Zbigniew Olszak

Almost cosymplectic real hypersurfaces in Kähler manifolds

Archivum Mathematicum, Vol. 18 (1982), No. 4, 187--192

Persistent URL: http://dml.cz/dmlcz/107143

ARCH. MATH. 4, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XVIII: 187—192, 1982

ALMOST COSYMPLECTIC REAL HYPERSURFACES IN KÄHLER MANIFOLDS*

ZBIGNIEW OLSZAK

(Received November 17, 1982)

It is well known that an almost contact metric structure is induced on an oriented real hypersurface in a Kähler manifold. Certain kinds of almost contact metric manifolds (for instance, Sasakian, normal, cosymplectic in the sense of Blair, etc.) can be obtained in this way. The purpose of the present paper is to prove that such an induced structure cannot be almost cosymplectic in the sense of Goldberg and Yano, if the ambient manifold is a Kähler manifold of non-zero constant holomorphic sectional curvature.

§ 1. Almost cosymplectic manifolds

Let M be an almost contact metric manifold, that is, it is a $(2n + 1)$-dimensional $(n \geqq 1)$ differentiable manifold endowed with an almost contact metric structure $(\emptyset, \xi, \eta, g)$. The almost contact metric structure $(\emptyset, \xi, \eta, g)$ is formed by tensor fields $\emptyset, \xi, \eta$ of type (1,1), (1,0), (0,1) respectively, and a Riemannian metric g such that

$$\emptyset^2 = -I + \eta \oplus \xi, \qquad \eta(\xi) = 1 \qquad \text{and} \qquad g(\emptyset X, \emptyset Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any $X, Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the Lie algebra of vector fields on M. Then also has $\emptyset\xi = 0$, $\eta \circ \emptyset = 0$, $\eta(X) = g(X, \xi)$ and the tensor field Φ of type (0,2) defined by $\Phi(X, Y) = g(\emptyset X, Y)$ is a 2-form on M.

M is said to be normal (cf. [6]), if the almost complex structure J, defined on $M \times \mathbf{R}$ (where $\mathbf{R}$ is the real line with coordinate t) by $J(X, \lambda d/dt) = (\emptyset X - \lambda\xi, \eta(X)\, d/dt)$, is integrable. As it is known M is normal if and only if $[\emptyset, \emptyset] + d\eta \otimes \xi = 0$, where $[\emptyset, \emptyset]$ is the Nijenhuis tensor field of $\emptyset$.

Following Goldberg and Yano [2] we say that M is almost cosymplectic if $d\Phi = 0$ and $d\eta = 0$, where d is the operator of exterior differentiation. And follow-

* (Delivered at the Joint Czech-Polish-G.D.R. Conference on Differential Geometry and its Applications, September 1980, Nové Město na Moravě, Czechoslovakia.)

ing Blair [1] we say that M is cosymplectic if it is normal and almost cosymplectic. It is known that an almost contact metric manifold is cosymplectic if and only if both $\nabla\eta$ and $\nabla\phi$ vanish, where ∇ is the covariant differentiation with respect to g.

Remark. Our almost cosymplectic manifolds are cosymplectic in the sense of Libermann [3]. However we use the terminology of Goldberg and Yano [2], and Blair [1]. Okumura [4] used the terminology of Libermann.

Curvature properties of almost cosymplectic manifolds were investigated by Goldberg and Yano [2], and by the author [5]. In the present paper we are interested on almost cosymplectic structures induced on oriented hypersurfaces in Kähler manifolds.

§ 2. Almost cosymplectic hypersurfaces in Kähler manifolds

Let $\bar{M}$ be a $(2n+2)$-dimensional ($n \geqq 1$) Kähler manifold and let (J, G) be its Kähler structure. Thus, J is an almost complex structure and G a Riemannian metric on $\bar{M}$ such that $J^2 = -I$, $G(J\bar{X}, J\bar{Y}) = G(\bar{X}, \bar{Y})$ and $\bar{\nabla}J = 0$, for any $\bar{X}, \bar{Y} \in \mathfrak{X}(\bar{M})$, where $\bar{\nabla}$ is the covariant differentiation with respect to G.

