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NOTE ON COMPATIBLE BINARY RELATIONS ON ALGEBRAS

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By the compatibility of a binary relation with an algebraic structure, is usually ment a generalization of the characteristic property of a congruence on an algebra, i.e. a relation should be a subalgebra of the direct square of the given algebra (see e.g. [2], [3] and other papers cited in the mentioned papers). This type of compatibility is called the *Substitution Property,* further briefly (SP) (in the agreement with e.g. [4]). Some of relation compatibility types on topological structures are transferable (using the notion of the algebraic closure space) to general algebras and special algebraic structures. The present note contains a certain comparison of some relation compatibility concepts on algebras in general, without the assumption of further special properties of relations.

The algebraic closure space on the algebra (A, F) (in the sense of [6]) will be denoted by (A, C_F). A mapping $f: A \to A$ is called a *closure endomorphism* of an algebra (A, F) if it is an endomorphism of the closure space (A, C_F), i.e. for every subset $X \subset A$ it holds $f(C_F X) = C_F f(X)$; cf. [1]. The semigroup of all endomorphisms of (A, F) will be denoted by End (A, F) and similarly End (A, C_F) stands for the semigroup of all endomorphisms of (A, C_F). In the agreement with the terminology of multivalued mappings between topological spaces (see e.g. [9]) we say that a relation $R \subset A \times A$ is *upper semicontinuous* in the closure space (A, C_F) if for every C_F-closed subset $X \subset A$ the set $R^{-1}(X)$ is also C_F-closed and the relation R is said to be *image closed* if $R(X)$ is C_F-closed for every C_F-closed subset $X \subset A$. (For $R \subset A \times A$, $X \subset A$ we write as usually $R(X) = \{b ; \exists\, a \in X, \langle a, b\rangle \in R\}$, $R^{-1}(X) = \{a : \exists\, b \in X, \langle a, b\rangle \in R\}$. A relation $R \subset A \times A$ is said to be domain full if $R^{-1}(A) = A$. Any mapping f is identified with its graph, i.e. $f = \{\langle a, b\rangle : f(a) = b\}$. The set of all positive integers is denoted by $\mathbf{N}$.

With respect to [6] § 8 Lemma 7 and § 9 Lemma 3 (ii) we get:

Proposition 1. *Let (A, F) be an algebra. A binary relation R on A with (SP) is upper semicontinuous and image closed in the algebraic closure space (A, C_F).*

The proof is straightforward and hence omitted.

Remark 1. The opposite assertion to the above one does not hold in general, as it follows from this simple example: Put $A = \{a, b, c\}$. Let $(A, \vee)$ be an upper semilattice, where $a \vee b = a\ c\ = b \vee c = c$ and $R = \{\langle a, a\rangle, \langle b, b\rangle, \langle c, a\rangle, \langle b, c\rangle\}$. Then the relation R is upper semicontinuous and image closed in the corresponding closure space but it does not possess (SP) for $\langle c, c\rangle \notin R$.

Professor Gerhard Grimeisen has introduced in [7] the notion of the continuity of relations between topological spaces analogical to the relational form of the axiom of choice. The introduced concept of the relation continuity, useful e.g. in the theory of general continuous systems, requires the existence of the covering of a relation by graphs of continuous mappings with the same domain (identical with the domain of the considered relation). This notion is evidently transferable to every concrete category.

Definition. A binary relation R on an algebra (A, F) or a closure space (A, C) respectively is said to be *homomorphic in the sense of Grimeisen,* briefly *G-homomorphic*, if for every pair of elements $a, b \in A$ with $\langle a, b\rangle \in R$ there exists $\varphi \in \mathrm{End}(A,F)$ or $\varphi \in \mathrm{End}\,(A, C)$ respectively such that $\varphi \subset R$ and $\varphi(a) = b$.

Remark 2. It is evident that a binary relation R on (A, F) (similarly for a closure space) is G-homomorphic iff there exists a set $\Phi \subset \mathrm{End}\,(A, F)$ such that $R = \bigcup_{\varphi\in\Phi} \varphi$.

