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ON CONNECTED UNARS WITH REGULAR ENDOMORPHISM MONOIDS

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A monounary algebra, i.e. a pair (A,f), where A is a non-void set and f a self-map of the set A, is briefly called a unar. This paper aims to give some conditions of the topological and algebraic character equivalent to the regularity of the endomorphism monoid of a connected unar. There are used the descriptions of unars with regular and inverse endomorphism monoids obtained by L. A. Skornjakov in [12] and results of papers [4], [6]. In the below stated characterizations we consider mostly endomorphism monoids which are not groups. For the characterization of unars whose endomorphism monoids are automorphism groups see [12] Theorem 3.

Fundamental used notions concerning monounary algebras can be found e.g. in papers [5], [8], [11], [12]. Let (A,f) be a connected unar. The set of all cyclic elements of (A,f) (i.e. such elements $a \in A$ that $f^n(a) = a$ for some integer $n \geqq 1$) will be denoted in regard with [8] by A^{∞_2} and further $A^{\infty_1} = \{x \in A \setminus A^{\infty_2}$: there is a sequence $\{x_i\}_{i \in \omega}$ such that $x_0 = x$ and $f(x_{i+1}) = x_i$ for each $i \in \omega\}$, $A_0 = \{x \in A: f^{-1}(x) = \emptyset\}$. A unar is called a cycle if $A = A^{\infty_2}$. The upper cone of an element a, i.e. the set $\{f^n(a): n = 0, 1, 2, \ldots\}$ will be denoted by $[a)_f$, the lower cone $\{x \in A: f^n(x) = a$ for some $n \in \omega\}$ by $(a]_f$. We agree on denoting the cardinality of a set A by $|A|$. A connected unar (A,f) with $|A| = \aleph_0$ and f – a permutation of A is called a line. A connected unar (A,f) is said to be a cycle with short tails or a line with short tails if it contains a cycle or a line C such that $f(x) \in C$ for every $x \in A$. If $|B^{\infty_2}| \leqq 1$ for each component (B, f_B) of a unar (A,f) we put $a \leqq_f b$ for $a, b \in A$ if there exists $n \in \omega$ with $f^n(a) = b$ and $a <_f b$ if $a \leqq_f b$, $a \neq b$. Further, we denote by $(\mathbf{A}, \mathbf{f})$ the factor-unar (i.e. the factor-algebra of a monounary algebra (A,f)) corresponding to the congruence $\equiv_f$ on (A,f) defined by $a \equiv_f b$ if $a = b$ or $a, b \in A^{\infty_2}$. The monoid of all endomorphisms of (A,f) is denoted by $E(A,f)$. For the definition of a regular and inverse semigroup see [3] § 1.9. A certain strengthening of the notion of a regular semigroup is the notion of an anti-regular semigroup (cf. [10]) called in [1] an anti-inverse semigroup. Let us recall the necessary definitions (see [1] and [10]): A semigroup S is said to be anti-inverse if for each element $a \in S$ there is an element $b \in S$ such that $aba = b$ and $bab = a$. The elements a and b are then called anti-inverses.

A saturated topological space called also quasi-discrete ([2] 26A) is a topological space (A, τ) with the completely additive topological closure operation τ i.e. each point of this space possesses the minimum neighbourhood (cf. [9]). A discrete space of Alexandrov is a saturated T_0-space. Compactness is meant in the sense of [2] 41A, i.e. quasi-compactness considered in [9]. A continuous closed self-map of a topological space (A, τ) will be called as usual a closed deformation of (A, τ) and the monoid of all closed deformations of this space will be denoted by $S(A, \tau)$ We say that a topological space (A, τ) has the fixed set property or briefly the FS-property (the fixed point property, briefly the FP-property) with respect to closed deformations if there exists a non-void proper subset $X \subset A$ (a point $x \in A$) with $f(X) = X (f(x) = x)$ for each $f \in S(A, \tau)$.

In what follows $\subseteq$ means the usual set inclusion and $A \subset B$ means $A \subseteq B$ $A \neq B$.

