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ON THE LATTICE OF CONVEXLY COMPATIBLE TOPOLOGIES ON A PARTIALLY ORDERED SET

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The notion of the convex compatibility and the convex weak compatibility of a topology with an ordering was introduced in [3]. Let $(A, \leqq)$ be a partially ordered set. The system of all topologies on A in the sense of Čech, which are convexly compatible and convexly weakly compatible with the ordering $\leqq$, will be denoted by $\alpha(A, \leqq)$ and $\beta(A, \leqq)$, respectively. If $\alpha(A, \leqq)$, $\beta(A, \leqq)$ are partially ordered in a natural way, both these systems turn to be lattices. In this note some properties of these lattices are investigated. Analogous problems for other systems of topologies on a fixed set are studied in papers [2], [5], [6].

1. PRELIMINARIES

For the sake of completeness let us recall some definitions introduced in [3].

Denote by 2^P the system of all subsets of a set P. We start with the basic definition.

1.1. Definition. *Let P be a given set. A mapping $u: 2^P \to 2^P$ is said to be a topology on P, if the following three axioms are satisfied:*

(1) $u\emptyset = \emptyset$,

(2) $M \subset P \Rightarrow M \subset uM$,

(3) $M_1 \subset M_2 \subset P \Rightarrow uM_1 \subset uM_2$.

If u is a topology on P, the pair (P, u) is called a topological space. The system of all topologies on P is denoted by $\mathscr{T}(P)$.

1.2. Definition. *A set $O \subset P$ is said to be a neighborhood of an element $x \in P$ in the space (P, u), if $x \notin u(P - O)$. The notation $D_u(x)$ is used for the system of all neighborhoods of x in (P, u).*

We shall often use the following statement which enables to introduce a topology into a set P (cf. [1], 4.1).

1.3. Theorem. 1. *Let (P, u) be a topological space, $x \in P$. The system $\mathbf{D}_u(x)$ has the following properties:*

(i) $\mathbf{D}_u(x) \neq \emptyset$,

(ii) $O \in \mathbf{D}_u(x) \Rightarrow x \in O$,

(iii) $O \subset O_1,\ O \in \mathbf{D}_u(x) \Rightarrow O_1 \in \mathbf{D}_u(x)$.

2. *Let P be an arbitrary set and let $\mathbf{D}(x)$ be a nonvoid family of subsets of P, assigned to each element $x \in P$, satisfying:*

(1) $O \in \mathbf{D}(x) \Rightarrow x \in O$,

(2) $O \subset O_1,\ O \in \mathbf{D}(x) \Rightarrow O_1 \in \mathbf{D}(x)$.

If we define a mapping $u : 2^P \to 2^P$ in such a way that $x \in uM$ $(M \subset P)$ iff $P - M \notin \mathbf{D}(x)$, then u is a topology on P and for each $x \in P$ it is $\mathbf{D}_u(x) = \mathbf{D}(x)$.

The following theorem was proved in [1].

1.4. Theorem. *If P is an arbitrary set, then the set $\mathscr{T}(P)$ of all topologies on P is a complete lattice with respect to the relation $\leqq$ defined as follows:*

$$u \leqq v \ \big(u, v \in \mathscr{T}(P)\big) \quad \textit{iff} \quad uM \subset vM \quad \textit{for every } M \subset P.$$

A topology u is an infimum of $\{u_i : i \in I\} \subset \mathscr{T}(P)$ if and only if one of the following two conditions is fulfilled:

(a) $uM = \cap\{u_i M : i \in I\}$ *for every* $M \subset P$,

(b) $\mathbf{D}_u(x) = \cup\{\mathbf{D}_{u_i}(x) : i \in I\}$ *for every* $x \in P$,

and dually for $v = \bigvee\{u_i : i \in I\}$.

The least element of $\mathscr{T}(P)$ is a topology u^0 such that $u^0 M = M$ for every $M \subset P$. The greatest topology u^1 satisfies $u^1(\emptyset) = \emptyset$, $u^1(M) = P$ for every $\emptyset \neq M \subset P$.

The algebraic characterization of the lattice $\mathscr{T}(P)$ is given in [4].

1.5. Theorem. *The lattice $\mathscr{T}(P)$ is isomorphic to a complete ring of sets.*

1.6. Definition. *Let $(A, \leqq)$ be a partially ordered set. A topology u on A will be said to be convexly compatible with the ordering $\leqq$, if it has the following property:*

(α) *If $a, b \in A$ and if U is a neighborhood of a with $b \notin U$, then there exists a convex neighborhood V of a such that $b \notin V$.*

1.7. Definition. *Let $(A, \leqq)$ be a partially ordered set. A topology u on A will be called convexly weakly compatible with the ordering $\leqq$, if it has the following property:*

(β) *If a and b are comparable elements of A and if U is a neighborhood of a with $b \notin U$, then there exists a convex neighborhood V of a such that $b \notin V$.*

Let $(X, \leqq)$ be a partially ordered set. If $a, b \in X$, $a \leqq b$, the interval $\{x \in X : a \leqq \leqq x \leqq b\}$ is denoted by $\langle a, b\rangle$. For the incomparability of $a, b \in X$ we use the notation $a \parallel b$. If M is a subset of X, the symbol $[M]$ is used for the convex hull of M in X. For the cardinality of a set Y we use the notation card Y.

2. THE PARTIAL ORDERING ON THE SETS $\alpha(A, \leqq)$, $\beta(A, \leqq)$

Let $(A, \leqq)$ be a partially ordered set. The set of all topologies on A which are convexly compatible and convexly weakly compatible with the ordering $\leqq$ will be denoted by $\alpha(A, \leqq)$ and $\beta(A, \leqq)$, respectively. Clearly $\alpha(A, \leqq) \subset \beta(A, \leqq)$ and both these sets are subsets of the complete lattice $\mathscr{T}(A)$. A question arises, whether $\alpha(A, \leqq)$, $\beta(A, \leqq)$ are sublattices of $\mathscr{T}(A)$.

2.1. Lemma. *Let $\{u_i : i \in I\}$ be a nonempty subset of the set $\alpha(A, \leqq)$, $u = \bigwedge\{u_i : i \in I\}$ in the complete lattice $\mathscr{T}(A)$. Then $u \in \alpha(A, \leqq)$.*

Proof. Take $a, b \in A$ such that there exists $U \in \boldsymbol{D}_u(a)$ with $b \notin U$. By 1.4 there is $U \in \boldsymbol{D}_{u_i}(a)$ for some $i \in I$. The assumption that $u_i \in \alpha(A, \leqq)$ yields the existence of a convex set $V \in \boldsymbol{D}_{u_i}(a)$ with $b \notin V$. Obviously $V \in \boldsymbol{D}_u(a)$.