Let M be an oriented (real) hypersurface in $\bar{M}$ ($\dim M = 2n+1$). Taking a unit normal vector field N defined along M, we define an almost contact metric structure (ϕ, ξ, η, g) on M by

$$JX = \phi X + \eta(X)N, \qquad \xi = -JN \qquad \text{and} \qquad g(X, Y) = G(X, Y),$$

for any $X, Y \in \mathfrak{X}(M)$. So M is an almost contact metric manifold.

We say that M is a cosympletic (resp. almost cosymplectic) hypersurface in $\bar{M}$ if M is cosymplectic (resp. almost cosymplectic) as the almost contact metric manifold.

Let h be the second fundamental form of M given by the Gauss and Weingarten equations

$$\bar{\nabla}_X = \nabla_X Y + h(X, Y)N \qquad \text{and} \qquad \bar{\nabla}_X N = -HX,$$

for $X, Y \in \mathfrak{X}(M)$, where $h(X, Y) = g(HX, Y)$ and ∇ is the covariant differentiation with respect to g. The covariant derivatives of the structure tensor fields are given by

$$\nabla_X\xi = \phi HX, \qquad (\nabla_X\eta)(Y) = g(\phi HX, Y), \tag{2.1}$$

$$(\nabla_X\phi)Y = -h(X, Y)\xi + \eta(Y)HX. \tag{2.2}$$

So, M is an almost cosymplectic (resp. cosymplectic) hypersurface in $\bar{M}$ if and only if $\phi H + H\phi = 0$ (resp. $H = \alpha\eta \otimes \xi$, where α is a scalar function on M) (cf. [4]). Hence we see that a totally geodesic hypersurface in a Köhler manifold is cosymplectic (cf. [4]).

In the sequel we assume that M is an almost cosymplectic hypersurface in $\bar{M}$. Then we have

$$\theta H = -H\theta, \tag{2.3}$$

whence $H\xi = \alpha\xi$, where $\alpha = h(\xi, \xi)$, and

$$g(\theta HX, Y) = g(\theta HY, X). \tag{2.4}$$

Moreover we suppose that $\bar{M}$ is of constant holomorphic sectional curvature. Under this assumption the Gauss and Codazzi equations of M take the following forms

$$\begin{aligned} R_{XY}Z = c\{&g(Y, Z)\, X - g(X, Z)\, Y + g(\theta Y, Z)\, \theta X - g(\theta X, Z)\, \theta Y - \\ &-2g(\theta X, Y)\, \theta Z\} + h(Y, Z)\, HX - h(X, Z)\, HY, \end{aligned} \tag{2.5}$$

$$(\nabla_X H)\, Y - (\nabla_Y H)\, X = c\{\eta(X)\, \theta Y - \eta(Y)\, \theta X - 2g(\theta X, Y)\, \xi\}, \tag{2.6}$$

where $R_{XY} = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$ is the curvature operator and $c = K/4$, $K =$ const. being the holomorphic sectional curvature of $\bar{M}$.

Okumura ([4], p. 68) proved that K must be non-positive, and if $K = 0$, then M is locally flat and cosymplectic.

The aim of the present paper is to prove the following theorem.

Theorem. *There are no almost cosymplectic hypersurfaces in Kähler manifolds of non-zero constant holomorphic sectional curvature.*

§ 3. Proof

As above, we assume that M is an almost cosymplectic hypersurface in a Kähler manifold of constant holomorphic sectional curvature.

