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# CONSTRUCTING INFINITELY DIVISIBLE CHARACTERISTIC FUNCTIONS

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**1. Introduction.** A characteristic function $a(u)$ is said to be infinitely divisible, if for every positive integer $n$, it is the $n$th power of some characteristic function. This means that there exists for every positive integer $n$ a characteristic function $a_n(u)$, such that

(1) $$a(u) = [a_n(u)]^n.$$

The function $a_n(u)$ is uniquely determined by $a(u)$, $a_n(u) = [a(u)]^{1/n}$, provided that one selects for the $n$th root the principal branch. The concept of infinite divisiblity is very important in probability theory, particularly in the study of limit theorems. The question, which characteristic functions are infinitely divisible, is answered completely – in a certain sense – by the Levy – Khinchine representation theorem for infinitely divisible characteristic functions [3].

The family of infinitely divisible characteristic functions is quite broad. It includes the class of stable characteristic functions, as well as the class $L$ of self-decomposable characteristic functions and the class $U$ of infinitely divisible characteristic functions. It includes the Poisson and those compound Poisson characteristic functions for which the distribution on the Poisson parameter is infinitely divisible. The family also includes the generalized Poisson characteristic functions as well as some other generalized characteristic functions. Characteristic functions which vanishes at some point on the real line and entire characteristic functions vanishing at some point in the complex plane are not infinitely divisible.

During the last two decades there has been an increasing interest in transformations of characteristic functions for the construction of infinitely divisible characteristic functions. These transformations give some interesting information concerning the structure of infinitely divisible characteristic functions. Naturally one investigates also the properties of the transformed infinitely divisible characteristic function. Lukacs [3] introduced a transformation which converts an arbitrary characteristic function into an infinitely divisible characteristic function. This transformation was studied recently by Artikis [1].

The present paper investigates a modified form of Lukacs' transformation. This maps the characteristic function of a distribution function with a single mode at the origin into an infinitely divisible characteristic function. Conditions are given for the transformed characteristic function to be imbedded in certain classes of infinitely divisible characteristic functions.

**2. Results.** A distribution function $F(x)$ is called unimodal with the mode at $x = a$ (or in short, (a) unimodal) if and only if $F(x)$ is convex in $(-\infty, a)$ and concave in $(a, \infty)$. In theorem 1 we establish a modified form of Lukacs' transformation.

**Theorem 1.** *Let* $\varphi(u)$ *be the characteristic function of a* (0) *unimodal distribution function. Then* $\gamma(u) = \exp\{-u\int_0^u \varphi(y)\,dy\}$ *is the characteristic function of an infinitely divisible distribution function having a finite second moment.*

Proof. The characteristic function $\varphi(u)$ can be written in the form $\varphi(u) = \frac{1}{u}\int_0^u \beta(y)\,dy$, where $\beta(u)$ is a characteristic function (see [3] p. 92). Furthermore from theorem 12.2.8 of [3] we have that

$$\gamma_1(u) = \exp\{-\int_0^u\int_0^y \varphi(x)\,dx\,dy\} \tag{2}$$

and

$$\gamma_2(u) = \exp\{-\int_0^u\int_0^y \beta(x)\,dx\,dy\} \tag{3}$$

are characteristic functions of infinitely divisible distribution functions having finite second moment. Since $\gamma(u) = \gamma_1(u)\,\gamma_2(u)$ and the family of infinitely divisible characteristic functions is closed under multiplication we conclude that $\gamma(u)$ is the characteristic function of an infinitely divisible distribution function having a finite second moment.

The family of distribution functions whose characteristic functions satisfy theorem 1 is quite broad. It includes the exponential distribution, as well as some gamma distributions, the uniform distribution having support (0,1) and some chi-square distributions. It includes the (0) symmetric distributions of the class $U$ as well as certain power mixtures of this class [4], [5]. The family also includes, every scale mixture of a (0) unimodal distribution.