2.2. Lemma. *Let $\{u_i : i \in I\}$ be a nonempty subset of the set $\beta(A, \leqq)$, $u = \bigwedge\{u_i : i \in I\}$ in the complete lattice $\mathscr{T}(A)$. Then $u \in \beta(A, \leqq)$.*

The proof is analogous to that of 2.1.

2.3. Theorem. *The set $\beta(A, \leqq)$ is a closed sublatice of the complete lattice $\mathscr{T}(A)$.*

Proof. In view of the foregoing lemma, to prove 2.3, it is sufficient to show that if $\emptyset \neq \{u_i : i \in I\} \subset \beta(A, \leqq)$, $v = \bigvee \{u_i : i \in I\}$ in $\mathscr{T}(A)$, then $v \in \beta(A, \leqq)$. Suppose that a, b are comparable elements of A such that there exists $U \in \boldsymbol{D}_v(a)$ not containing b. By 1.4 it is $U \in \boldsymbol{D}_{u_i}(a)$ for each $i \in I$. Since all u_i are convexly weakly compatible with the ordering $\leqq$, we can find for every $i \in I$ a convex set $V_i \in \boldsymbol{D}_{u_i}(a)$ that does not contain b. Put $V = \cup \{V_i : i \in I\}$. Obviously $V \in \boldsymbol{D}_v(a)$ which implies that the convex hull $[V]$ of V also belongs to $\boldsymbol{D}_v(a)$. Assume $b \in [V]$. Then there exist elements $x \in V_{i_1}$, $y \in V_{i_2}$ such that $x < b < y$. If $a < b$, from the relations $a < b < y$, $a, y \in V_{i_2}$ and from the convexity of V_{i_2} we get $b \in V_{i_2}$, a contradiction. The inequality $a > b$ yields a contradiction analogously. Therefore $b \notin [V]$ and the proof of 2.3 is complete.

It can be shown by examples that the join of two topologies from $\alpha(A, \leqq)$ in $\mathscr{T}(A)$ does not belong to $\alpha(A, \leqq)$ in general. Hence the set $\alpha(A, \leqq)$ need not be a closed sublattice of the complete lattice $\mathscr{T}(A)$. But since the finest topology and the coarsest one on A are convexly compatible with every ordering on A, in view of 2.1 the set $\alpha(A, \leqq)$ is a complete lattice.

By 2.1 the meet of a nonempty subset $\{u_i : i \in I\}$ of the set $\alpha(A, \leqq)$ in the complete lattice $\alpha(A, \leqq)$ is the same as in the complete lattice $\mathscr{T}(A)$ and we shall denote it by $\bigwedge \{u_i : i \in I\}$. The join of the set $\{u_i : i \in I\}$ in $\mathscr{T}(A)$ will be denoted by $\bigvee \{u_i : i \in I\}$ while for the join of this set in $\alpha(A, \leqq)$ there will be used the notation $\bigvee^\alpha \{u_i : i \in I\}$.

We are going to describe $\bigvee^\alpha \{u_i : i \in I\}$ for an arbitrary subset $\{u_i : i \in I\}$ of the set $\alpha(A, \leqq)$.

If $v \in \mathscr{T}(A)$, $a \in A$, we denote by $c_v(a)$ the set $\cap \{[V] : V \in \boldsymbol{D}_v(a)\}$.

2.4. Lemma. *Let* $v \in \mathscr{T}(A)$, $a \in A$. *The system* $\boldsymbol{D}(a) = \{O \in \boldsymbol{D}_v(a) : c_v(a) \subset O\}$ *has the following properties:*

(i) $\boldsymbol{D}(a) \neq \emptyset$,

(ii) $O \in \boldsymbol{D}(a) \Rightarrow a \in O$,

(iii) $O_1 \supset O \in \boldsymbol{D}(a) \Rightarrow O_1 \in \boldsymbol{D}(a)$.

Proof. The assertion (iii) is trivial. Since $A \in \boldsymbol{D}(a)$, it holds (i). The validity of (ii) follows from $\boldsymbol{D}(a) \subset \boldsymbol{D}_v(a)$.

2.5. Theorem. *Let* $v \in \mathscr{T}(A)$ *and let* $\bar{v}$ *be a topology on* A *such that* $\boldsymbol{D}_{\bar{v}}(a) = \{O \in \boldsymbol{D}_v(a) : c_v(a) \subset O\}$ *for every* $a \in A$. *Then*

(1) $\bar{v} \geqq v$,

(2) $\bar{v} \in \alpha(A, \leqq)$,

(3) $u \in \alpha(A, \leqq)$, $u \geqq v$ *implies* $u \geqq \bar{v}$.

Proof. The existence of the topology $\bar{v}$ with the above-mentioned systems of neighborhoods follows from 2.4 and 1.3. It is evident that $\boldsymbol{D}_{\bar{v}}(a) \subset \boldsymbol{D}_v(a)$ for every $a \in A$. Hence (1) holds. To prove (2), suppose that for some $a, b \in A$ there exists a set $U \in \boldsymbol{D}_{\bar{v}}(a)$ not containing b. Since $c_v(a) \subset U$, it must be $b \notin c_v(a)$. Thus $b \notin [O]$ for some $O \in \boldsymbol{D}_v(a)$. Evidently $[O] \in \boldsymbol{D}_v(a)$, $c_v(a) \subset [O]$, hence $[O] \in \boldsymbol{D}_{\bar{v}}(a)$. We have found a convex set $[O] \in \boldsymbol{D}_{\bar{v}}(a)$ not containing b, as desired.

Let the assumptions of (3) hold. It is sufficient to prove that $\boldsymbol{D}_u(a) \subset \boldsymbol{D}_{\bar{v}}(a)$ for each $a \in A$. Let $O \in \boldsymbol{D}_u(a)$. Then evidently $O \in \boldsymbol{D}_v(a)$. Suppose that there exists an element $b \in c_v(a) - O$. Since $O \in \boldsymbol{D}_u(a)$, $b \notin O$, $u \in \alpha(A, \leqq)$, there exists a convex set $U \in \boldsymbol{D}_u(a)$ not containing b. From $b \in c_v(a)$, $U \in \boldsymbol{D}_v(a)$ we obtain $b \in [U] = U$, a contradiction. Therefore $c_v(a) \subset O$ which implies $O \in \boldsymbol{D}_{\bar{v}}(a)$.

2.6. Remark. *Let* v *be a topology on* A. *In what follows the symbol* $\bar{v}$ *will be used for the topology fulfilling* (1)–(3) *of* 2.5.

2.7. Theorem. *Let* $\{u_i : i \in I\}$ *be a nonempty subset of the set* $\alpha(A, \leqq)$ *and let* $v = \bigvee \{u_i : i \in I\}$, $w = \bigvee^\alpha \{u_i : i \in I\}$. *Then* $w = \bar{v}$.

