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RELATIONAL CLASSES AND THEIR CHARACTERIZATIONS

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The concept of a congruence class can be generalized in two different ways for arbitrary binary relations on an algebra $\mathfrak{A}$. The first is the concept of a block of the relation (see [3]) and the second is the so-called relational class. Characterizations of relational blocks of relations (on algebras) satisfying a combination of properties: reflexivity, symmetry, transitivity, are contained in [1]. The paper [2] is a continuation of [1] and gives the conditions under which a system of subsets of a given algebra is a system of all blocks of some relation on this algebra with the prescribed combination of properties: reflexivity, symmetry and transitivity. The characterizations in [1] and [2] are based on polynomials and algebraic functions of a given algebra (see [6]). The concept of relational class is advantageous for some investigations of relations on algebra, see e.g. [8]. The aim of this paper is to give characterizations of relational classes in the similar way as in [1] and [2] for blocks.

Definition 1. Let R be a binary relation on a set A and $z \in A$. Call $[z]_R = \{a \in A; \langle a, z \rangle \in R\}$ an *R-class.*

Definition 2. Let R be a binary relation on a set A and $\emptyset \neq B \subseteq A$. Call B a *block of* R if $B \times B \subseteq R$ (i.e. $x, y \in B$ implies $\langle x, y \rangle \in R$) and B is a maximal subset with respect to this property.

We will study only relations with the Substitution Property on algebras (in [1], [2], [3], [4] the so called *compatible relations*), namely:

Definition 3. Let R be a binary relation on a set A and $\mathfrak{A} = (A, F)$ be an algebra. R has the *Substitution Property,* briefly (SP), if for each n-ary $f \in F$ and arbitrary $\langle a_i, b_i \rangle \in R$ ($a_i, b_i \in A$; $i = 1, \ldots, n$) we have $\langle f(a_1, \ldots, a_n), f(b_1, \ldots, b_n) \rangle \in R$.

Remark. It is clear that both blocks of R and R-classes coincide with congruence classes of R whenever R is a congruence on $\mathfrak{A}$. Hence they both are generalizations of congruence classes.

Notation. Let p be an n-ary polynomial and φ an n-ary algebraic function of $\mathfrak{A} = (A, F)$ (see e.g. [6]) and $B \subseteq A$. Denote by $p(B) = \{p(b_1, \dots, n_n); b_i \in B\}$, $\varphi(B) = \{\varphi(b_1, \dots, b_n); b_i \in B\}$.

Theorem 1. *Let $\mathfrak{A} = (A, F)$ be an algebra, $z \in A$ and $\emptyset \neq B \subseteq A$. The following conditions are equivalent:*

(a) *$B = [z]_R$ for some binary relation R with (SP) on $\mathfrak{A}$;*

(b) *for every integer $n > 0$ and every n-ary polynomial p over $\mathfrak{A}$ with $p(z, \dots, z) = z$ we have $p(B) \subseteq B$.*

Proof. (a) $\Rightarrow$ (b): Let $B = [z]_R$ for some R with (SP) on $\mathfrak{A}$ and p be an n-ary polynomial with $p(z, \dots, z) = z$. If $a_1, \dots, a_n \in B$, then $\langle a_i, z \rangle \in R$, and by (SP), also

$$\langle p(a_1, \dots, a_n), p(z, \dots, z) \rangle = \langle p(a_1, \dots, a_n), z \rangle \in R.$$

Hence $p(a_1, \dots, a_n) \in [z]_R = B$, i.e. $p(B) \subseteq B$.

(b) $\Rightarrow$ (a): Let us construct R as a set of all pairs $\langle x, y \rangle$, to which there exists an n-ary polynomial p over $\mathfrak{A}$ and elements $a_1, \dots, a_n \in B$ such that $y = p(z, \dots, z)$ and $x = p(a_1, \dots, a_n)$. Clearly R has (SP) on $\mathfrak{A}$ (it is easy to show by induction over the rank of polynomial, see [6]). It remains to prove $B = [z]_R$. Let $x \in [z]_R$. Then $\langle x, z \rangle \in R$ and, by the definition, there exists a polynomial p and $a_i \in B$ such that $x = p(a_1, \dots, a_n)$, $z = p(z, \dots, z)$. By (b), $x \in B$, i.e. $[z]_R \subseteq B$. Conversely, if $y \in B$, then $\langle y, z \rangle \in R$ (e.g. for $p(x) = x$). Hence $y \in [z]_R$ proving $B \subseteq [z]_R$.

Theorem 2. *Let $\mathfrak{A} = (A, F)$ be an algebra, $\emptyset \neq B \subseteq A$ and $z \in B$. The following conditions are equivalent:*

(a) *$B = [z]_R$ for some reflexive binary relation R with (SP) on $\mathfrak{A}$.*

(b) *For every n-ary algebraic function φ over $\mathfrak{A}$ with $\varphi(z, \dots, z) = z$ we have $\varphi(B) \subseteq B$.*

The proof is analogous to that of Theorem 1. It can also be derived directly from Theorem 1 by adding all elements from A as nullary operations to $\mathfrak{A}$. The polynomials of this new algebra are algebraic functions of the original one.

