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Archivum Mathematicum, Vol. 19 (1983), No. 4, 209--214

Persistent URL: `http://dml.cz/dmlcz/107175`

ARCH. MATH. 4, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XIX: 209—214, 1983

A TYPE OF CONTINUOUS PROJECTIONS

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(Received November 26, 1981)

1. Introduction

Let S be a nonempty set and $V \subseteqq S$. A mapping $E : S \to V$ satisfying $E(S) = V$ and $E^2 = E$ is said to be projection from S onto V. If S is a topological space, V a subspace of S and E a continuous mapping, then E is called continuous projection. Continuous projections in function spaces can be viewed as approximations of given functions in function subspaces. For instance, the orthogonal projection onto a closed subspace V of a Banach space is the best approximation with respect to V (see, e.g., [2]).

In practice we can comparatively easily solve problems of linear approximations. In this paper we show that a type of operators defined by means of linear approximations are continuous projections. This can be used for parameters estimation. We present the following examples in which f denotes a given function (experimental data) to be fitted by a function g using the least squares method (i.e., $\int_\alpha^\beta (f - g)^2 = \min$)

1. $g = \dfrac{1}{ax^2 + bx + c}$; An approximation of the exact solution can be obtained solving the problem

$$f_1 = \frac{1}{f}, \qquad g_1 = ax^2 + bx + c,$$

which is linear with respect to the parameters a, b, c.

2. $g = ae^{bx}$;

$$f_1 = \ln f, \qquad g_1 = bx + \ln a$$

3. $g = ae^{bx} + c$;

$$f_2 = \frac{df}{dx}, \qquad g_2 = by - d.$$

Solving of this problem determines $b^0 \neq 0$, d^0. We put $b^0 c^0 = d^0$ and solve the problem

$$f_1 = f, \qquad g_1 = a e^{b^0 x} + c^0.$$

Solving of this linear problem determines a^0. From the main theorem of this paper follows that the mapping

$$f \mapsto a^0 e^{b^0 x} + c^0$$

is a continuous projection in a space of sufficiently smooth functions.

Parameters estimations of such types were used in optimization programs package OPTIPACK [3] which was developed in Institute of Physical Metallurgy Computing Department of Czechoslovak Academy of Sciences.

Let R be a normed space, $V \subsetneqq S \subsetneqq R$. Then a mapping E from S onto V is a continuous projection from S onto V iff for every $z \in V$ the following condition holds

$$\lim_{\|y-z\| \to 0} \| E(y) - z \| = 0$$

i.e., for every $\varepsilon > 0$ there exists $\delta > 0$ such that for $y \in S$ satisfying $\| y - z \| < \delta$ it holds $\| E(y) - z \| < \varepsilon$.

2. Preliminary Lemmas

Notations. Throughout the following text we shall use the symbol R for a normed linear space over the field T of all real numbers. The norm in R is denoted by $\| . \|$. Further we shall consider the norm $[.]$ in T^n defined by

$$[(a_1, \ldots, a_n)] = \max \{| a_1 |, \ldots, | a_n |\}.$$

For $y_1, \ldots, y_n, y_0 \in R$ and $\delta > 0$ we put

$$\langle y_1, \ldots, y_n, y_0, \delta \rangle = \{(a_1, \ldots, a_n) \in T^n; \| a_1 y_1 + \ldots + a_n y_n + y_0 \| < \delta\}.$$

Lemma 1. $\langle y_1, \ldots, y_n, 0, \delta \rangle$ is a convex subset of T^n which is bounded iff $y_1, \ldots, y_n$ are linearly independent.

Notation. For the sake of simplicity we shall use the following notation:

$$\sup \langle y_1, \ldots, y_n, y_0, \delta \rangle = \sup \{[x]; x \in \langle y_1, \ldots, y_n, y_0, \delta \rangle\}.$$

If V is a finite-dimensional subspace of R and $x \in R$, we denote

$$\varrho_V(x) = \min_{y \in V} \| y - x \|.$$

Lemma 2. Let $y_1, \ldots, y_n$ be linearly independent elements in R. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sup \langle y_1, \ldots, y_n, 0, \delta \rangle < \varepsilon.$$

Lemma 3. Let $y_1, \ldots, y_n$ be linearly independent elements from R, $\Delta_1, \ldots, \Delta_n \in R$ and $\delta > 0$. Let us denote

$$A_1 = \langle y_1 + \Delta_1, \ldots, y_n + \Delta_n, \Delta_0, \delta \rangle,$$
$$A_2 = \langle y_1, \ldots, y_n, 0, \delta \rangle.$$

Then for every $\varepsilon > 0$ there exists $\sigma > 0$ such that $\| \Delta_i \| < \sigma$ for every i $(1 \leqq i \leqq n)$ implies

$$\sup A_1 - \sup A_2 < \varepsilon.$$

Proof. Suppose that there exists $\varepsilon_0 > 0$ such that for every $\sigma > 0$ from $\| \Delta_i \| \leqq \sigma$ $(1 \leqq i \leqq n)$ it follows

$$\sup A_1 - \sup A_2 \geqq \varepsilon_0 .$$

Let us denote:

$$\varepsilon_k = k \frac{\varepsilon_0}{n+2}$$

$$s_k = \sup A_2 + \varepsilon_k$$

for $k = 1, \ldots, n + 1$.