This theorem follows immediately from 2.5.

3. DISTRIBUTIVITY OF THE LATTICE $\alpha(A, \leqq)$

It was proved that the lattice $\beta(A, \leqq)$ is a closed sublattice of the lattice $\mathscr{T}(A)$. Hence by 1.5 the lattice $\beta(A, \leqq)$ is distributive. On the other hand the lattice $\alpha(A, \leqq)$ is not distributive in general. The purpose of this section is to described directed sets $(A, \leqq)$ for which the lattice $\alpha(A, \leqq)$ is distributive.

3.1. Theorem. *If $(A, \leqq)$ is a chain, then the lattice $\alpha(A, \leqq)$ is distributive.*

Proof. It is evident that if $(A, \leqq)$ is a chain, then $\alpha(A, \leqq) = \beta(A, \leqq)$. The lattice $\beta(A, \leqq)$ is by 1.5 distributive.

3.2. Definition. *A partially ordered set $(A, \leqq)$ will be said to have the property* (mnd), *if A has the least element o, the greatest element i and $A - \{o, i\}$ is an antichain.*

In what follows we denote by o and i the least and the greatest element of $(A, \leqq)$, respectively, if such an element exists.

3.3. Lemma. *Let $(A, \leqq)$ be a directed set which is not a chain and has not the property* (mnd). *Then the lattice $\alpha(A, \leqq)$ is not modular.*

Proof. Since $(A, \leqq)$ has not the property (mnd), there exist noncomparable elements $a, b \in A$ such that there are either at least two elements which are less than b or at least two elements which are greater than b. Suppose that the first case occurs. In the second case we should proceed analogously as in the first one. Let $c < a$, $c < b$, $d > a$, $d > b$, $e < b$, $e \neq c$. Without loss of generality we can suppose that $e \nless c$. Define topologies u, v, w as follows:

$$\begin{aligned} \boldsymbol{D}_u(a) &= \{O \subset A : O \supset \langle c, a\rangle \quad \text{or } O \supset [\{a, e\}]\},\\ \boldsymbol{D}_v(a) &= \{O \subset A : O \supset \langle a, d\rangle\},\\ \boldsymbol{D}_w(a) &= \{O \subset A : O \supset \langle c, a\rangle\},\\ \boldsymbol{D}_u(z) = \boldsymbol{D}_v(z) &= \boldsymbol{D}_w(z) = \{A\} \quad \text{for every } z \in A,\ z \neq a. \end{aligned}$$

Evidently $u, v, w \in \alpha(A, \leqq)$, $u < w$. We shall prove $\boldsymbol{D}_{(u \vee^\alpha v) \wedge w}(a) \neq \boldsymbol{D}_{u \vee^\alpha (v \wedge w)}(a)$ by showing that $[\{a, e\}] \cup \langle a, d\rangle \in \boldsymbol{D}_{u \vee^\alpha (v \wedge w)}(a) - \boldsymbol{D}_{(u \vee^\alpha v) \wedge w}(a)$. It is $\boldsymbol{D}_{u \vee^\alpha (v \wedge w)}(a) = \{O \in \boldsymbol{D}_{u \vee (v \wedge w)}(a) : O \supset c_{u \vee (v \wedge w)}(a)\}$. It is clear that $\boldsymbol{D}_{u \vee (v \wedge w)}(a) = \boldsymbol{D}_u(a) \cap (\boldsymbol{D}_v(a) \cup \boldsymbol{D}_w(a)) = \{O \subset A : O \supset \langle c, a\rangle \text{ or } O \supset [\{a, e\}] \cup \langle a, d\rangle\}$, thus $[\{a, e\}] \cup \langle a, d\rangle$ belongs to $\boldsymbol{D}_{u \vee (v \wedge w)}(a)$. Further we have to prove that $[\{a, e\}] \cup \langle a, d\rangle$ contains $c_{u \vee (v \wedge w)}(a) = \langle c, a\rangle \cap [[\{a, e\}] \cup \langle a, d\rangle]$. Let $c \leqq s \leqq a$, $x \leqq s \leqq y$, where $x, y \in [\{a, e\}] \cup \langle a, d\rangle$. Distinguish two cases:

1) $x \in \langle a, d\rangle$; Then $a \leqq x \leqq s \leqq a$, from where we get $s = a \in [\{a, e\}] \cup \langle a, d\rangle$.

2) $x \in [\{a, e\}]$; If $x \geqq a$, we proceed as in 1). If $x \ngeqq a$, we have $x \geqq e$, which implies $e \leqq x \leqq s \leqq a$. Hence $s \in [\{a; e\}] \subset [\{a, e\}] \cup \langle a, d\rangle$. Consequently $[\{a, e\}] \cup \langle a, d\rangle \in \boldsymbol{D}_{u \vee^\alpha (v \wedge w)}(a)$.

It remains to show that $[\{a, e\}] \cup \langle a, d\rangle \notin \boldsymbol{D}_{(u \vee^\alpha v) \vee w}(a)$. It is $\boldsymbol{D}_{u \vee v}(a) = \boldsymbol{D}_u(a) \cap$

$\cap D_v(a) = \{O \subset A : O \supset \langle c, a\rangle \cup \langle a, d\rangle$ or $O \supset [\{a, e\}] \cup \langle a, d\rangle\}$, $D_{u \vee^\alpha v}(a) = \{O \in D_{u \vee v}(a) : O \supset [\langle c, a\rangle \cup \langle a, d\rangle] \cap [[\{a, e\}] \cup \langle a, d\rangle]\}$, $D_{(u \vee^\alpha v) \wedge w}(a) = D_{u \vee^\alpha v}(a) \cup D_w(a)$. Obviously $[\{a, e\}] \cup \langle a, d\rangle \notin D_{u \vee^\alpha v}(a)$, since $b \notin [\{a, e\}] \cup \langle a, d\rangle$, $b \in [\langle c, a\rangle \cup \langle a, d\rangle] \cap [[\{a, e\}] \cup \langle a, d\rangle]$. Finally, $[\{a, e\}] \cup \langle a, d\rangle \notin D_w(a)$, as $c \notin [\{a, e\}] \cup \langle a, d\rangle$.