Theorem 1. *Let (A, f) be a connected unar whose endomorphism monoid is not a group. Then $E(A, f)$ is regular if and only if there exists a discrete topology of Alexandrov τ on the set A such that $E(A, f) = S(A, \tau)$ and the space (A, τ) has the FS-property with respect to closed deformations.*

Proof. Let (A, f) be a connected unar satisfying the assumption of the theorem. Since $(\mathbf{A}, \mathbf{f})$ contains at most one cyclic element, by Theorem 3.3 [4] there exists a discrete topology of Alexandrov τ with $E(\mathbf{A}, \mathbf{f}) = S(\mathbf{A}, \tau)$ if and only if the unar $(\mathbf{A}, \mathbf{f})$ has one of the following forms:

(i) $\mathbf{f}^2 = \mathbf{f}$,

(ii) $A = A^{\infty 1} \cup A^0$, where either $A^0 = \emptyset$ or $(A^{\infty 1}, \leq_f)$ is a chain of the type $\omega^* \oplus \omega$ and $A^0 = \emptyset$ (i.e. (A, f) is a line with short tails),

(iii) $A = A^0 \cup A_1$, where $(A_1, \leq_f)$ is a chain of the type ω with the first element c and $f(a) = c$ for each $a \in A^0$.

Suppose $A = A^{\infty 1}$ and simultaneously $(A^{\infty 1}, f)$ is not a line. Admit there exists a non-void set $B \subset A$ with $g(B) = B$ for each $g \in E(A, f)$. Since $f^k \in E(A, f)$ for every $k \in \omega$, the ordered set $(B, \leq_f)$ does not contain any minimal and maximal element and $[b)_f \subseteq B$ for each $b \in B$. There exists a pair of elements $a, b \in A$ such that $a \in A \setminus B$, $b \in B$ and $f^n(a) = f^n(b)$ for some $n \in \omega$. Since elements a, b form a pair of h-elements in the sense of [8] Definition 1.22 and xii [8] there exists $g \in E(A, f)$ such that $g(b) = a$. We get a contradiction, hence in the considered case for every non-void subset $B \subset A$ there exists an endomorphism g of (A, f) with $g(B) \neq B$. Consequently $(A^{\infty 1}, f)$ is a line in the considered case. Since the existence of a non-void subset $B \subseteq A$ with the property $g(B) = B$ for each $g \in E(A, f)$ implies the inclusion $B \subseteq$ $\subseteq A^{\infty 1} \cup A^{\infty 2}$ we have that the case (iii) is eliminated. On the other hand if (A, f) is a connected unar with $f^2 = f$ and $| A | \geq 2$ or (A, f) is a line with short tails then A contains an $E(A, f)$-invariant non-void proper subset. (A singleton formed by the cyclic element in the first case and the carrier of the line in the second one). Therefore there exists a discrete topology of Alexandrov τ on **A** with $S(\mathbf{A}, \tau) = E(\mathbf{A}, \mathbf{f})$ and the space $(\mathbf{A}, \tau)$ has the FS-property with respect to closed deformations if and only if (A, f) is either a cycle with short tails or a line with short tails. Now, from Theorem 1 [12] there follows the assertion, q.e.d.

In the following proposition $LT(A)$ means the left zeros subsemigroup of the full transformation monoid $T(A)$ on the set A. Recall that a unar is said to be nested if the system of all its subunars ordered by means of set inclusion forms a chain.

Proposition 1. *Let (A, f) be a connected unar. The following conditions are equivalent:*

1° *$E(A, f)$ is regular and $LT(A) \cap E(A, f^k) \neq \emptyset$ for some $k \in \omega$.*

2° *There exists a compact saturated topology τ on **A** with the property $E(\mathbf{A}, \mathbf{f}) = S(\mathbf{A}, \tau)$.*

3° *There exists a saturated topology τ on A with $E(\mathbf{A}, \mathbf{f}) = S(\mathbf{A}, \tau)$ and the space $(\mathbf{A}, \tau)$ has the FP-property with respect to closed deformations.*

Proof. 1° $\Rightarrow$ 2°: Since for some positive integer $k \in \omega$ there exists a constant self-map g of A with $g \in E(A, f^k)$ we have by [12] Theorem 1 (A, f) is a cycle with short tails (or without tails). If we define a topology τ on the set **A** by putting a τ-closure of a subset $X \subseteq \mathbf{A}$ as $\tau X = X \cup \mathbf{f}(X)$, Condition 2° is satisfied.