Theorem 3. *Let $\mathfrak{A} = (A, F)$ be an algebra, $z \in A$ and $\emptyset \neq B \subseteq A$. The following conditions are equivalent:*

(a) *$B = [z]_R$ for some symmetric relation with (SP) on $\mathfrak{A}$.*

(b) *For every $(n + m)$-ary polynomial p over $\mathfrak{A}$ we have: if $p(b_1, \dots, b_n, z, \dots, z) = z$ for some $b_i \in B$, then $p(z, \dots, z, a_1, \dots, a_m) \in B$ for each $a_1, \dots, a_m \in B$.*

Proof. (a) $\Rightarrow$ (b): let $B = [z]_R$ and p be an $(n + m)$-ary polynomial over $\mathfrak{A}$ with $p(b_1, \dots, b_n, z, \dots, z) = z$ for some $b_i \in B$ and $a_1, \dots, a_m \in B = [z]_R$. Then $\langle b_i, z \rangle \in R$, $\langle z, a_j \rangle \in R$ and, by (SP), the assertion (b) is evident.

(b) $\Rightarrow$ (a): let R be a set of all pairs $\langle x, y \rangle$ such that $x = p(z, \dots, z, a_1, \dots, a_m)$, $y = p(b_1, \dots, b_n, z, \dots, z)$ for some $(n + m)$-ary polynomial p and some $a_j, b_i \in B$.

Clearly R is symmetric and it has (SP). It remains to prove $B = [z]_R$. Let $x \in [z]_R$. Then $\langle x, z \rangle \in R$, i.e. $x = p(z, \ldots, z, a_1, \ldots, a_m)$, $z = p(b_1, \ldots, b_n, z, \ldots, z)$. By (b), $x \in B$ proving $[z]_R \subseteq B$. The converse implication is trivial.

Theorem 4. *Let $\mathfrak{A} = (A, F)$ be an algebra, $\emptyset \neq B \subseteq A$, $z \in B$. The following conditions are equivalent*:

(a) *$B = [z]_R$ for some reflexive and symmetric relation R with* (SP) *on $\mathfrak{A}$* (*i.e. for some tolerance on $\mathfrak{A}$, see* [4]);

(b) *for every $(n + m)$-ary algebraic function φ over $\mathfrak{A}$, if $\varphi(b_1, \ldots, b_n, z, \ldots, z) = z$ for some $b_i \in B$, then $\varphi(z, \ldots, z, a_1, \ldots, a_m)$ B for every $a_1, \ldots, a_m \in B$.*

The argumentation of the proof is the same as in the proof of Theorem 2.

Now, we will investigate transitive relations:

Theorem 5. *Let $\mathfrak{A} = (A, F)$, $\emptyset \neq B \subseteq A$ and $z \in B$. The following conditions are equivalent*:

(a) *$B = [z]_R$ for some reflexive and transitive relation R with* (SP) *on $\mathfrak{A}$* (*i.e. for R compatibe quasiorder,* [1]);

(b) *for each n-ary algebraic function φ we have*

$$\varphi(z, \ldots, z) \in B \text{ implies } \varphi(B) \subseteq B.$$

Proof. (a) $\Rightarrow$ (b): Let $B = [z]_R$, φ be an algebraic function over $\mathfrak{A}$ and $\varphi(z, \ldots, z) \in B$. Since R is reflexive and has (SP), we obtain $\langle \varphi(a_1, \ldots, a_n), \varphi(z, \ldots, z \rangle \in R$ when $a_1, \ldots, a_n \in B$. As $\varphi(z, \ldots, z) \in B = [z]_R$, also $\langle \varphi(z, \ldots, z), z \rangle \in R$. By the transitivity of R, we obtain $\langle \varphi(a_1, \ldots, a_n), z \rangle \in R$, and thus $\varphi(a_1, \ldots, a_n) \in [z]_R = B$. This proves the inclusion $\varphi(B) \subseteq B$.

(b) $\Rightarrow$ (a): Let R^* be the set of all pairs $\langle x, y \rangle$ such that $x = \varphi(a_1, \ldots, a_n)$, $y = \varphi(z, \ldots, z)$ for some algebraic function φ over $\mathfrak{A}$ and elements $a_1, \ldots, a_n \in B$. Let R be the transitive hull of R^*. Clearly R^* is reflexive and has (SP) on $\mathfrak{A}$ (by induction over the rank of polynomial generating φ). By Theorem 6 in [5], R is reflexive, transitive and has (SP) on $\mathfrak{A}$. It remains to prove $B = [z]_R$. If $y \in B$, then by the definition of R^*, $\langle y, z \rangle \in R^*$. Since $R^* \subseteq R$, also $\langle y, z \rangle \in R$, thus $y \in [z]_R$ and $B \subseteq [z]_R$. Prove the converse inclusion. Let $x \in [z]_R$, i.e. $\langle x, z \rangle \in R$. Then there exist elements $x_0, \ldots, x_m \in A$ such that $x_0 = x$, $x_m = z$ and $\langle x_{i-1}, x_i \rangle \in R^*$ for $i = 1, \ldots, m$, i.e. there exist n_i-ary algebraic functions φ_i and $a_1^i, \ldots, a_{n_i}^i \in B$ such that

$$x_{i-1} = \varphi_i(a_1^i, \ldots, a_{n_i}^i),$$
$$x_i = \varphi_i(z, \ldots, z).$$