By our assumptions for every $\sigma > 0$ there exists $(a_1^\sigma, \ldots, a_n^\sigma) \in A_1$ such that

(1) $$[(a_1^\sigma, \ldots, a_n^\sigma)] - \sup A_2 > \varepsilon_{n+1}$$

Let us denote V_i the linear subspace generated by the set $\{y_1, \ldots, y_n\} - \{y_i\}$. Then it holds $\varrho_{V_i}(s_k y_i) \geqq \delta$. Clearly, there exists $s = s_m$ satisfying

(2) $$\varrho_{V_i}(s_m y_i) > \delta$$

for every i $(1 \leqq i \leqq n)$.

Then from (1) it follows

$$[(a_1^\sigma, \ldots, a_n^\sigma)] - \sup A_2 > \varepsilon_m \; \forall \, \sigma > 0$$

and hence

(3) $$s/[(a_1^\sigma, \ldots, a_n^\sigma)] = (\sup A_2 + \varepsilon_m)/[(a_1^\sigma, \ldots, a_n^\sigma)] > 1.$$

We put

$$\varrho = \min \{\varrho_{V_1}(s y_1), \ldots, \varrho_{V_n}(s y_n)\}.$$

In view of (2) we have $\varrho > \delta$. Let us choose $\varkappa$ such that

$$0 < \varkappa < \varrho - \delta.$$

Now we put $\sigma = \min (\varkappa/3, \varkappa/3ns)$. Let $\| \Delta_i \| < \sigma$ $(1 \leqq i \leqq n)$ and let $(a_1^\sigma, \ldots, a_n^\sigma) \in A_1$ satisfying (1). Further we put

$$K = s/[(a_1^\sigma, \ldots, a_n^\sigma)].$$

Then it holds

(4) $$K a_i^\sigma \leqq s$$

for every i ($1 \leqq i \leqq n$) and in view of (3)

$$K < 1. \tag{5}$$

Because of

$$[Ka_1^\sigma, \ldots, Ka_n^\sigma] = K[(a_1^\sigma, \ldots, a_n^\sigma)] = s,$$

we have

$$\| \sum_i Ka_i y_i \| \geqq \varrho v_j(sy_j) \geqq \varrho > \delta + \varkappa,$$

wherein $a_j = [(a_1^\sigma, \ldots, a_n^\sigma)]$.

Hence

$$\| \sum_i a_i^\sigma y_i \| > \frac{1}{K}(\sigma + \varkappa). \tag{6}$$

Further we obtain

$$\| \sum_i a_i^\sigma \Delta_i \| \leqq \frac{1}{K} \sum_i Ka_i^\sigma \| \Delta_i \| \leqq \frac{1}{K} \sum_i s \| \Delta_i \| \leqq \frac{1}{K} \frac{\varkappa}{3}. \tag{7}$$

Because of $\| \Delta_0 \| \leqq \frac{\varkappa}{3}$ and using (5) we obtain

$$\| \sum_i a_i^\sigma y_i + \sum_i a_i^\sigma \Delta_i + \Delta_0 \| \geqq (\delta + \varkappa)\frac{1}{K} - \frac{1}{K}\frac{\varkappa}{3} - \frac{\varkappa}{3} > \delta + \frac{\varkappa}{3} > \delta$$

contradicting the assumption $(a_1^\sigma, \ldots, a_n^\sigma) \in A_1$.

3. Main Theorem

Theorem. Let T_1 be an open subset of T^n, $G_0, G_1, \ldots, G_m$ continuous mappings from T^n into R, m, n natural numbers satisfying $m + n \geqq 1$ and

$$V = \{ \sum_{i=1}^{m} b_i G(a_1, \ldots, a_n) + G_0(a_1, \ldots, a_n); (b_1, \ldots, b_m) \in T^m, (a_1, \ldots, a_n) \in T_1 \}.$$

Suppose that there exist continuous operators $F_0, F_1, \ldots, F_n$ ($F_i : R \to R$) satisfying

$$x = \sum_{i=1}^{m} b_i G_i(a_1, \ldots, a_n) + G_0(a_1, \ldots, a_n) \Rightarrow F_0(x) + \sum_{i=1}^{n} a_i F_i(x) = 0$$

for every $(a_1, \ldots, a_n) \in T_1$ and for every $(b_1, \ldots, b_m) \in T^m$. Further suppose

1. $\{F_i(y)\}_{i=1}^n$ is linearly independent set for every $y \in V$.
2. $\{G_j(a_1, \ldots, a_n)\}_{j=1}^m$ is linearly independent set for every $(a_1, \ldots, a_n) \in T_1$.