3.4. Theorem. *Let* $(A, \leqq)$ *be a partially ordered set with the property* (mnd) *containing at least* 5 *elements. Then the lattice* $\alpha(A, \leqq)$ *is not modular.*

Proof. Take arbitrary various elements $a, b, c \in A - \{o, i\}$. Consider topologies u, v, w on A such that

$$D_u(a) = \{O \subset A : \{a, o\} \subset O\},$$
$$D_v(a) = \{O \subset A : \{a, i\} \subset O\},$$
$$D_w(a) = \{O \subset A : \{a, c, o\} \subset O \text{ or } \{a, b, o\} \subset O\},$$
$$D_u(z) = D_v(z) = D_w(z) = \{A\} \text{ for every } z \in A,\ z \neq a.$$

Evidently $u, v, w \in \alpha(A, \leqq)$ and $u < w$. We shall prove $u \vee^\alpha (v \wedge w) \neq (u \vee^\alpha v) \wedge w$ by showing that $\{a, o, i\} \in D_{u \vee^\alpha (v \wedge w)}(a) - D_{(u \vee^\alpha v) \wedge w}(a)$. It is $D_{u \vee (v \wedge w)}(a) = D_u(a) \cap (D_v(a) \cup D_w(a)) = \{O \subset A : \{a, o, i\} \subset O$ or $\{a, c, o\} \subset O$ or $\{a, b, o\} \subset O\}$, $D_{u \vee^\alpha (v \wedge w)}(a) = \{O \in D_{u \vee (v \wedge w)}(a) : O \supset [\{a, o, i\}] \cap [\{a, c, o\}] \cap [\{a, b, o\}]\} = D_{u \vee (v \wedge w)}(a)$. Therefore $\{a, o, i\} \in D_{u \vee^\alpha (v \wedge w)}(a)$. It is easy to show that $D_{u \vee^\alpha v}(a) = \{A\}$, hence $D_{(u \vee^\alpha v) \wedge w}(a) = D_w(a)$. Thus $\{a, o, i\} \notin D_{(u \vee^\alpha v) \wedge w}(a)$, completing the proof.

3.5. Theorem. *Let* $(A, \leqq)$ *be the Boolean algebra containing four elements. Then the lattice* $\alpha(A, \leqq)$ *is distributive.*

Proof. Let $A = \{o, i, a, b\}$. It is sufficient to prove that for every $x \in A$ and topologies $u, v, w \in \alpha(A, \leqq)$ it is $D_{(u \wedge v) \vee^\alpha (u \wedge w)}(x) \subset D_{u \wedge (v \vee^\alpha w)}(x)$. Pick an element $x \in A$ and suppose that $O \in D_{(u \wedge v) \vee^\alpha (u \wedge w)}(x)$, i.e. $O \in D_{(u \wedge v) \vee (u \wedge w)}(x)$, $O \supset c_{(u \wedge v) \vee (u \wedge w)}(x)$. It holds $D_{(u \wedge v) \vee (u \wedge w)}(x) = (D_u(x) \cup D_v(x)) \cap (D_u(x) \cup D_w(x)) = D_u(x) \cup (D_v(x) \cap D_w(x))$ and this implies that either $O \in D_u(x)$ or $O \in D_v(x) \cap D_w(x) = D_{v \vee w}(x)$. If the first possibility occurs, then evidently $O \in D_{u \wedge (v \vee^\alpha w)}(x)$. Assume $O \notin D_u(x)$. Then $O \in D_{v \vee w}(x)$ and it remains to show that $O \supset c_{v \vee w}(x)$. If O is convex, it is nothing to prove. Suppose that O is not convex. Then $O = \{o, i, a\}$ or $O = \{o, i, b\}$ or $O = \{o, i\}$.

Analyse the first possibility. In the second case we should proceed analogously. We need eliminate the relation $b \in c_{v \vee w}(x)$. Assume $b \in c_{v \vee w}(x)$. It is easy to show that $c_{(u \wedge v) \vee (u \wedge w)}(x) = c_u(x) \cap c_{v \vee w}(x)$. Using the assumption $O \supset c_{(u \wedge v) \vee (u \wedge w)}(x)$ we obtain $b \notin c_u(x)$. Thus b does not belong to some convex set $V \in D_u(x)$. Then $V \subset O$, which implies $O \in D_u(x)$, a contradiction.

Finally let $O = \{o, i\}$. Without loss of generality we can suppose that $x = o$. As $\{o, i\} \in D_{v \vee w}(o) = D_v(o) \cap D_w(o)$ and v, w are convexly compatible with the ordering $\leqq$ on A, it is $\{o\} \in D_v(o)$ or $\{o\} \notin D_v(o)$ but $\{o, i\}$, $\{o, a\}$, $\{o, b\} \in D_v(o)$ and analogously for w. From $O \notin D_u(o)$ we obtain $\{o\} \notin D_u(o)$, hence $D_u(o) \subset D_{v \vee w}(o)$. We conclude that $c_{v \vee w}(o) = c_{(u \wedge v) \vee (u \wedge w)}(o) \subset O$, completing the proof.

From 3.3, 3.4, 3.5 we have immediately:

3.6. Theorem. *Let $(A, \leqq)$ be a directed set, which is not a chain. The lattice $\alpha(A, \leqq)$ is distributive if and only if $(A, \leqq)$ is the Boolean algebra with four elements. If A contains more than four elements, the lattice $\alpha(A, \leqq)$ is not even modular.*

4. RELATIVE COMPLEMENTS IN THE LATTICES $\mathscr{T}(P)$, $\alpha(A, \leqq)$, $\beta(A, \leqq)$

Let v, u, w be topologies of the lattice $\mathscr{T}(P)$ and $\alpha(A, \leqq)$ and $\beta(A, \leqq)$, respectively, such that $v \leqq u \leqq w$. In the following there are investigated conditions under which the topology u has a relative complement in the interval $\langle v, w\rangle$ of $\mathscr{T}(P)$ and $\alpha(A, \leqq)$ and $\beta(A, \leqq)$, respectively.

4.1. Theorem. *Let v, u, w be topologies on a set P with $v \leqq u \leqq w$. Then u has a relative complement in the interval $\langle v, w\rangle$ of the lattice $\mathscr{T}(P)$ if and only if the following condition is satisfied:*

(r) *If $x \in P$ and $O \in D_u(x) - D_w(x)$, then for every subset U of O containing x either $U \in D_u(x)$ or $U \notin D_v(x)$ holds.*

Proof. Let the condition (r) be satisfied. Set $D(x) = D_w(x) \cup (D_v(x) - D_u(x))$ for every $x \in P$. Evidently $D(x) \neq \emptyset$ and each set from $D(x)$ contains x. Suppose $O_1 \supset \supset O \in D(x)$. We shall show that $O_1 \in D(x)$. If $O_1 \in D_w(x)$, it is nothing to prove. Assume that $O_1 \notin D_w(x)$. Then $O \notin D_w(x)$ and it follows that $O \in D_v(x) - D_u(x)$. The last relation implies $O_1 \in D_v(x)$. Further $O_1 \notin D_u(x)$, for otherwise $O \in D_u(x)$ or $O \notin D_v(x)$ by (r), which is a contradiction. In view of 1.3 there exists a topology u' on P such that $D_{u'}(x) = D(x)$ for every $x \in P$. It is easy to verify that u' is a complement of u in the interval $\langle v, w\rangle$.