2° $\Rightarrow$ 3°: Let τ be a compact saturated topology on the set **A** such that $E(\mathbf{A}, \mathbf{f}) = S(\mathbf{A}, \tau)$. By [4] Theorem 3.3 the unar $(\mathbf{A}, \mathbf{f})$ has one of the forms (i)–(iii) listed in the proof of Theorem 1. For each $a \in A$ there exists a nested subunar (B, f_B) of (A, f), an element $b \in B$ and a surjective homomorphism $g : (A, f) \to (B, f_B)$ such that $g(a) = b$ and the equality $f^m(a) = f^n(b)$ with integers m, n minimal with respect to this property implies $m = n$. Since $\mathbf{f}^n \in S(\mathbf{A}, \tau)$ for each $n \in \omega$ we have that for each $a \in \mathbf{A}$ the closure $\tau\{a\}$ is a right cofinal subset of $[a)_f$ and has the following property: If $x, y, z \in \tau\{a\}$, $x <_\mathbf{f} y <_\mathbf{f} z$ then from $\mathbf{f}^n(x) = y$, $\mathbf{f}^m(y) = z$ with minimal m, n it follows either $n = m$ or $z = \mathbf{f}(y)$. Then the least τ-neighbourhood of a (i.e. the closure of $\{a\}$ in the saturated topology dual to τ) is a left cofinal subset of $(a]_\mathbf{f}$. Since the space $(\mathbf{A}, \tau)$ is compact by [9] Proposition 1 the unar $(\mathbf{A}, \mathbf{f})$ contains a cyclic element, say e. Hence $\mathbf{f}^2 = \mathbf{f}$ and $g(e) = e$ for each $g \in S(\mathbf{A}, \tau)$.

3° $\Rightarrow$ 1°: Since $\mathbf{f} \in S(\mathbf{A}, \tau)$ and the unar $(\mathbf{A}, \mathbf{f})$ is connected there exists exactly one element $e \in A$ with $\mathbf{f}(e) = e$. By [4] Theorem 3.3 $\mathbf{f}^2 = \mathbf{f}$. Condition 1° follows easy with respect to [12] Theorem 1, q.e.d.

Corollary. *Let (A, f) be a connected unar. The following conditions are equivalent:*

1° *Each element of $E(A, f)$ has a unique anti-inverse element in $E(A, f)$.*

2° *The set A is either a singleton or (A, τ) is the Sierpinski-space for each topology τ on A with the property $S(A, \tau) = E(A, f)$.*

3° *There exists a discrete topology of Alexandrov τ on A such that (A, τ) is a tower space and $E(A, f) = S(A, \tau)$.*

4° *There exists a saturated tower topology τ on A such that the space (A, τ) has the FP-property with respect to closed deformations and $E(A, f) = S(A, \tau)$.*

Proof follows immediately from Theorem 1, Proposition 1 and [10] Theorem 2.

The other characterization is expressed in terms of the groupoid theory. Similarly to [5] § 1 we associate a groupoid with every connected unar (A, f). For $a, b \in A$, denote by m, n the smallest non-negative integers such that $f^n(a) \in [b)_f$, $f^m(b) \in [a)_f$. We put $\delta(a, b) = m - n$. Evidently $\delta(a, b) + \delta(b, a) = 0$ for each pair $a, b \in A$ and $\delta(a, b) = \delta(b, a) = 0$ for each pair $a, b \in A^{\infty 2}$. Further we put $a\varepsilon_f b = f(b)$ if $\delta(a, b) \geqq 0$ and $a\varepsilon_f b = f(a)$ if $\delta(a, b) < 0$. It is to be noted that the groupoid (A, ε_f) associated in this way with a unar (A, f) is neither associative nor commutative in general. In papers [5], [6] the binary operation ε_f is denoted by ∇_f.

The following statement is contained in [5] Proposition 1.2.

Proposition 2. *Let (A, f) be a connected unar such that either $A^{\infty 2} = \emptyset$ or $f^2 = f$. Then $E(A, f) = E(A, \varepsilon_f)$.*

Proposition 3. *Let (A, f) be a connected unar with the regular endomorphism monoid $E(A, f)$. Then there exists a commutative binary operation $\circ$ on the set* **A** *such that* $E(\mathbf{A}, \mathbf{f}) = E(\mathbf{A}, \circ)$.

Proof. According to [12] Theorem 1 the unar $(\mathbf{A}, \mathbf{f})$ has one of the following forms:

(1) it is trivial (i.e. $|\mathbf{A}| = 1$),

(2) $\mathbf{f}^2 = \mathbf{f}$,

(3) $(\mathbf{A}, \mathbf{f})$ is a line,

(4) $(\mathbf{A}, \mathbf{f})$ is a line with short tails.

If one of cases (3), (4) occurs, then $\mathbf{A} = A$, $\mathbf{f} = f$. Putting for each pair $a, b \in \mathbf{A}$: $a \circ b = a\varepsilon_f b$, we get evidently a commutative groupoid $(\mathbf{A}, \circ)$ satisfying the condition $E(\mathbf{A}, \mathbf{f}) = E(\mathbf{A}, \circ)$ with respect to Proposition 2, q.e.d.