Consider a non-degenerate distribution function $F(x)$ having a mode at $x = 0$, and let $f(x)$ be the density of $F(x)$. Then $F(x) - xf(x)$ is the distribution function which corresponds to the characteristic function $\beta(u)$ in (3), see Lukacs [3] p. 94.

In theorem 2 we establish sufficient conditions for the transformed characteristic function $\gamma(u)$ to be member of the class $L$.

**Theorem 2.** *Let $\varphi(u)$ be the characteristic function of a distribution function $F(x)$ whose density $f(x)$ is twice differentiable. If $f(x)$ is convex on $(-\infty, 0) \cup (0, \infty)$ then $\gamma(u) = \exp\{-u \int_0^u \varphi(y)\,\mathrm{d}y\}$ is a self-decomposable characteristic function (or, of class L).*

Proof. Distribution functions having convex densities are (0) unimodal. Hence $\gamma(u) = \exp\{-u \int_0^u \varphi(y)\,\mathrm{d}y\}$ is the characteristic function of an infinitely divisible distribution function having a finite second moment. Theorem 12.2.8 of [3] and the correspondence between the Levy and Kolmogorov canonical representations of infinitely divisible characteristic functions (see p. 118 of [3]) imply that the Levy spectral functions of $\gamma(u)$ are given by

$$M(x) = \int_{-\infty}^{x} \frac{1}{y^2}\,\mathrm{d}[2F(y) - yf(y)] \qquad \text{for } x < 0$$

and

$$N(x) = -\int_{x}^{\infty} \frac{1}{y^2}\,\mathrm{d}[2F(y) - yf(y)] \qquad \text{for } x > 0.$$

Let $f'(x)$ be the left or right-hand derivative of $f(x)$. The convexity of $f(x)$ in $(-\infty, 0)$ implies that $xM'(x) = [f(x) - xf'(x)]/x$ is non-increasing in $(-\infty, 0)$. In a similar way we can prove that $xN'(x)$ is non-increasing in $(0, \infty)$. Hence $\gamma(u)$ is a self-decomposable characteristic function (see [3] theorem 5.11.2).

**Corollary 1.** Let $\varphi(u)$ be the characteristic function of a distribution function $F(x)$ having a finite second moment. Then

$$\gamma(u) = \exp\left\{-u \int_0^u \frac{2[\varphi(y) - \varphi'(0)\,y - 1]}{\varphi''(0)\,y^2}\,\mathrm{d}y\right\}$$

is a self-decomposable characteristic function.

Proof. Theorem 1 of [6] implies that $2[\varphi(u) - \varphi'(0)\,u - 1]/\varphi''(0)\,u^2$ is the characteristic function of a distribution function having a convex density. Hence from theorem 2 we conclude the self-decomposability of $\gamma(u)$.

An infinitely divisible characteristic function, $\psi(u)$ is said to belong to the class $U$ if, there exists an infinitely divisible characteristic function $\psi_1(u)$ such that

$$\psi(u) = \exp\left\{\frac{1}{u} \int_0^u \log \psi_1(y)\,\mathrm{d}y\right\}.$$

In theorem 3 we establish the connections of characteristic functions of the form $\gamma(u) = \exp\{-u \int_0^u \varphi(y)\,\mathrm{d}y\}$ with the characteristic functions of certain (0) unimodal (0) symmetrical distribution functions.