Definition. A binary relation R on an algebra (A, F) is said to be *G-closure homomorphic* if for every pair of elements $a, b \in A$ with $\langle a, b\rangle \in R$ there exists $\varphi \in \mathrm{End}\,(A, C_F)$ such that $\varphi \subset R$ and $\varphi(a) = b$, i.e. if it is G-homomorphic on (A, C_F).

Evidently G-homomorphic and G-closure homomorphic relations are domain full. In what follows all considered relations are domain full. In the agreement with the notion of the Substitution Property we shall also use terms: the G-homomorphness and the G-closure homomorphness.

Proposition 2. *Every G-homomorphic relation on any algebra is G-closure homomorphic.*

Proof. Let R be a G-homomorphic relation on an algebra (A, F), i.e. $R = \bigcup_{\varphi\in\Phi} \varphi$, where $\Phi \subset \mathrm{End}\,(A, F)$. The assertion follows from the inclusion $\mathrm{End}\,(A, F) \subset \subset \mathrm{End}\,(A, C_F)$ which can be easily verified: Consider $\varphi \in \mathrm{End}\,(A, F)$ and $X \subset A$, $X \neq \emptyset$. Let $a \in \varphi(C_F X)$ be an arbitrary element. There exist an integer $n \in \mathbf{N} \cup \{0\}$, an n-ary polynomial p over (A, F) and an n-tupple $\langle x_0, \dots, x_{n-1}\rangle \in X^n$ with $a = \varphi(p(x_0, \dots, x_{n-1}))$. Suppose f is an n_γ-ary operational symbol and $\mathbf{p}_0, \dots, \mathbf{p}_{n_\gamma-1}$ n-ary polynomial symbols with $\mathbf{p} = \mathbf{f}(\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_{n_\gamma-1})$. From $\varphi(p_j(x_0, \dots, x_{n-1})) = = p_j(\varphi(x_0), \dots, \varphi(x_{n-1}))$, where $j \in \{0, 1, \dots, n_\gamma - 1\}$ there follows $a = = \varphi(p(x_0, \dots, x_{n-1}) = f(\varphi(p_0(x_0, \dots, x_{n-1})), \dots, \varphi(p_{n_\gamma-1}(x_0, \dots, x_{n-1}))) =$

$= p(\varphi(x_0), \ldots, \varphi(x_{n-1})) \in C_F\varphi(X)$. Since $X \subset C_F X$, thus $\varphi(X) \subset \varphi(C_F X)$ and $C_F\varphi(X) \subset C_F\varphi(C_F X) = \varphi(C_F X)$, we have $\varphi(C_F X) = C_F\varphi(X)$, i.e. $\varphi \in \mathrm{End}\,(A, C_F)$.

Remark 3. The G-closure homomorphness implies neither (SP) nor the G-homomorphness, (SP) implies neither the G-homomorphness nor the G-closure homomorphness and the G-homomorphness does not imply (SP) in general. Indeed, let us consider a 1-unary algebra $(\mathbf{N}, f)$, where $f(n) = n + 1$ for each $n \in \mathbf{N}$. Define mappings φ, ψ of the set $\mathbf{N}$ into itself by putting $\varphi(1) = 1$, $\varphi(n) = n - 1$ for $n \geqq 2$, $\psi(1) = \psi(2) = 1$ and $\psi(n) = n - 2$ for $n \geqq 3$. Put $R = \varphi \cup \psi$. Then R is evidently a G-homomorphic relation on $(\mathbf{N}, C_f)$, where $C_f X = \bigcup_{0 \leqq k < \omega} f^k(X)$ for every subset $X \subset \mathbf{N}$, i.e. R is G-closure homomorphic on $(\mathbf{N}, f)$. It holds $\langle 1, 1\rangle \in R$ but $\langle f(1), f(1)\rangle = \langle 2, 2\rangle \notin R$, thus R does not possess (SP). Since for any mapping $g : \mathbf{N} \to \mathbf{N}$ with the property $g \subset R$ there is $g(1) = g(2) = 1$, consequently $g \notin \mathrm{End}\,(\mathbf{N}, f)$, the relation R is not G-homomorphic.