Okumura [4] obtained the following relations for M

$$H^2 X = -cX + (\alpha^2 + c)\, \eta(X)\, \xi, \tag{3.1}$$

$$X\alpha = \beta\eta(X), \qquad \text{where} \qquad \beta = \xi\alpha, \tag{3.2}$$

$$trH = \alpha, \qquad trH^2 = \alpha^2 - 2nc. \tag{3.3}$$

Note that we also have

$$X\beta = (\xi\beta)\, \eta(X). \tag{3.4}$$

Indeed, one always has $X(Y\alpha) - Y(X\alpha) = [X, Y]\, \alpha$. Hence by (3.2), (2.1) and (2.4) we find $(X\beta)\, \eta(Y) - (Y\beta)\, \eta(X) = 0$, which gives (3.4).

Lemma. *For the second fundamental form of M we have*

$$\begin{aligned} (\nabla_X H)\, Y = &-c\{\eta(Y)\, \theta X + g(\theta X, Y)\, \xi\} + \\ &+ \alpha\{\eta(X)\, \theta HY + \eta(Y)\, \theta HX + g(\theta HX, Y)\, \xi\} + \beta\eta(X)\, \eta(Y)\, \xi. \end{aligned} \tag{3.5}$$

Proof: At first, using (2.6) and $\eta(HX) = \alpha\eta(X)$ we find

$$(\nabla_{HX}h)(Y, Z) = g((\nabla_{HX}H)Z, Y) = g((\nabla_Z H)HX, Y) + \\ + c\{\alpha\eta(X)g(\theta Z, Y) - \eta(Z)g(\theta HX, Y) - 2\eta(Y)g(\theta HX, Z)\}.$$

The symmetrization of the above equality with respect to X, Y and the use of (2.4) give

$$(3.6) \qquad (\nabla_{HX}h)(Y, Z) + (\nabla_{HY}h)(X, Z) = \\ = g((\nabla_Z H^2)X, Y) + c\{\alpha\eta(X)g(\theta Z, Y) + \alpha\eta(Y)g(\theta Z, X)\} - \\ - 2c\{\eta(X)g(\theta HY, Z) + \eta(Y)g(\theta HX, Z) + \eta(Z)g(\theta HX, Y)\}.$$

But by (3.1), (3.2) and (2.1) we have

$$(\nabla_Z H^2)X = 2\alpha\beta\eta(Z)\eta(X)\xi + (\alpha^2 + c)g(\theta HZ, X)\xi + (\alpha^2 + c)\eta(X)\theta HZ,$$

which together with (2.4) used in (3.6) yields

$$(3.7) \qquad (\nabla_{HX}h)(Y, Z) + (\nabla_{HY}h)(X, Z) = \\ = 2\alpha\beta\eta(Z)\eta(X)\eta(Y) + c\alpha\{\eta(X)g(\theta Z, Y) + \eta(Y)g(\theta Z, X)\} + \\ + (\alpha^2 - c)\{\eta(X)g(\theta HZ, Y) + \eta(Y)g(\theta HZ, X)\} - 2c\eta(Z)g(\theta HX, Y).$$

The antisymmetrization of (3.7) with respect to X, Z and the use of (2.4) give us

$$(\nabla_{HX}h)(Z, Y) - (\nabla_{HZ}h)(X\ Y) = \\ = c\alpha\{\eta(X)g(\theta Z, Y) - \eta(Z)g(\theta X, Y) - 2\eta(Y)g(\theta X, Z)\} + \\ + (\alpha^2 + c)\{\eta(X)g(\theta HZ, Y) - \eta(Z)g(\theta HX, Y)\}.$$

This compared with (3.7) implies with the help of (2.4)

$$(3.8) \quad (\nabla_{HX}h)(Y, Z) = \alpha\beta\eta(Z)\eta(X)\eta(Y) - c\alpha\{\eta(Y)g(\theta X, Z) + \eta(Z)g(\theta X, Y)\} + \\ + \alpha^2\eta(X)g(\theta HY, Z) - c\{\eta(Y)g(\theta HX, Z) + \eta(Z)g(\theta HX, Y)\}.$$

