, dt.$$

Lemma 1. ([2] p. 1 or [3] p. 160)
Let $(p(t) > 0,\ q(t)) \in C_j^0 j$ *Define*

$$\psi(t) = \int_{t_0}^{t} \frac{d\sigma}{p(\sigma)}, \qquad t_0 \in j = (a, b).$$

Let γ *be the inverse to* ψ. *The transformation* $t = \gamma(\xi)$, $y(t) = w(\xi)$ *transforms the equation* (p, q) $(p(t) \,.\, y'(t))' = q(t) \,.\, y(t)$ *into the equation*

$$\text{(Q)} \qquad \frac{d^2 w}{d\xi^2} - Q(\xi)\, w = 0,$$

where $Q(\xi) = p(\gamma(\xi)) \,.\, q(\gamma(\xi))$.
The interval $j = (a, b)$ *is transformed to* $(\psi(a), \psi(b))$.

Remark 1. Equations (Q) and (p, q) have the same character. In case that the equation (p, q) is the Euler's equation of the functional (p^-, q) we put $t_0 = 0$, i.e $\psi(t) = \int_0^t \frac{d\sigma}{p(\sigma)}$.
Since the value of L-integral $\int_0^t \frac{d\sigma}{p(\sigma)}$ for $t \in (0, b]$ is not dependent on the value of the function $\frac{1}{p(t)}$ at the point zero we define $\frac{1}{p(0)} = 0$.

Lemma 2. *Let the functional of type* (p^-, q) *is given and* $B = \int_0^b \frac{dt}{p(t)}$. *Let* $Y(\xi)$ *are A-admissible functions on* $[0, B]$. *Then there exists the functional of type* $(\bar{Q})$ *which is defined on* $[0, B]$ *with this property. There exists a one–to–one maping of the set of A-admissible functions* $Y(\xi)$ *on* $[0, B]$ *on to the set of A-admissible functions* $y(t)$ *on* $[0, b]$ *such that for each two corresponding A-admissible functions* $Y(\xi)$ *and* $y(t)$ *it holds*:

$$\text{(III)} \quad \liminf_{e \to 0_+} \int_e^b [p(t)\, y'^2 + q(t)\, y^2(t)]\, dt = \liminf_{E \to 0_+} \int_E^B [\dot{Y}^2(\xi) + Q(\xi)\, Y^2(\xi)]\, d\xi.$$

Proof. According to Lemma 1 and Remark 1 we construct the transformation $\xi = \int_0^t \frac{d\sigma}{p(\sigma)}$. Firstly we prove that the function $Y\left(\int_0^t \frac{d\sigma}{p(\sigma)}\right)$ is an A-admissible function on $[o, b]$, if the function $Y(\xi)$ is A-admissible function on $[0, B]$.

i) If $Y(\xi) \in C^0_{[0,B]}$, $Y(0) = Y(B) = 0$ then also $Y\left(\int_0^t \frac{d\sigma}{p(\sigma)}\right) \in C^0_{[0,b]}$ and $Y\left(\int_0^0 \frac{d\sigma}{p(\sigma)}\right) = Y\left(\int_0^b \frac{d\sigma}{p(\sigma)}\right) = 0$.

ii) If $Y(\xi)$ is absolutely continuous and $\dot{Y}^2(\xi)$ is L-integrable on each closed subinterval of the interval $(0, B]$ then $Y\left(\int_0^t \frac{d\sigma}{p(\sigma)}\right)$ is absolutely continuous and $Y'^2\left(\int_0^t \frac{d\sigma}{p(\sigma)}\right)$ is L-integrable on each closed subinterval of the interval $(o, b]$. We see that

$$Y'\left(\int_0^t \frac{d\sigma}{p(\sigma)}\right) = \dot{Y}(\xi)/p\ t)$$

on each closed interval of the interval $(o, b]$ almost everywhere. Consequently the function $Y\left(\int_0^t \frac{d\sigma}{p(\sigma)}\right)$ is here absolutely continuous.