To prove the converse, assume that there exists a topology u' on P such that $u \wedge u' = v$, $u \vee u' = w$. Further let $U \subset O \in D_u(x) - D_w(x)$, $x \in U$ for some $x \in P$. From $D_w(x) = D_u(x) \cap D_{u'}(x)$ we obtain $O \notin D_{u'}(x)$. Now if $U \in D_v(x) = D_u(x) \cup \cup D_{u'}(x)$, then $U \in D_u(x)$, as desired.

4.2. Remark. *Since in view of* 1.5 *the lattice $\mathscr{T}(P)$ is distributive, the topology u has in the interval $\langle v, w\rangle$ ($u, v, w \in \mathscr{T}(P)$, $v \leqq u \leqq w$) at most one relative complement.*

4.3. Corollary. *A topology $u \in \mathscr{T}(P)$ has a complement in the lattice $\mathscr{T}(P)$ if and only if for each $x \in P$ either $D_u(x) = \{P\}$ or $D_u(x) = \{O \subset P : x \in O\}$ holds.*

4.4. Corollary. *Complemented elements of the lattice $\mathcal{T}(P)$ form a complete Boolean algebra.*

Proof. By 1.5 the lattice $\mathcal{T}(P)$ is completely distributive, hence also infinitely distributive. Complemented elements of an arbitrary infinitely distributive complete lattice form a closed sublattice.

4.5. Lemma. *Let $(A, \leqq)$ be a partially ordered set and let $v, w \in \alpha(A, \leqq)$, $v \leqq w$. If for topologies $u, u' \in \mathcal{T}(A)$ the equalities $u \wedge u' = v$, $u \vee u' = w$ hold, then $u, u' \in \alpha(A, \leqq)$.*

Proof. We prove that u is convexly compatible with the ordering $\leqq$. Take $a, b \in A$ such that there exists $O \in \boldsymbol{D}_u(a)$ not containing b. Then $A - \{b\} \in \boldsymbol{D}_u(a)$. Since by 4.1 and 4.2 it is $\boldsymbol{D}_u(a) = \boldsymbol{D}_w(a) \cup (\boldsymbol{D}_v(a) - \boldsymbol{D}_{u'}(a))$, we have $A - \{b\} \in \boldsymbol{D}_w(a)$ or $A - \{b\} \in \boldsymbol{D}_v(a) - \boldsymbol{D}_{u'}(a)$. In the first case there exists a convex set $U \in \boldsymbol{D}_w(a) \subset \boldsymbol{D}_u(a)$ not containing b. If $A - \{b\} \in \boldsymbol{D}_v(a) - \boldsymbol{D}_{u'}(a)$, then $b \in V$ for every $V \in \boldsymbol{D}_{u'}(a)$. Since $v \in \alpha(A, \leqq)$, there exists a convex set $U_1 \in \boldsymbol{D}_v(a)$ not containing b. But then $U_1 \notin \boldsymbol{D}_{u'}(a)$ and as $\boldsymbol{D}_v(a) = \boldsymbol{D}_u(a) \cup \boldsymbol{D}_{u'}(a)$, we get $U_1 \in \boldsymbol{D}_u(a)$. Therefore $u \in \alpha(A, \leqq)$. Analogously it can be shown that $u' \in \alpha(A, \leqq)$.

4.6. Lemma. *Let $v, w \in \beta(A, \leqq)$, $v \leqq w$. If for topologies $u, u' \in \mathcal{T}(A)$ the equalities $u \wedge u' = v$, $u \vee u' = w$ hold, then $u, u' \in \beta(A, \leqq)$.*

The proof of this lemma is analogous to that of 4.5.

The following theorem is a direct consequence of 2.3 and 4.6.

4.7. Theorem. *Let $(A, \leqq)$ be a partially ordered set and let $v, u, w \in \beta(A, \leqq)$, $v \leqq u \leqq w$. A topology u' is a relative complement of u in the interval $\langle v, w \rangle$ of the lattice $\beta(A, \leqq)$ if and only if the same holds in the lattice $\mathcal{T}(A)$.*

Using 2.7 we obtain the following theorem.

4.8. Theorem. *Let $(A, \leqq)$ be a partially ordered set and let $v, u, w \in \alpha(A, \leqq)$, $v \leqq u \leqq w$. A topology $u' \in \alpha(A, \leqq)$ is a relative complement of u in the interval $\langle v, w \rangle$ of the lattice $\alpha(A, \leqq)$ if and only if u' is a relative complement of u in the interval $\langle v, t \rangle$ of $\mathcal{T}(A)$ for some $t \in \mathcal{T}(A)$ with $u \leqq t$, $\bar{t} = w$.*

5. THE CONSTRUCTION OF THE SET $\{v \in \mathcal{T}(A) : \bar{v} = u\}$ FOR A GIVEN TOPOLOGY $u \in \alpha(A, \leqq)$

In connection with searching for relative complements to a topology of the lattice $\alpha(A, \leqq)$ in a fixed interval of $\alpha(A, \leqq)$, a question arises, in which way we can construct all the topologies $v \in \mathcal{T}(A)$ with the property $\bar{v} = u$, for a given $u \in \alpha(A, \leqq)$.

If $u \in \mathscr{T}(A)$, $a \in A$, we denote by $s_u(a)$ the set $\cap \{O : O \in \boldsymbol{D}_u(a)\}$.

5.1. Lemma. *Let $(A, \leqq)$ be a partially ordered set and let u be a topology on A convexly compatible with the ordering $\leqq$. Take $a \in A$ and an arbitrary fixed system $S'(a)$ of sets $O_i - B_i$, indexed by I, such that $O_i \in \boldsymbol{D}_u(a)$, $\emptyset \neq B_i \subset s_u(a)$, $a \notin B_i$. Let $S(a) = \{O - B : O \supset O_i,\ \emptyset \neq B \subset B_i$ for some $i \in I\}$. Then the system $\boldsymbol{D}(a) = \boldsymbol{D}_u(a) \cup S(a)$ has the following properties:*

(i) $\boldsymbol{D}(a) \neq \emptyset$,

(ii) $U \in \boldsymbol{D}(a) \Rightarrow a \in U$,

(iii) $U_1 \supset U \in \boldsymbol{D}(a) \Rightarrow U_1 \in \boldsymbol{D}(a)$.