For the proof of the second assertion put $G = \{a, b, c\}$ and denote by $\mathfrak{G}$ a commutative groupoid (a semigroup in fact) $(G, \circ)$, where the binary operation $\circ$ is given by this rule: For $x, y \in G$ we put $x \circ y = b$ iff $x = y = a$ and $x \circ y = c$ otherwise. Let P be a reflexive relation on G such that $\langle b, a\rangle \in P$ and $\langle c, b\rangle \in P$. Then P has (SP) with respect to $\mathfrak{G}$, but any mapping $\varphi \subset P$ with $\varphi(b) = a$ is not an endomorphism of $\mathfrak{G}$ for $\varphi(a) = a$, $\varphi(a \circ a) = a \neq b = \varphi(a) \circ \varphi(a)$. Further (denoting by C the corresponding algebraic closure operation on $\mathfrak{G}$) we have $C\{a\} = G$, $C\{b\} = \{b, c\}$, $C\{c\} = \{c\}$, hence the considered mapping φ is not a closure homomorphism of the space (G, C), as well.

Finally, consider the additive semigroup of positive integers $(\mathbf{N}, +)$ and a reflexive relation $R = \left\{\langle m, n\rangle : m, n \in \mathbf{N}, \dfrac{n}{m} \in \{1, 2, 3\}\right\}$. Denoting by φ_1 the identity mapping $\mathrm{id}_{\mathbf{N}}$ and by $\varphi_k : \mathbf{N} \to \mathbf{N}$, $k = 2, 3$ mappings defined by $\varphi_k(n) = k \,.\, n$ for $k = 2, 3$ and each $n \in \mathbf{N}$, we have $R = \bigcup_{k=1}^{3} \varphi_k$ and simultaneously φ_k, $k = 1, 2, 3$ are endomorphisms of $(\mathbf{N}, +)$, thus R is G-homomorphic. On the other hand, $\langle 1, 2\rangle \in R$, $\langle 1, 3\rangle \in R$ but $\langle 2, 5\rangle \notin R$. It is to be noted that the finiteness of $\mathfrak{G}$ in the first part of the above proof is not essential for the given counterexample. It can be replaced by the groupoid $\mathfrak{G}_1 = (G_1, \circ)$ defined as follows: Let $(G_1, \leqq)$ be a chain of the type ω^* with the greatest element s and for $a, b \in G_1$ we put $a \circ b = \min\{x : \max\{a, b\} < x\}$ if $a \neq s \neq b$ and $a \circ b = s$ in the opposite case. Further, $\langle a, b\rangle \in R$ if $a = b$ or a is a predecessor of the element b.

If we restrict ourselves to unary algebras (with arbitrary many unary operations) we get:

Proposition 3. *Every G-homomorphic relation on a unary algebra has (SP).*

Proof. Let R be a G-homomorphic relation on a unary algebra (A, F). For

$a, b \in A$ such that $\langle a, b\rangle \in R$ there exists $\varphi \subset R$, $\varphi \in \operatorname{End}(A, F)$ with $\varphi(a) = b$. Then for any operation $f \in F$ we have $\varphi(f(a)) = f(\varphi(a)) = f(b)$, i.e. $\langle f(a), f(b)\rangle \in R$ for each $f \in F$.

Remark 4. The opposite assertion is not valid as it follows from this example of a 1-unary algebra (A, f), where $A = \{a, b, c\}, f(a) = b, f(b) = f(c) = c$ and $R = \{\langle a, a\rangle, \langle b, b\rangle, \langle c, c\rangle, \langle b, a\rangle, \langle c, b\rangle\}$.