Since $x_m = z \in B$, we have $\varphi_m(z, \ldots, z) \in B$ and, by (b), also $x_{m-1} = \varphi_m(a_1^m, \ldots, a_{n_m}^m) \in B$. However, $x_{m-1} = \varphi_{m-1}(z, \ldots, z)$, thus $\varphi_{m-1}(z, \ldots, z) \in B$ and, by (b), also $x_{m-2} \in B$. After m steps we obtain $x = x_0 \in B$. Hence $[z]_R \subseteq B$.

The following casses remain:

(1) R is *transitive* with (SP) on $\mathfrak{A}$ (but not necessarily reflexive).

(2) R is *transitive and symmetric* with (SP) on $\mathfrak{A}$.

(3) R is a *congruence* on $\mathfrak{A}$.

In the case (1), no characterization of $[z]_R$ is known. This problem remains open. In the casses (2) and (3), R-classes and blocks of R coincide, i.e. they are solved in [1, Theorems 3 and 4]. Especially, the case (3) is a classical result of A. I. Mal'cev in [7].

Now, we can describe the relationship between blocks of R and R-classes in the case where R is reflexive, symmetric and has (SP) on $\mathfrak{A}$, i.e. R is a *tolerance* on $\mathfrak{A}$, see [4].

Theorem 6. *Let $\mathfrak{A} = (A, F)$, $\emptyset \neq B \subseteq A$ and R be a reflexive and symmetric binary relation with (SP) on $\mathfrak{A}$. The following conditions are equivalent:*

(a) *B is a block of R;*

(b) $B = \cap\{[z]_R;\ z \in B\}$.

Proof. (a) $\Rightarrow$ (b): Let B be a block of R. If $z \in B$, clearly $\langle x, z\rangle \in R$ for each $x \in B$, i.e. $B \subseteq [z]_R$. Hence $B \subseteq \cap\{[z]_R;\ z \in B\}$. Conversely, if $a \in \cap\{[z]_R;\ z \in B\}$, then $\langle a, z\rangle \in R$ for each $z \in B$. Since R is reflexive and symmetric, also $\langle z, a\rangle \in R$, $\langle a, a\rangle \in R$, $\langle z, z\rangle \in R$, and thus $C \times C \subseteq R$ for $C = B \cup \{a\}$. However, B is a block of R, i.e. it is a maximal subset with $B \times B \subseteq R$. Hence $C = B$, i.e. $a \in B$ proving the inclusion.

(b) $\Rightarrow$ (a): Let $B = \cap\{[z]_R;\ z \in B\}$ and $a, b \in B$. Thus $a \in [b]_R$, i.e. $\langle a, b\rangle \in R$. Analogously, $\langle b, a\rangle \in R$ and, by the reflexivity of R, also $\langle a, a\rangle \in R$, $\langle b, b\rangle \in R$, i.e. $\{a, b\} \times \{a, b\} \subseteq R$. Since a, b are arbitrary of B, also $B \times B \subseteq R$. By Zorn's lemma, there exists a block C of R such that $B \subseteq C$. Let $a \in C$. Then $\langle a, z\rangle \in R$ for each $z \in B$. Hence $a \in \cap\{[z]_R;\ z \in B\} = B$, i.e. $B = C$, and so B is a block of R.

Theorem 7. *Let $\mathfrak{A} = (A, F)$, $z \in B \subseteq A$ and R be a reflexive and symmetric relation with (SP) on $\mathfrak{A}$. The following conditions are equivalent:*

(a) $B = [z]_R$;

(b) *B is a set-union of all blocks of R containing z.*

Proof. (a) $\Rightarrow$ (b): Let $x \in [z]_R$. Then $\langle x, z\rangle \in R$ and clearly also $\langle z, x\rangle \in R$, $\langle x, x\rangle \in R$, $\langle z, z\rangle \in R$. Hence $\{x, z\} \times \{x, z\} \subseteq R$ and, by Zorn's lemma, there exists a block C of R with $x, z \in C$. Thus $[z]_R \subseteq D$, where D is the set-union of all blocks of R containing z. Conversely, if $y \in D$, then $\langle y, z\rangle \in R$, i.e. $y \in [z]_R$, thus $D \subseteq [z]_R$.

The implication (b) $\Rightarrow$ (a) is obvious.

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