Proof. The assertions (i), (ii) can be easy verified. Let $U_1 \supset U \in \boldsymbol{D}(a)$. If $U_1 \in \boldsymbol{D}_u(a)$, then $U_1 \in \boldsymbol{D}(a)$. Hence we can suppose that $U_1 \notin \boldsymbol{D}_u(a)$. Then also $U \notin \boldsymbol{D}_u(a)$, which implies $U \in S(a)$. Consequently, $U = O - B$, where $O \supset O_i$, $\emptyset \neq B \subset B_i$ for some $i \in I$. Now $U_1 \not\supset s_u(a)$, for otherwise $U_1 \supset O$, contrary to $U_1 \notin \boldsymbol{D}_u(a)$. Hence $s_u(a) - U_1 \neq \emptyset$ and obviously $U_1 = (U_1 \cup s_u(a)) - (s_u(a) - U_1)$. Since $U_1 \cup s_u(a) \supset O_i$ and $\emptyset \neq s_u(a) - U_1 \subset s_u(a) - U = B \subset B_i$, it is $U_1 \in S(a)$.

5.2. Theorem. *Let $u \in \alpha(A, \leqq)$ and let for every $a \in A$ $\boldsymbol{D}(a)$ be the system defined in the foregoing lemma derived from a system $S'(a)$ fulfilling in addition to the assumptions of* 5.1 *also the condition:*

(t) *If $b \in B_i$, then there exist elements $o_1, o_2 \in O_i - B_i$ with $o_1 < b < o_2$.*
Let v be a topology on A such that $\boldsymbol{D}_v(a) = \boldsymbol{D}(a)$ for every $a \in A$. Then $\bar{v} = u$.

Proof. The existence of a topology v on A with $\boldsymbol{D}_v(a) = \boldsymbol{D}(a)$ for every $a \in A$ follows from 1.3 and 5.1. To prove the equality $\bar{v} = u$, by 2.5, it suffices to show that $U \in \boldsymbol{D}_u(a)$ iff $U \in \boldsymbol{D}_v(a)$ and $c_v(a) \subset U$. Hence let $U \in \boldsymbol{D}_u(a)$. Then obviously $U \in \boldsymbol{D}(a) = \boldsymbol{D}_v(a)$. Suppose that there exists $b \in c_v(a) - U$. Since the topology u is convexly compatible with the ordering $\leqq$, there exists a convex set $W \in \boldsymbol{D}_u(a)$ with $b \notin W$. As $W \in \boldsymbol{D}_u(a) \subset \boldsymbol{D}_v(a)$, it is $b \in [W] = W$, a contradiction.

Conversely, let $U \in \boldsymbol{D}_v(a)$, $c_v(a) \subset U$. Suppose that $U \notin \boldsymbol{D}_u(a)$. Then $U = O - B$, where $O \supset O_i$, $\emptyset \neq B \subset B_i$ for some $i \in I$. Clearly, $s_u(a) \not\subset U$. We prove that $s_u(a) \subset [W]$ for every $W \in \boldsymbol{D}_v(a)$. If $W \in \boldsymbol{D}_u(a)$, it is $s_u(a) \subset W \subset [W]$. Let $W \in S(a)$, $W = O' - B'$, $O' \supset O_j$, $\emptyset \neq B' \subset B_j$ for some $j \in I$. Then for every $b \in s_u(a) - W$ it is $b \in B'$, which implies, by (t), $b \in [W]$. Therefore $s_u(a) \subset c_v(a)$. As $c_v(a) \subset U$, we get $s_u(a) \subset U$, a contradiction.

5.3. Corollary. *Let $u \in \alpha(A, \leqq)$. Then $u = \bar{v}$ for some $v \in \mathscr{T}(A) - \alpha(A, \leqq)$ if and only if there exist elements $a \in A$ and $b \in s_u(a)$, $b \neq a$, such that b is neither maximal nor minimal element of A.*

Proof. Suppose that $u = \bar{v}$ for some $v \in \mathscr{T}(A) - \alpha(A, \leqq)$. Then $v < u$, so that there exists $a \in A$ with $\boldsymbol{D}_u(a) \subsetneqq \boldsymbol{D}_v(a)$. Let $O \in \boldsymbol{D}_v(a) - \boldsymbol{D}_u(a)$. Then $c_v(a) \not\subset O$. Take an arbitrary element $b \in c_v(a) - O$. Using 2.5, it is not hard to see that $c_v(a) = s_u(a)$.

Hence $b \in s_u(a)$ and obviously $b \neq a$. As $b \in c_v(a) \subset [O]$ and $b \notin O$, b is neither maximal nor minimal.

Conversely, suppose that for some $a \in A$ there exists $b \in s_u(a) - \{a\}$ such that b is neither maximal nor minimal of A. Keeping notations as in 5.1, put $S'(a) = \{A - \{b\}\}$, $S'(x) = \emptyset$ for $x \neq a$. Then $D(a) = D_u(a) \cup \{A - \{b\}\}$, $D(x) = D_u(x)$ for $x \neq a$. Let v be a topology on A such that $D_v(z) = D(z)$ for every $z \in A$. By 5.2, $\bar{v} = u$ and obviously $v < u$.

5.4. Theorem. *Let $u \in \alpha(A, \leqq)$. The construction described in 5.2 gives all topologies $v \in \mathscr{T}(A)$ with $\bar{v} = u$.*

Proof. Let v be a topology on A with $\bar{v} = u$. First we show that if $U \in D_v(a) - D_{\bar{v}}(a)$, then U can be expressed in the form $O - B$, where $O \in D_{\bar{v}}(a)$, $\emptyset \neq B \subset s_{\bar{v}}(a)$, $a \notin B$ and for every $b \in B$ there exist elements $x, y \in U$ with $x < b < y$. Denote $O = U \cup s_{\bar{v}}(a)$, $B = s_{\bar{v}}(a) - U$. Trivially, $U = O - B$. Since $O \supset U \in D_v(a)$, $s_{\bar{v}}(a) \subset O$, using 2.5, we get $O \in D_{\bar{v}}(a)$. Obviously $B \subset s_{\bar{v}}(a) - \{a\}$. Further B is nonempty, for otherwise $s_{\bar{v}}(a) \subset U$, which implies, using $U \in D_v(a)$ and 2.5, $U \in D_{\bar{v}}(a)$, a contradiction. If we take an arbitrary element $b \in B$, then $b \in s_{\bar{v}}(a) = c_v(a) \subseteq [U]$, $b \notin U$ which implies the existence of elements $x, y \in U$ with $x < b < y$.

It remains to show that if $O' \supset O$, $\emptyset \neq B' \subset B$ (O, B have the same meaning as above), then $O' - B' \in D_v(a) - D_{\bar{v}}(a)$. Since $O' - B' \supset O - B$, it is $O' - B' \in D_v(a)$. Suppose $O' - B' \in D_{\bar{v}}(a)$. Then $s_{\bar{v}}(a) = c_v(a) \subset O' - B'$, a contradiction.