Then

$$Y'^2\left(\int_0^t \frac{d\sigma}{p(\sigma)}\right) = \frac{\dot{Y}^2(\xi)}{P^2\ t)}$$

and this function is L-integrable on each closed subinterval of interval $(o, b]$ (see [3] p. 312). Similarly we can prove, that for each A-admissible function $y(t)$ on $[o, b]$ is the function $y(\gamma(\xi))$ A-admissible function on $[0, B]$, where $\gamma(\xi)$ is inverse to $\psi(t) = \int_0^t \frac{d\sigma}{p(\sigma)}$. Immediately it holds

$$\int_e^b [p(t)\, y'^2(t) + q(t)\, y^2(t)]\, dt = \int_e^b \left\{p(t)\left[Y\left(\int_0^t \frac{d\sigma}{p\ \sigma)}\right)\right]'^2 + q\ t)\, Y^2\left(\int_0^t \frac{d\sigma}{p(\sigma)}\right)\right\} dt.$$

Substituting

$$\xi = \int_0^t \frac{d\sigma}{(\sigma)}, \quad d\xi = \frac{dt}{p(t)}, \quad t = \gamma(\xi).$$

This substitution satisfies all properties for substitution of L-integrals. Further if $e \to 0_+$ then also $E = \int_0^e \frac{d\sigma}{p(\sigma)} \to 0_+$. Putting $Q(\xi) = p(\gamma(\xi)) \,.\, q(\gamma(\xi))$, we get the equation (III) and the proof is complete.

Theorem 4. *The functional of the type* (p^-, q) *satisfies the relation*

$$\text{(IV)} \qquad \liminf_{e \to 0_+} \int_e^b [p(t)\, y'^2(t) + q(t)\, y^2(t)]\, dt \geqq 0$$

for each A-admissible function $y(t)$ *on* $[o, b]$ *if and only if the associated E-equation*

$$\text{(p, q)} \qquad (p(t) \,.\, y'(t))' = q(t) \,.\, y(t)$$

is disconjugate on the open interval $(0, b)$, *i.e. if and only if there is an increasing* *phase* $A(\xi)$ *of the equation* (Q) *such that*

$$\lim_{\xi\to 0+} A(\xi) = 0, \qquad \lim_{\xi\to B-} A(\xi) \leqq \pi.$$

If $\lim_{\xi\to B-} A(\xi) = \pi$, *then the each function along which the functional* (p^-, q) *achieves its absolute minimum equal to zero exactly on the functions:*

$$y(t) = k \sin A\left(\int_0^t \frac{d\sigma}{p(\sigma)}\right) \Big/ \sqrt{A'\left(\int_0^t \frac{d\sigma}{p(\sigma)}\right)}$$

where $k \in R$.

If $\lim_{\xi\to B-} A(\xi) < \pi$, *then the equality in* (IV) *occurs just for* $y(t) \equiv 0$. *If* $\lim_{\xi\to B-} A(\xi) > \pi$, *then the relation* (IV) *is not satisfied for all A-admissible functions* $y(t)$ *on* $[0, b]$.

Proof: follows immediately from Lemma 2 and Theorem 2.

Theorem 5. *Let the equation* (p, q) *be the Euler's associated equation of the functional of type* (p^-, q) *and let the equation* (Q) *be the global transformation the equation* (p, q) *(see Lemma* 1). *Then the value of the integral*

$$\int_0^B Q(\xi)\, d\xi$$

is not dependent on the function $p(t) > 0$ *and is equal to* $\int_0^b q(t)\, dt$.