Then there exists an open subset $S \subseteq R$ satisfying $S \supseteq V$ such that each operator $E : S \to V$ of the form

$$E(y) = \sum_{i=1}^{m} b_i^y G_i(a_1^y, \ldots, a_n^y) + G_0(a_1^y, \ldots, a_n^y)$$

with the following properties

a) $\| F_0(y) + \sum_i a_i^y F(y) \| = \min_{a_i} \| F_0(y) + \sum_i a_i F_i(y) \|$

b) $\| G_0(a_1^y, \ldots, a_n^y) + \sum_i b_i^y G_i(a_1^y, \ldots, a_n^y) - y \| =$

$= \min_{b_j} \| G_0(a_1^y, \ldots, a_n^y) + \sum_i b_j G_j(a_1^y, \ldots, a_n^y) - y \|$

is a continuous projection from S onto V.

Now we shall prove Lemmas 4, 5, 6, from which the assertion of Theorem follows easily.

Notation. In what follows we shall use the following notation. For an arbitrarily chosen $y \in R$ we put

$$y^* = E(y) = \sum_i b_i^* G_i(a_1^*, \ldots, a_n^*) + G_0(a_1^*, \ldots, a_n^*) \in V$$

$$F_i(y) = F_i(y^*) + \Delta F_i(y).$$

Lemma 4. For every $\varepsilon > 0$ there exists $\delta > 0$ such that $\| \Delta F_i(y) \| < \delta$ $(1 \leqq i \leqq n)$ implies

$$\min_{a_i} \| F_0(y) + \sum_i a_i F_i(y) \| < \varepsilon.$$

Proof. The assertion follows easily from the following relation

$$\min_{a_i} \| F_0(y) + \sum_i a_i F_i(y) \| \leqq \| F_0(y) + \sum_i a_i^* F_i(y) \| =$$

$$= \| F_0(y^*) + \Delta F_0(y) + \sum_i a_i^* F_i(y) + \sum_i a_i^* F_i(y) \| = \| \Delta F_0(y) + \sum_i a_i^* F_i(y) \|.$$

Lemma 5. For every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\| y - y^* \| < \delta \Rightarrow [(a_1^*, \ldots, a_n^*) - (a_1^y, \ldots, a_n^y)] < \varepsilon.$$

Proof. The assertion follows easily from Lemmas 2, 3, 4, and from the continuity of operators F_i.

Lemma 6. For every $\varepsilon > 0$ there exist $\delta_1 > 0$, $\delta_2 > 0$ such that $\| y^* - y \| < \delta_2$, and $[(a_1^y, \ldots, a_n^y) - (a_1^*, \ldots, a_n^*)] < \delta$ implies

(***) $$\| G_0(a_1^y, \ldots, a_n^y) + \sum_i b_i^y G_i(a_1^y, \ldots, a_n^y) - y \| < \varepsilon.$$

Proof. Let us choose δ_1 so that $\{G_j(a_1, \ldots, a_n)\}_{j=1}^m$ is linearly independent set and every n-tuple $(a_1^y, \ldots, a_n^y)$ satisfying $[(a_1^y, \ldots, a_n^y) - (a_1, \ldots, a_n)] < \delta$ belongs to T_1. Now the assertion follows easily from the relation

$$\min_{b_j} \| G_0(a_1^y, \ldots, a_n^y) + \sum_j b_j G_j(a_1^y, \ldots, a_n^y) - y \| \leqq$$

$$\leqq \| G_0(a_1^y, \ldots, a_n^y) + \sum_j b_j^* G_j(a_1^y, \ldots, a_n^y) - y \| =$$

$$= \| G_0(a_1^*, \dots, a_n^*) + G_0(a_1^y, \dots, a_n^y) + \sum_j b_j^* G_j(a_1^*, \dots, a_n^*) +$$
$$+ \sum_j b_j^* G_j(a_1^y, \dots, a_n^y) - y^* + y^* - y \| \leqq$$
$$\leqq \| G_0(a_1^y, \dots, a_n^y) + \sum_j b_j G_j(a_1^y, \dots, a_n^y) \| + \delta_2.$$

Proof of Theorem. Let $\varepsilon > 0$ be arbitrarily chosen. We chose $\delta_1 > 0$ and $\delta_2 > 0$ so that the condition (***) is satisfied. Furhter we choose $\delta_3 > 0$ in such a way that

$$\| y^* - y \| < \delta_3 \Rightarrow [(a_1^y, \dots, a_n^y) - (a_1^*, \dots, a_n^*)] < \delta_1$$

(using Lemma 5) and put $\delta = \min(\delta_2, \delta_3)$. Then in view of Lemma 6

$$\| y^* - y \| < \delta \Rightarrow \| E(y) - y^* \| < \varepsilon.$$

Now we put $S = \bigcup_{y^* \in V} o_{y^*}$, where o_{y^*} is point y^* δ-neibourghood constructed as above. Then E is a continuous projection from S onto V, and S is an open set satisfying $S \supseteqq V$.

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