Taking into account (2.6), $H\xi = \alpha\xi$, (3.2), (2,1), (2.3) and (3.1) we get

$$(\nabla_\xi H)Y = (\nabla_Y H)\xi + c\theta Y = (Y\alpha)\xi + \alpha\nabla_Y\xi - H\nabla_Y\xi + c\theta Y = \\ = \beta\eta(Y)\xi + \alpha\theta HY + \theta H^2 Y + c\theta Y = \beta\eta(Y)\xi + \alpha\theta HY,$$

whence

$$(3.9) \qquad (\nabla_\xi h)(Y, Z) = \beta\eta(Y)\eta(Z) + \alpha g(\theta HY, Z).$$

Now substituting HX instead of X into (3.8) and using (3.1) and (3.9) we obtain

$$c(\nabla_X h)(Y, Z) = -c^2\{\eta(Y)g(\theta X, Z) + \eta(Z)g(\theta X, Y)\} + \\ + c\alpha\{\eta(X)g(\theta HY, Z) + \eta(Y)g(\theta HX, Z) + \eta(Z)g(\theta HX, Y)] + c\beta\eta(X)\eta(Y)\eta(Z).$$

If $c \neq 0$, the above equation gives (3.5). Let us assume that $c = 0$. Then (3.1) and $H\xi = \alpha\xi$ imply $HY = \alpha\eta(Y)\xi$. This, by covariant differentiation and the using of (2.1), (3.2) and $\theta H = 0$, yields (3.5). This completes the proof of the lemma.

Fix a point $m \in M$. Let $X \in M_m$ and let $\{E_0, \ldots, E_{2n}\}$ be an orthonormal basis in M_m. For the sake of simplicity we extend $X, E_0, \ldots, E_{2n}$ to local vector fields, denoted by the same letters and defined in a neighborhood of m, so that $\{E_0, \ldots, E_{2n}\}$ is a local orthonormal basis and $(\nabla_W X)(m) = 0$, $(\nabla_W E_i)(m) = 0$, for any $W \in M_m$ and $i = 0, 1, \ldots, 2n$.

To prove our Theorem we compute $\Sigma_i(R_{XE_i}H)E_i$ in the point m by two methods.

The first method. Note that

$$\Sigma_i(R_{XE_i}H)E_i = \Sigma_i R_{XE_i}HE_i - H\Sigma_i R_{XE_i}E_i.$$

Applying (2.5) to the right hand side of the above relation we get

$$\Sigma_i(R_{XE_i}H)E_i = -(trH)H^2X + [trH^2 - (2n+1)c]HX + \\ + 3c(H\theta - \theta H)\theta H + c(trH)X.$$

This equality, in virtue of (3.1), (3.3) and (2.3) takes the following from

$$\Sigma_i(R_{XE_i}H)E_i = [\alpha^2 - (4n+7)c]HX + 2c\alpha X + (5c\alpha - \alpha^3)\eta(X)\xi. \tag{3.10}$$

The second method. Because of $[X, E_i] = 0$ in the point m, we have

$$\Sigma_i(R_{XE_i}H)E_i = \Sigma_i(\nabla_X \nabla_{E_i}H - \nabla_{E_i}\nabla_X H)E_i. \tag{3.11}$$

But from (3.5) we get

$$\Sigma_i(\nabla_{E_i}H)E_i = \beta\xi. \tag{3.12}$$

Hence we derive, with the help of (3.4) and (2.1),

$$\Sigma_i(\nabla_X \nabla_{E_i}H)E_i = \nabla_X\Sigma_i(\nabla_{E_i}H)E_i = (\xi\beta)\eta(X)\xi + \beta\theta HX. \tag{3.13}$$