In Remark 3 it has been shown that the G-closure homomorphness does not imply the Substitution Property even in the case of 1-unary algebras. Now, we are going to prove that the implication G-closure homomorphness $\Rightarrow$ (SP) enforces very simple form of the corresponding 1-unary algebra. Using Theorems 3.2 and 4.1 from [5], we get the below stated results.

The notions, the component of a 1-unary algebra, the one-way infinite chain, the two-way infinite chain are taken from [10]; see also [5]. If (A_ι, f_ι) is a component of (A, f), we put as usually $A_\iota^0 = \{x \in A : f^{-1}(x) = \emptyset\}$, $A_\iota^{\infty 1}$ is the set of all elements $x \in A_\iota$ with this property: There exists an infinite sequence $\{a_n\}_{0 \leqq n < \omega} \subset A_\iota$ such that $a_n \neq a_m$ whenever $n \neq m, f(a_{n+1}) = f(a_n)$ and $a_0 = x$. Finally, by $A_\iota^{\infty 2}$ we denote the cycle (called also a core — cf. [8], section 3.1) of the component (A_ι, f_ι). An algebra (A, f) is called involutory if $f^2 = \mathrm{id}_A$.

Theorem 1. *Let (A, f) be a 1-unary algebra, $\{(A_\iota, f_\iota) ; \iota \in I\}$ the system of all its components. The following conditions are equivalent:*

1° $\iota \in I$ implies (A_ι, f_ι) is either idempotent or involutory.

2° Every G-closure homomorphic relation on (A, f) has (SP).

Proof. 1° $\Rightarrow$ 2°: Let R be a G-closure homomorphic relation on a 1-unary algebra (A, f) which satisfies the condition 1°. Suppose $a, b \in A$, $\langle a, b\rangle \in R$. There exists $g \in \operatorname{End}(A, C_f)$ with $g \subset R$, $g(a) = b$. Since $\{g(a), gf(a)\} = g(C_f\{a\}) = C_f\{g(a)\} = \{g(a), fg(a)\}$, we have $gf(a) = fg(a) = f(b)$, thus $\langle f(a), f(b)\rangle \in R$.

2° $\Rightarrow$ 1°: We have $\operatorname{End}(A, C_f) \subset \operatorname{End}(A, f)$. Admit there exists $\iota \in I$ such that (A_ι, f_ι) has at least three-element cycle $A_\iota^{\infty 2}$. Let $a \in A_\iota^{\infty 2}$ be an arbitrary element. Define a mapping $g : A \to A$ by: $g(a) = f(a)$, $g(f(a)) = a$ and $g(x) = x$ for each $x \in A \setminus \{a, f(a)\}$. Then evidently $g \in \operatorname{End}(A, C_f)$ but $g \in \operatorname{End}(A, f)$. Consequently $\operatorname{card} A_\iota^{\infty 2} \leqq 2$ for every $\iota \in I$. Suppose $A_\iota \setminus A_\iota^{\infty 2} \neq \emptyset$ and simultaneously $A_\iota^{\infty 2} = \{a, b\}$ for some $\iota \in I$. Using the mapping g defined above, i.e. $g(a) = b$, $g(b) = a$ and $g(x) = x$ for $x \in A \setminus \{a, b\}$, we get a contradiction again. Thus $\operatorname{card} A_\iota^{\infty 2} = 2$ implies $A_\iota^{\infty 2} = A_\iota$. Now we are going to verify that $A_\iota^{\infty 2} \neq \emptyset$ for each $\iota \in I$. Consider the congruence ϱ_ι on components (A_ι, f_ι) with $\operatorname{card} A_\iota^{\infty 2} \leqq 1$ defined in this way: $\langle x, y\rangle \in \varrho_\iota$ for $x, y \in A_\iota$ if from the equality $f_\iota^n(x) = f_\iota^m(y)$ with the least $m, n \in \mathbf{N} \cup \{0\}$ there follows $n = m$. Now admit there is $\varkappa \in I$ with $A_\varkappa^{\infty 1} = \emptyset$. Let $a \in A_\varkappa \setminus A_\varkappa^0$ be an arbitrary element. Define a selfmap φ_a of A as follows: For each $x \in A_\varkappa \setminus \bigcup_{0 \leqq k < \omega} \varrho_\varkappa(f^k(a))$ we put $\varphi_a(x) = f(x)$ and $\varphi_a(x) = x$ for every element $x \in \bigcup_{\substack{\iota \in I \\ \iota \neq \varkappa}} A_\iota \cup$