6. ATOMS, DUAL ATOMS OF THE LATTICES $\alpha(A, \leqq)$, $\beta(A, \leqq)$

In this section the atoms and the dual atoms of the lattices $\alpha(A, \leqq)$ and $\beta(A, \leqq)$ are described and the conditions on a partially ordered set $(A, \leqq)$ are investigated, under which these lattices are weakly atomic, atomic, weakly dually atomic, dually atomic, in the sense of the definitions given below. Throughout this section we suppose **card** $A \geqq 2$.

6.1. Definition. *A partially ordered set $(X, \leqq)$ with the least element o is said to be weakly atomic, if for every $x \in X$, $x \neq o$ there exists an atom $a \leqq x$.*

The weakly dually atomic partially ordered set is defined dually.

6.2. Definition. *The lattice L with the least element o is said to be atomic, if every element $x \in L$, $x \neq o$ is a join of a nonempty set of atoms of L.*

The dually atomic lattice is defined dually.

6.3. Lemma. *Let a topology v be an atom of the lattice $\alpha(A, \leqq)$ or $\beta(A, \leqq)$. Then there exists $a \in A$ such that $\{a\} \notin D_v(a)$ and for every $x \in A$ different from a it is $\{x\} \in D_v(x)$.*

Proof. If v is an atom, then v is not the least topology, hence there exists $a \in A$ with $\{a\} \notin D_v(a)$. Suppose that $\{a_1\} \notin D_v(a_1)$ and $\{a_2\} \notin D_v(a_2)$ for some $a_1, a_2 \in A$, $a_1 \neq a_2$. Consider a topology u defined as follows:

$$D_u(a_1) = D_v(a_1),$$
$$D_u(z) = \{O \subset A : z \in O\} \qquad \text{for every } z \in A,\ z \neq a_1.$$

If v is convexly compatible with the ordering $\leqq$, so is u. If $v \in \beta(A, \leqq)$, it is also $u \in \beta(A, \leqq)$. Obviously $u < v$, u is not the least topology, a contradiction.

Consider the following conditions for an element a of a partially ordered set $(A, \leqq)$:

(1) *a is neither the least nor the greatest element of A;*

(2) *a is the greatest element of A but there does not exist a dual atom of A comparable with every element of A;*

(2′) *the dual of* (2);

(3) *a is the greatest element of A and there exists a dual atom b of A comparable with every element of A;*

(3′) *the dual of* (3).

Evidently each element of A fulfils just one of these conditions.

6.4. Theorem. *Let $\alpha_0(A, \leqq)$ and $\beta_0(A, \leqq)$ be the set of all atoms of the lattice $\alpha(A, \leqq)$ and $\beta(A, \leqq)$, respectively. Then $\alpha_0(A, \leqq) = \beta_0(A, \leqq) = \{v(a) : a \in A\}$, where $v(a)$ is a topology described as follows:*

If a fulfils one of the conditions (1), (2), (2′), *then $D_{v(a)}(a) = \{O \subset A : a \in O,$ **card** $O \geqq 2\}$, $D_{v(a)}(z) = \{O \subset A : z \in O\}$ for each $z \in A$, $z \neq a$.*

If a fulfils (3) *or* (3′), *then $D_{v(a)}(a) = \{O \subset A : \{a, b\} \subset O\}$, $D_{v(a)}(z) = \{O \subset A : z \in O\}$ for each $z \in A$, $z \neq a$.*

Hence the number of atoms of the lattices $\alpha(A, \leqq)$ and $\beta(A, \leqq)$ is **card** *A.*

Proof. Let a be an arbitrary fixed element of A. First we prove that the topology $v(a)$ is convexly compatible with the ordering $\leqq$. Let $U \in D_{v(a)}(x)$, $y \notin U$. If $x \neq a$, then $\{x\}$ is a convex neighborhood of x not containing y. Hence we can suppose that $x = a$. Assume that a fulfils (1). Then there exist $x_1, x_2 \in A$ with $a \ngeqq x_1$, $a \nleqq x_2$. We have three possibilities: (i) $a < y$, (ii) $a > y$, (iii) a, y are noncomparable. In the first and second case $[\{a, x_2\}]$ and $[\{a, x_1\}]$, respectively, is a convex neighborhood of a not containing y. If (iii) occurs, pick an arbitrary $c \in U$, $c \neq a$. The set $[\{a, c\}]$ is a convex neighborhood of a that does not contain y. Further assume that a fulfils (2). Then there exists $c \in A$, $c \neq a$ with $c \nleqq y$. It is $[\{a, c\}] \in D_{v(a)}(a)$, $y \notin [\{a, c\}]$. If a fulfils (2′), we use the dual consideration. Finally, if a fulfils (3) or (3′), then $\{a, b\}$ is a convex neighborhood of a not containing y.

Evidently the topology $v(a)$ is not the least one. If a fulfils one of the conditions (1), (2), (2′), the topology $v(a)$ is an atom of the lattice $\mathcal{T}(A)$, hence $v(a)$ is an atom of the lattices $\alpha(A, \leqq)$, $\beta(A, \leqq)$ as well. Assume that a fulfils (3) or (3'). Let $v < v(a)$ for some $v \in \beta(A, \leqq)$. We need to show that v is the least topology. The inequality $v < v(a)$ implies $\boldsymbol{D}_{v(a)}(a) \subsetneqq \boldsymbol{D}_v(a)$. Hence there exists $U \in \boldsymbol{D}_v(a)$ not containing b. As a, b are comparable and the topology v is convexly weakly compatible with the ordering $\leqq$, there exists a convex set $V \in \boldsymbol{D}_v(a)$ that does not contain b. It must be $V = \{a\}$.

We complete the proof of 6.4 by showing that if w is an arbitrary topology with the property (β), different from the least one, then there exists $a \in A$ such that $v(a) \leqq \leqq w$. If the topology w is not the least one, there exists $a \in A$ with $\{a\} \notin \boldsymbol{D}_w(a)$. Obviously $\boldsymbol{D}_w(x) \subset \boldsymbol{D}_{v(a)}(x)$ for every $x \in A$, $x \neq a$. It is easy to see that if a fulfils (1), (2) or (2'), then $\boldsymbol{D}_w(a) \subset \boldsymbol{D}_{v(a)}(a)$. If a fulfils (3) or (3') and $O \in \boldsymbol{D}_w(a)$, it must be $b \in O$. Suppose this is not the case. Then there exists a convex set $V \in \boldsymbol{D}_w(a)$ with $b \notin V$, hence $V = \{a\} \in \boldsymbol{D}_w(a)$, a contradiction. The proof of 6.4 is complete.

During the proof of 6.4 we also proved the following theorem.

6.5. Theorem. *The lattices $\alpha(A, \leqq)$, $\beta(A, \leqq)$ are weakly atomic.*

Now we will be concerned with the atomicity of the lattices $\alpha(A, \leqq)$, $\beta(A, \leqq)$.