By an ideal of a groupoid $(A, .)$ we mean a both-side ideal, i.e. a non-empty subset $J \subseteq A$ such that $a \in J$, $b \in A$ implies $a . b \in J$ and $b . a \in J$. The principal ideal generated by an element a is denoted by $J(a)$. An ideal J is said to be trivial if $|J| = 1$. If $(A, .)$ is a groupoid and J an ideal of this groupoid then the corresponding Rees factor-groupoid is denoted by $(A/J, ._J)$; cf. [3] and [7]. A groupoid $(A, .)$ is called distributive if for each triad $a, b, c \in A$ equalities $a.(b.c) = (a.b).(a.c)$, $(a.b).c = (a.c).(b.c)$ hold and it is called a BD-groupoid (in accordance with [7]) if it satisfies one of the following equivalent conditions (see [7] Proposition 1.2):

(i) $(A, .)$ is distributive and the set of all its idempotents contains just one element,

(ii) there is an element $e \in A$ with $a.e = e = e.a$ (a zero of $(A, .)$) and $a.(b.c) = e = (a.b).c$ for all $a, b, c \in A$.

Theorem 2. *Let (A, f) be a connected unar such that the endomorphism monoid $E(A, f)$ is not a group. $E(A, f)$ is regular if and only if $(\mathbf{A}, \varepsilon_\mathbf{f})$ is a commutative groupoid containing the least proper ideal J with the following properties:*

(i) *The factor-groupoid* $(A / J, \varepsilon_J)$ *is a BD-groupoid.*

(ii) *If J is principal or contains an idempotent then it is trivial.*

Proof. Let (A,f) be a connected unar with the regular endomorphism monoid $E(A,f)$ being not a group. With respect to [12] Theorem 1 and the definition of the binary operation ε_f we have that (A, ε_f) is a commutative groupoid. If $f^2 = f$ we put $J = \{e\}$, where $e = A^{\infty 2}$. If (A,f) is a line with short tails, i.e. $A = A^{\infty 1} \cup A^0$ with $(A^{\infty 1},f)$ a line—we put $J = A^{\infty 1}$. It can be easily shown (see the third part of the proof of Theorem 3.8 [6]) that in this case J is the least proper ideal of the groupoid (A, ε_f). Since the Rees factor-groupoid $(A/J, \varepsilon_J)$ of the groupoid (A, ε_f) is associated with a connected idempotent unar $(A/J, F)$ (which is a factor unar of (A, f)) we have by [5] Lemma 1.3 that $(A/J, \varepsilon_J)$ is a BD-groupoid. The ideal J is principal if it contains an idempotent of (A, ε_f), i.e. if $J = \{e\}$, where e is the only cyclic element of (A, f).

Suppose (A,f) is a connected unar such that $E(A,f)$ is not a group and such that (A, ε_f) is a commutative groupoid the least proper ideal J of which satisfies the above assumptions. From the commutativity of ε_f it follows that for each pair $a, b \in A$, the equality $\delta(a, b) = 0$ implies $f(a) = f(b)$. Since $E(A,f)$ is not a group, (A,f) is neither a cycle nor a line. If $A^{\infty 1} \neq \emptyset$ then it can be easily verified (in the same way as in the proof of Theorem 3.8 [6] p. 150) that the least ideal of (A, ε_f) coincides with the least subunar of (A, f) containing the set $A^{\infty 1} (= A^{\infty 1})$. This ideal is non-trivial hence (A,f) is a line with short tails. If $A^{\infty 1} = \emptyset$ then there exists an element $a \in A$ with $\delta(a, x) \leqq 0$ for each $x \in A$. Then $\{f^k(a) : k = 1, 2, \ldots\}$ is the least ideal of (A, ε_f) and since it is principal we have $f^2 = f$. Hence (A,f) is a cycle with short tails. Applying [12] Theorem 1 we get $E(A,f)$ is regular, q.e.d.

In [12] Theorem 2 there are given necessary and sufficient conditions under which the endomorphism monoid of a unar is an inverse semigroup. In fact these conditions strengthen those which are necessary and sufficient for the regularity of $E(A,f)$. In the case of a connected unar $E(A,f)$ is an inverse semigroup if and only if $|f^{-1}(a)| \leqq 2$ for each $a \in A$ and (A,f) is either a cycle with short tails or a line with short tails (cf. [12] Theorem 2). From this result, using the binary operation ε_f, we get the below stated characterization analogical to Theorem 3.9 [6].