On the other hand, we have from (3.5)

$$\Sigma_i(\nabla_{E_i}\nabla_X H)E_i = \Sigma_i \nabla_{E_i}(\nabla_X H)E_i = \\ = \Sigma_i \nabla_{E_i}[-c\{\eta(E_i)\theta X + g(\theta X, E_i)\xi\} + \\ + \alpha\{\eta(X)\theta HE_i + \eta(E_i)\theta HX + g(\theta HX, E_i)\xi\} + \beta\eta(X)\eta(E_i)\xi].$$

To simplify the above relation at first we note that:

by (2.1), $\Sigma_i(\nabla_{E_i}\eta)(E_i) = tr(\theta H) = 0$, $\nabla_\xi\eta = 0$ and $\nabla_\xi\xi = 0$;

by (2.2) and (3.3), $\Sigma_i(\nabla_{E_i}\theta)E_i = -(trH)\xi + H\xi = 0$, $\nabla_\xi\theta = 0$ and $\Sigma_i(\nabla_{E_i}\theta)HE_i = -(trH^2)\xi + H^2\xi = 2nc\xi$;

by (3.5) and (3.3), $(\nabla_\xi H)X = \alpha\theta HX + \beta\eta(X)\xi$ and $\Sigma_i(\nabla_{E_i}H)\theta E_i = c(tr\theta^2)\xi - \alpha tr(\theta^2 H)\xi = -2nc\xi$.

Using all these identities, (3.12) and (3.2) we find

$$\Sigma_i(\nabla_{E_i}\nabla_X H)E_i = -c\nabla_{\varphi X}\xi + \beta\theta HX + \alpha\Sigma_i(\nabla_{E_i}\eta)(X)\theta HE_i + \\ + \alpha^2\theta^2 HX + 4nc\alpha\eta(X)\xi + \alpha\nabla_{\theta HX}\xi + (\xi\beta)\eta(X)\xi.$$

Hence in virtue of (2.1), (2.3), (2.4) and (3.1) one can obtain

$$\Sigma_i(\nabla_{E_i}\nabla_X H)E_i = \beta\theta HX + (\xi\beta)\eta(X)\xi - (\alpha^2 + c)HX - 2c\alpha X + \\ + [\alpha^3 + (4n + 3)c\alpha]\eta(X)\xi.$$

This together with (3.13) used in (3.11) gives

$$\Sigma_i(R_{XE_i}H)E_i = (\alpha^2 + c)HX + 2c\alpha X - [\alpha^3 + (4n + 3)c\alpha]\eta(X)\xi.$$

The last equation compared with (3.10) yields $cHX = c\alpha\eta(X)\xi$. Let us assume that $X \neq 0$ is orthogonal to ξ. Then $cHX = 0$ and $cH^2X = 0$, or by (3.1) $c^2X = 0$. This gives $c = 0$, completing the proof of our theorem.

REFERENCES

[1] D. E. Blair: *The theory of quasi-Sasakian structures,* J. Differential Geometry 1 (1967), 331—345.

[2] S. I. Goldberg and K. Yano: *Integrability of almost cosymplectic structures,* Pacific J. Math. 31 (1969), 373—382.

[3] P. Libermann: *Sur les automorphismes infinitésimaux des structures symplectiques et des structures de contact,* Coll. de géom. diff. globale, CBRM (1959), 37—59.

[4] M. Okumura: *Cosymplectic hypersurfaces in Kaehlerian manifold of constant holomorphic sectional curvatures,* Kodai Math. Sem. Rep. 17 (1965), 63—73.

[5] Z. Olszak: *On almost cosymplectic manifolds,* in print.

[6] S. Sasaki and Y. Hatakeyama: *On differentiable manifolds with certain structures which are closely related to almost contact structure II,* Tôhoku Math. J. 13 (1961), 281—294.

Z. Olszak
Institute of Mathematics, Technical University,
Wybrzeże Wyspiańskiego 27, 50-370 Wrocław
Poland