$\cup \bigcup_{0 \leq k < \omega} \varrho_x(f^k(a))$. Then for every $x \in A$ we have $\varphi_a(C_f\{x\}) = \{\varphi_a f^k(x)\colon k = 0, 1, 2, \ldots\} = \{f_k \varphi_a(x)\colon k = 0, 1, 2, \ldots\} = C_f\{\varphi_a(x)\}$ thus $\varphi_a \in \operatorname{End}(A, C_f)$. But for $b \in A_x$ such that $f(b) = a$ there holds $\varphi_a(f(b)) = \varphi_a(a) = a \neq f(a) = f(\varphi_a(b))$, hence $\varphi_a \notin \operatorname{End}(A, f)$, which is a contradiction. It remains to show that $\iota \in I$, $f_\iota^2 \neq \mathrm{id}_{A_\iota}$ implies $f_\iota^2 = f_\iota$. Admit, similarly as above, that for some $\varkappa \in I$ the component $(A_\varkappa, f_\varkappa)$ contains different elements a, b, c such that $f_\varkappa(b) = a, f_\varkappa(a) = f_\varkappa(c) = c$. Considering the above defined mapping φ_a ; $A \to A$, we get a contradiction again. Therefore for an arbitrary component (A_ι, f_ι) either $f_\iota^2 = f_\iota$ or $f_\iota^2 = \mathrm{id}_{A_\iota}$.

Theorem 2. *Let (A,f) be a 1-unary algebra, $\{(A_\iota, f_\iota);\ \iota \in I\}$ the system of all its components, $I_0 = \{\iota \in I;\ \operatorname{card} A_\iota > 1$, $F = \{f^2, f^3\}$. Suppose that there is satisfied exactly one of these conditions:*

1° $\iota \in I$ implies either $f_\iota^2 = f_\iota$ or $f_\iota^2 = \mathrm{id}_{A_\iota}$,

2° $\iota \in I_0$ implies $A_\iota = A_\iota^{\infty 1}$,

3° $\iota \in I_0$ implies $A_\iota = A_\iota^{\infty 1} \cap A_\iota^0$ where $(A_\iota^{\infty 1}, f_\iota)$ is a two-way infinite chain and $A_\iota^0 \neq \emptyset$,

4° $\iota \in I_0$ implies either (A_ι, f_ι) is a two-way infinite chain or $A_\iota = B_\iota \cap D_\iota$ where $(D_\iota, f_\iota \mid D_\iota)$ is a one-way infinite chain with the first element d, $B_\iota = A_\iota^0$ and $f(x) = d$ for each $x \in B_\iota$.

Then every G-closure homomorphic relation on the 2-unary algebra (A, F) has (SP) with respect to (A,f) and (A, F).

Proof. Let $a, b \in A$ be a pair of elements with the property $\langle a, b\rangle \in R$. There exists an endomorphism g of (A, C_F) such that $g(a) = b$ and $\langle x, g(x)\rangle \in R$ for every $x \in A$. Since for each $a \in A$ it holds $C_F\{a\} = \{f^k(a)\colon k = 0, 2, 3, \ldots\}$ and $C_F X = \bigcup_{x \in X} C_F\{x\}$ for $\emptyset \neq X \subset A$, we have by Theorems 3.2,4.1 from [5] $g \in \operatorname{End}(A,f)$, thus $g(f(a)) = f(g(a)) = f(b)$. Hence $\langle f^k(a), f^k(b)\rangle \in R$ for $k = 1, 2, 3$, q.e.d.