Proof. It holds $Q(\xi) = p(\gamma(\xi)) \cdot q(\gamma(\xi))$, from Lemma 1 and Remark 1. Further we recall that $B = \psi(b)$, $0 = \psi(0)$. Substitute $t = \gamma(\xi)$, which satisfies all properties of the substitution method for L-integral. We get $\gamma(0) = 0$, $\gamma(B) = b$ and $d\xi = \frac{dt}{p(t)}$. We get immediately $\int_0^B Q(\xi)\, d\xi = \int_0^B p(\gamma(\xi)) \cdot q(\gamma(\xi))\, d\xi = \int_{\gamma(0)}^{\gamma(B)} p(t) \cdot q(t) \times \frac{dt}{p(t)} = \int_0^b q(t)\, dt$. The proof is complete.

Lemma 3. ([4] p. 161)

Let Qd denote the class of all functions $q(t) \in C^0_{(-\infty,\infty)}$ *such that the equation* (q) $y''(t) = q(t) \cdot y(t)$ *have every nontrivial integral with zeros in equidistances equal to* d $(d > 0$, *const.). Then* $\min \int_0^d q(t)\, dt = -\pi^2/d$ *for* $q \in Qd$ *,the minimum being reached only for* $q = -\pi^2/d^2$.

Remark 2. Let $p(t) > 0$, $p(t) \in C^0_{(0,B]}$. Further extend its definition as follows: $p(t + b) = p(t)$. We put

$$B = \int_0^b \frac{d\sigma}{p(\sigma)}, \qquad T = \int_0^t \frac{d\sigma}{p(\sigma)}.$$

Then the differential equations

(Q_1)

$$Y''(T) = \left\{-\frac{\pi^2}{B^2} + \left[f''\left(\frac{\pi}{B}T\right) + f'^2\left(\frac{\pi}{B}T\right) + 2f'\left(\frac{\pi}{B}T\right)\operatorname{cotg}\frac{\pi}{B}T\right]\frac{\pi^2}{B^2}\right\} Y(T),$$

where f satisfies the properties of Theorem 3, are exactly all differential equation of the 2-nd order of type (Q) with the same basic central dispersion Φ of the first kind: $\Phi(T + B) = \Phi(T)$. Define the function

$$q_1(t) = Q_1\left(\int_0^t \frac{d\sigma}{p(\sigma)}\right)\Big/ p(t).$$

Theorem 6

All equations of type (p, q_1) have the same basic central dispersion φ of the first kind satisfying $\varphi(t + b) = \varphi(t)$ and for all functionals of type (p^-, q_1) it holds:

$$\lim_{e\to 0+} \operatorname{nf} \int_e^b [p(t)\, y'^2(t) + q_1(t)\, y^2(t)]\, dt \geqq 0$$

for each A-admissible function $y(t)$ on $[0, b]$. These functionals (p^-, q_1) achieve the absolute minimum equal to zero exactly along the functions

$$y(t) = k\, e^{f\left(\frac{\pi}{B}\int_0^t \frac{d\sigma}{p(\sigma)}\right)} . \sin\left(\frac{\pi}{B}\int_0^t \frac{d\sigma}{p(\sigma)}\right),$$

where $k \in R$.
Further the function $q_1(t)$ satisfies the relations

(N) $$-\frac{\pi^2}{b} \leqq \int_0^b q_1(t)\, dt \quad \text{and} \quad \frac{q_1(t_i)}{p(t_i)} = -\frac{\pi^2}{B^2}.$$

for at least four different values $t_i, \in (O, B]$.

Proof: follows immediately from Theorem 3, Remark 1, Theorem 4, Theorem 5, Lemma 3, Remark 2 and ([2] p. 136).

Remark 3. Theorem 6 gives the new necessary condition for the functionals of type (p^-, q) to achieve absolute mimum equaled zero in nontrivial A-admissible functions on the interval $[o, b]$.

Remark 4. F. Neuman [5] extended Borůvka's global transformation of linear differential equations of the second order to equations of the n-th order, $n \geqq 2$. Using it, it would be possible to extend this unified method to investigating extremal properties of more general functionals whose associated Euler's equation is a linear differential equation of the $2n$-th order.

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