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ON TOLERANCE ANALYSIS

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We study below a special case of a natural stability problem in the theory of tolerance relations: What theorems of classical calculus remain valid, when equality is substituted by a tolerance relation? We shall consider a wellknown theorem, stating, that every polynomial-equation $p(x) = 0$ of degree $n \geqq 1$ has at most n solutions.

Substituting some $R_\varepsilon = \{(x, y) \in \mathbf{R}^2 : | x - y | < \varepsilon\}$, $\varepsilon > 0$, for $=$, then, in general, the set $\{x \in \mathbf{R} : p(x)\ R_\varepsilon 0\}$ of ε-solutions is infinite. The cardinal number of the connectednesscomponents of this set, however, has the degree n of p as an upper bound, according to the next theorem. As $\{x \in \mathbf{R}: p(x) = 0\}$ trivially has at most n components, this theorem yields an extension of its classical version.

Theorem. Let $p: \mathbf{R} \to \mathbf{R}$ be a polynomial of degree $n \geqq 1$ and let $\varepsilon > 0$. The set $\{x \in \mathbf{R}: p(x)\ R_\varepsilon 0\} = \{x \in \mathbf{R}: | p(x) | < \varepsilon\}$ of ε-solutions has at most n components.

Proof: If $p^{-1}(-\varepsilon, \varepsilon) = \emptyset$, the theorem is trivial; assume therefore $p^{-1}(-\varepsilon, \varepsilon) \neq \emptyset$. As the polynomial p is not constant, the set $p^{-1}(-\varepsilon, \varepsilon)$ is bounded. It furthermore is an open set the components of which are intervals (a, b), $a \in \mathbf{R}$, such that

$$(1) \qquad \{p(a), p(b)\} \subseteq \{+\varepsilon, -\varepsilon\},$$

because p is continuous. As the sets of solutions of the equations $p(x) = \varepsilon$, $p(x) = -\varepsilon$ are finite, $p^{-1}(-\varepsilon, \varepsilon)$ has only finitely many components $C = (a_c, b_c)$, which are pairwise disjoint and linearely ordered: $C < D$ iff $b_C \leqq a_D$; let $C_i = (a_i, b_i)$ be an enumeration of the components such that $C_i < C_{i+1}$, $i \leqq m - 1$, m the number of components.

For $i \leqq m - 1$ $p(b_i) = p(a_{i+1})$: If $b_i = a_{i+1}$, this is trivial; assume therefore $b_i < a_{i+1}$. If $p(b_i) \neq p(a_{i+1})$ then $p(b_i) = -p(a_{i+1})$ by (1). Therefore there is an x, $b_i < x < a_{i+1}$, such that $p(x) = 0$ and a component C_j containing x; necessarily $C_i < C_j < C_{i+1}$, thus yielding a contradiction.

Furthermore we note that there is an x, $b_i \leqq x \leqq a_{i+1}$, such that the derivative $p'(x) = 0$: If $b_i < a_{i+1}$ this is from Rolle's theorem. If $b_i = a_{i+1}$, then $p(b_i)$ is an extremum of p in (a_i, b_{i+1}) and therefore $p'(b_i) = 0$.

So there are at least $m - 1$ distinct points $x_i, b_i \leqq x_i \leqq a_{i+1} < b_{i+1} \leqq x_{i+1}$, such that $p'(x_i) = 0$. As the polynomial p' has degree at most $n - 1$, we get: $m - 1 \leqq \leqq n - 1$, $m \leqq n$: $p^{-1}(-\varepsilon, \varepsilon)$ has $m \leqq n$ components.

Corollary: Let $p: \mathbf{R} \to \mathbf{R}$, $q: \mathbf{R} \to \mathbf{R}$ be polynomials of degrees n and m respectively: The set $\{x \in \mathbf{R}: p(x)\ R_\varepsilon q(x)\}$ has at most max $\{n, m, 1\}$ components.

The same is true for the relations $\mathbf{R} \times \mathbf{R}$, $\Delta = \{(x, x): x \in \mathbf{R}\}$, $R_\varepsilon(I) = R_\varepsilon \cup \cup \{(x, y),: |x - y| = \varepsilon, x \in I\}$, $\varepsilon > 0$, $I \subseteq \mathbf{R}$ connected. If on thecontrary $R \subseteq \mathbf{R} \times \mathbf{R}$ is a tolerance relation, such that for each pair (p, q) of polynomials of degrees $n = m = 1$ the set $\{x \in \mathbf{R}: p(x)\ Rq(x)\}$ has only one component, then R is one of these relations.

Also $\{x \in \mathbf{R}: \exists\, y\,.\, xR_\zeta y\ \&\ p(y)\ R_\varepsilon q(y).\}$ has at most max $\{n, m, 1\}$ components.

REFERENCE

[1] I. Chajda: *Systems of equations and tolerance relations*, Czech. Math. J. 25 (1975), 302—308.

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