From the above theorem we can obtain some corollaries as e.g. the following one:

Corollary. *Let $\mathfrak{A} = (A, \{f_1, f_2\})$ be a 2-unary algebra such that f_1, f_2 are permutations of the set A and*

1° $f_k^n h \neq h f_k^n$ for each $n \in \mathbf{N}$, any constant selfmap h of A and $k = 1, 2$,

2° $f_1 f_2 = f_2 f_1$,

3° $f_1^3 = f_2^2$.

Then every G-closure homomorphic relation on the algebra $\mathfrak{A}$ has (SP).

Proof. Let $a \in A$ be an arbitrary element. There exists $b \in A$ with $f_1(b) = a$. Put $g(a) = f_2(b)$. Evidently g is a selfmap of the set A. For each $x \in A$ there exists $y \in A$ such that $f_1(y) = x$ and there exists $z \in A$ such that $f_1(z) = f_2(y)$. Then $g^2(x) = g(g(x)) = g(f_2(y)) = f_2(z)$ and $f_1^2(x) = f_1^3(y) = f_2^2(y) = f_2 f_1(z) = f_1 f_2(z) = f_1(g^2(x))$, i.e. $f_1^2 = f_1 g^2$. Since the mapping f_1 is one-to-one, we have $f_1 = g^2$. Further, $g^3(x) = g^2 g(x) = f_1 f_2(y) = f_2 f_1(y) = f_2(x)$, i.e. $f_2 = g^3$. Consequently every component (A_ι, g_ι) of the 1-unary algebra (A, g) is a two-way infinite chain and the assertion follows from Theorem 2.

Remark 5. It is to be noted that every congruence on arbitrary algebra is by Proposition 1 upper semicontinuous and image closed and every G-homomorphic congruence is G-closure homomorphic by Proposition 2. But quite trivial examples show that algebras with G-homomorphic congruences have very simple forms. Put e.g. $A = \{a, b, c, d, e\}, f(a) = b, f(b) = f(c) = f(d) = d, f(e) = e$. Then the congruence on (A, f) given by the decomposition of (A, f) into components is not G-homomorphic.

In this connection there arises a lot of questions concerning e.g. characterizations of algebras and relations for which the above mentioned compatibility types are comparable, questions of the description of algebras with the prescribed semigroups of G-compatible relations with further additional properties, etc.

REFERENCES

[1] Burris, S.: *Closure homomorphisms*, J. Algebra 15 (1970), 68—71.

[2] Chajda, I., Zelinka, B.: *Compatible relations on algebras*, Čas. pěst. matem. 100 (1975), 355—360.

[3] Chajda, I.: *Two constructions of compatible relations*, Czech. Math. J. 28 (1978), 439—444.

[4] Chajda, I.: *Relational classes and their characterizations*, Arch. Math. (Brno), 16 (1980), 199—204.

[5] Chvalina, J.: *Set transformations with centralizers formed by closed deformations of quasi-discrete topological spaces*, Proc. Fourth Prague Top. Sym. 1976, Part B, Soc. Czech. Math. Phys. Prague 1977, 83—89.

[6] Grätzer, G.: *Universal Algebra*, D. van Nostrand, New York 1968.

[7] Grimeisen, G.: *Continuous relations*, Math. Zeit. 127 (1972), 35—44.

[8] Jónsson, B.: *Topics in Universal Algebra*, Lect. Notes Math. 250, Berlin, Springer 1972.

[9] Smithson, R. E.: *Multifunctions*, Niew Arch. Wisk. 20 (1972), 31—55.

[10] Varlet, J. C.: *Endomorphisms and fully invariant congruences in unary algebras* $\langle A, \Gamma \rangle$, Bull. Soc. Royale Sci. Liège 39 (1970), 575—589.

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