For each element a of a groupoid $(A, .)$ we put $\sqrt{a} = \{x \in A : x.x = a\}$. Every element $b \in \sqrt{a}$ is called a square root of the element a in the groupoid $(A, .)$. If $|\sqrt{a}| = 1$ we say the element a possesses the unique square root in $(A, .)$. Especially, $\sqrt{a} = f^{-1}(a)$ for each element a of the groupoid (A, ε_f) and thus evidently $E(A,f)$ is a group (in the case of connected (A,f)) if and only if each element of (A, ε_f) possesses the unique square root.

Proposition 4. *Let (A,f) be a connected unar. $E(A,f)$ is an inverse semigroup if and only if either $|\sqrt{a}| = 1$ holds for each element a of (A, ε_f) or $|\sqrt{a}| \leqq 2$ for* *every $a \in (A, \varepsilon_f)$ and (A, ε_f) contains the least ideal J each element of which possesses the unique square root in (J, ε_f).*

Proof. Let (A,f) be a connected unar. For $E(A,f)$ being a group the assertion is evident. Thus we assume $E(A,f)$ is not a group. If $E(A,f)$ is an inverse semigroup then by [12] Theorem 2 for each $a \in A$ we have $|f^{-1}(a)| \leqq 2$ and either $A = A^0 \cup \cup A^{\infty_1}$ (where (A^{∞_1},f) is a line) or $A = A^0 \cup A^{\infty_2}$. Putting $J = A^{\infty_1}$ in the first case and $J = A^{\infty_2}$ in the second one, we obtain the assertion with respect to $A^0 \neq \emptyset$.

Assume the groupoid (A, ε_f) is satisfying conditions from the above proposition. Since each element of J possesses the unique square root in (J, ε_f) and $\sqrt{a} = f^{-1}(a)$ for each $a \in A$, the subunar (J, f_J) is either a cycle or a line. Since J is the least ideal of (A, ε_f) we have $A \setminus J = A^0$. Thus (A,f) is either a cycle with short tails or a line with short tails and $|f^{-1}(a)| = |\sqrt{a}| \leqq 2$ for each $a \in A$. Consequently $E(A,f)$ is an inverse semigroup, q.e.d.

The requirement of the anti-inversibility of $E(A,f)$ enforced a very simple structure of the unar (A,f).

Proposition 5. *Let (A,f) be a connected unar. $E(A,f)$ is an anti-inverse semigroup if and only if (A,f) is a cycle of the cardinality 1 or 2 with at most one short tail.*

Proof. Suppose (A,f) has one of the required form. If $E(A,f)$ is non-trivial, then either $E(A,f) = \{\mathrm{id}_A, f\}$ or $E(A,f) = \{\mathrm{id}_A, f, f^2\}$. Since $E(A,f)$ is commutative, by [10] Theorem 4 (i) and (ii) it is anti-inverse. It is to be noted that as the multiplicative table for $E(A,f) \setminus \{\mathrm{id}_A\} = \{f, f^2\}$ can serve the table 3) from [1] Example 2.1.

Let (A,f) be a connected unar such that $E(A,f)$ is an anti-inverse semigroup. Since $E(A,f)$ is regular by [10] Theorem 1 or [1] Corollary 2.1 (i), we have in virtue of [12] Theorem 1 and [1] Theorem 2.1 (A,f) is a cycle of the cardinality at most 4 (except 3) with at most short tails. Admit $|A^0| \geqq 2$. Assume $a, b \in A^0, a \neq b$. Since there exists $g \in E(A,f)$ such that $g(a) = b$, $g(b) \in A^{\infty_2}$, we have $g^5 \neq g$ thus in regard with [1] Theorem 2.1 $E(A,f)$ is not anti-inverse. Hence (A,f) is a cycle with at most one short tail. Then $E(A,f)$ is a commutative monoid. Admitting $|A^{\infty_2}| = 4$, we have $f^3 \neq f$, which is a contradiction in virtue of [10] Theorem 4. Consequently $|A^{\infty_2}| \leqq 2$, q.e.d.

Remark. It is easy to verify that $E(A,f)$ is anti-inverse for a connected unar (A,f) with $|A| > 1$ if and only if the groupoid (A, ε_f) has one of the following multiplicative table (or the other formed by a permutation of elements):

ε_f	a	b
a	a	a
b	a	a

ε_f	a	b
a	b	a
b	b	a

ε_f	a	b	c
a	b	b	b
b	b	c	b
c	b	c	b

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