**Theorem 3.** *Let $\varphi(u)$ be the characteristic function of a* (0) *unimodal* (0) *symmetrical distribution function $F(x)$. Then*:

(i) $\gamma(u)\,\varphi(u)$ *belongs to a* (0) *unimodal* (0) *symmetrical distribution function.*

(ii) $\gamma(u)\exp\left\{\frac{1}{u}\int_0^u \varphi(y)\,y^2\,dy\right\}$ *is the characteristic function of a* (0) *unimodal* (0) *symmetrical distribution function of the class $U$.*

(iii) $\exp\left\{-u\int_0^u \frac{\varphi'(y)}{\varphi''(0)\,y}\,dy\right\}$, *with* $-\varphi''(0)=\mu_2<\infty$, *is the characteristic function of a* (0) *unimodal* (0) *symmetrical distribution of the class $U$.*

Proof. (i) Theorem 2 and remark 3 of [1] imply that $\gamma_1(u)=\exp\{-\int_0^u\int_0^y \varphi(x)\,dx\,dy\}$ is the characteristic function of a (0) unimodal, (0) symmetrical and self-decomposable distribution function.

The characteristic function $\gamma_2(u)=\exp\{-\int_0^u\int_0^y \beta(x)\,dx\,dy\}$ belongs to a distribution function having a finite second moment and hence

$$\frac{\gamma_2'(u)}{\gamma_2''(0)\,u}=\frac{1}{u}\int_0^u \frac{\gamma_2''(y)}{\gamma_2''(0)}\,dy$$

is the characteristic function of a (0) unimodal (0) symmetrical distribution function. Since $\gamma(u)\,\varphi(u)=\gamma_1(u)\,[\gamma_2'(u)/\gamma_2''(0)\,u]$ and the class of (0) unimodal (0) symmetrical distribution functions is closed under convolution, we conclude that $\gamma(u)\,\varphi(u)$ belongs to a (0) unimodal (0) symmetrical distribution function.

(ii) It easily follows that

$$\gamma(u)\exp\left\{\frac{1}{u}\int_0^u \varphi(y)\,y^2\,dy\right\}=\exp\left\{2\frac{1}{u}\int_0^u \log\gamma(y)\,dy\right\} \tag{4}$$

and hence (4) belongs to a (0) unimodal (0) symmetrical distribution function of the class $U$ (see [4] and [5]).

(iii) The characteristic function $\delta_1(u)=\exp\left\{\frac{1}{\mu_2}(\varphi(u)-1)\right\}$ belongs to a (0) unimodal (0) symmetrical distribution function of the class $U$. Furthermore $\delta_2(u)=\exp\left\{-\int_0^u\int_0^y \frac{\varphi'(x)}{\varphi''(0)\,x}\,dx\,dy\right\}$ belongs to a (0) unimodal (0) symmetrical and self-decomposable distribution function [see (i) of this theorem]. Since $\delta_1(u)\delta_2(u)=$ $=\exp\left\{-u\int_0^u \frac{\varphi'(y)}{\varphi''(0)\,y}\,dy\right\}$ and the class $L$ is contained in the class $U$ [5], we conclude that $\exp\left\{-u\int_0^u \frac{\varphi'(y)}{\varphi''(0)\,y}\,dy\right\}$ is the characteristic function of a (0) unimodal (0) symmetrical distribution function of the class $U$.

Let $F(x)$ be a distribution function and let $f(x)$ be its density. Ibragimov [2] called a distribution function $F(x)$ strongly unimodal if its convolution with every unimodal distribution function is unimodal. He found that $F(x)$ is strongly unimodal if and only if $\log f(x)$ is concave.

Let $F(x)$ be a strongly unimodal (0) symmetrical distribution function having a finite second moment $\mu_2$ and let $\varphi(u)$ be its characteristic function. Then $G(x) = \int_0^x y^2 \, \mathrm{d}F(x)/\mu_2$ is a strongly unimodal, (with mode at $x = 0$), (0) symmetrical distribution function and $\varphi''(u)/\varphi''(0)$ is its characteristic function. Hence

$$\gamma(u) = \exp\left\{-u\int_0^u \frac{\varphi''(y)}{\varphi''(0)} \, \mathrm{d}y\right\} = \exp\left\{\frac{1}{\mu_2} u\varphi'(u)\right\}$$

is the characteristic function of an infinitely divisible distribution function having a finite second moment.

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