6.6. Lemma. *Let a topology $w \in \mathcal{T}(A)$ be a join of a nonempty set of atoms of the lattices $\alpha(A, \leqq)$, $\beta(A, \leqq)$ in the lattice $\mathcal{T}(A)$. Then w is convexly compatible with the ordering $\leqq$ and it can be described as follows: There exists a nonempty subset A_1 of A such that for every $a \in A$ it holds:*

$$\boldsymbol{D}_w(a) = \begin{cases} \{O \subset A : a \in O\} \text{ if } a \notin A_1; \\ \{O \subset A : a \in O, \operatorname{card} O \geqq 2\} \text{ if } a \in A_1 \text{ and } a \text{ fulfils one of the conditions} \\ \qquad (1), (2), (2'); \\ \{O \subset A : \{a, b\} \subset O\} \text{ if } a \in A_1 \text{ and } a \text{ fulfils (3) or (3').} \end{cases}$$

This statement is an immediate consequence of 1.4 and 6.4.

6.7. Theorem. *The lattices $\alpha(A, \leqq)$, $\beta(A, \leqq)$ are atomic if and only if* card $A = 2$.

Proof. The sufficiency is clear. To prove the necessity, consider the greatest topology u^1. It is $u^1 = \vee \{v(a) : a \in A\}$, hence $\{A\} = \cap \{\boldsymbol{D}_{v(a)}(x) : a \in A\} = \boldsymbol{D}_{v(x)}(x)$ for every $x \in A$. The system $\boldsymbol{D}_{v(x)}(x)$ contains a two-element set, hence it must be **card** $A = 2$.

The results of the remaining part of this paper deal with the questions of the dual atomicity of the lattices $\alpha(A, \leqq)$, $\beta(A, \leqq)$.

The proof of the following lemma is analogous to that of 6.3.

6.8. Lemma. *Let a topology v be a dual atom of the lattice $\alpha(A, \leqq)$ or $\beta(A, \leqq)$. Then there exists $a \in A$ such that $\boldsymbol{D}_v(a) \neq \{A\}$ and for every $x \in A$, $x \neq a$ it is $\boldsymbol{D}_v(x) = = \{A\}$.*

Denote by A^0 and A^1 the set of all minimal and maximal elements of the partially ordered set $(A, \leqq)$, respectively.

Let a, b be arbitrary fixed elements of A, $a \neq b$. Denote by $v(a, b)$ a topology on A defined as follows:

$$D_{v(a,b)}(a) = \{A - \{b\}, A\},$$
$$D_{v(a,b)}(z) = \{A\} \quad \text{for every } z \in A,\ z \neq a.$$

The following statement holds true.

6.9. Theorem. *Let $\alpha_1(A, \leqq)$ and $\beta_1(A, \leqq)$ be the set of all dual atoms of the lattice $\alpha(A, \leqq)$ and $\beta(A, \leqq)$, respectively. Then $\alpha_1(A, \leqq) = \{v(a, b) : a \in A, b \in A^0 \cup A^1\}$, $\beta_1(A, \leqq) = \{v(a, b) : a, b \in A, a \parallel b \text{ or } b \in A^0 \cup A^1\}$.*

Proof. If $a, b \in A$, $a \neq b$, then the topology $v(a, b)$ is obviously a dual atom of the lattice $\mathscr{T}(A)$. It is easy to see that if $b \in A^0 \cup A^1$, then $v(a, b) \in \alpha(A, \leqq) \subset \beta(A, \leqq)$ and if $a \parallel b$, then $v(a, b) \in \beta(A, \leqq)$.

Now let $w \in \alpha_1(A, \leqq)$. We will prove that $w = v(a, b)$ for some $a, b \in A$, $a \neq b$, $b \in A^0 \cup A^1$. By 6.8, there exists $a \in A$ such that $D_w(a) \neq \{A\}$, $D_w(z) = \{A\}$ for every $z \in A$, $z \neq a$. Since $D_w(a) \neq \{A\}$, there exists $b \in A$, $b \neq a$ with $A - \{b\} \in D_w(a)$. If $b \in A^0 \cup A^1$, then trivially $w = v(a, b)$. Suppose $b \notin A^0 \cup A^1$. Then $A - \{b\}$ is not a convex set and $w \in \alpha(A, \leqq)$ implies the existence of a convex set $W \in D_w(a)$ not containing b. Then either $W \subset \{x \in A : x \ngeqq b\}$ or $W \subset \{x \in A : x \nleqq b\}$. Analyse, e.g., the first possibility. As b is not a maximal element, there exists $c \in A$, $c > b$. Define $D_{w_1}(a) = \{O \subset A : O \supset \{x \in A : x \ngeqq c\}\}$, $D_{w_1}(z) = \{A\}$ for each $z \in A$, $z \neq a$. Evidently w_1 is a topology which is different from the greatest one and convexly compatible with the ordering $\leqq$. Since $\{x \in A : x \ngeqq c\} \in D_w(a)$ and $W \notin D_{w_1}(a)$, we have $w < w_1$, a contradiction.

Finally, let $w \in \beta_1(A, \leqq)$. Then there exist elements $a, b \in A$, $a \neq b$ such that $A - \{b\} \in D_w(a)$, $D_w(z) = \{A\}$ for every $z \in A$, $z \neq a$. If a, b are noncomparable or $b \in A^0 \cup A^1$, it is nothing to prove. Suppose that $b \notin A^0 \cup A^1$ and a, b are comparable elements. Then $A - \{b\}$ is not a convex set and using the assumption $w \in \beta(A, \leqq)$, we infer a contradiction analogously as above.

The following theorem shows that the lattices $\alpha(A, \leqq)$, $\beta(A, \leqq)$ are not weakly dually atomic, in general.

6.10. Theorem. *The following conditions are equivalent:*

(1) *The lattice $\alpha(A, \leqq)$ is weakly dually atomic.*
(2) *The lattice $\beta(A, \leqq)$ is weakly dually atomic.*
(3) *For every $b \in A$ there exist elements $c \in A^1$, $d \in A^0$ with $d \leqq b \leqq c$.*

Proof. Let the condition (1) be fulfilled. We prove that (2) holds. Take a topology $w \in \beta(A, \leqq)$ different from the greatest one. Then there exist elements $a, b \in A$ with

$A - \{b\} \in D_w(a)$. If a, b are noncomparable, then $v(a, b) \in \beta_1(A, \leqq)$ and obviously $w \leqq v(a, b)$. Therefore suppose that as soon as $A - \{b\} \in D_w(a)$ for some $a, b \in A$, the elements a, b are comparable. Then $w \in \alpha(A, \leqq)$ and using (1) we obtain $w \leqq \leqq v(a', b')$ for some $v(a', b') \in \alpha_1(A, \leqq) \subset \beta_1(A, \leqq)$.

Further we prove that (2) implies (3). Take $b \in A$. We will show that there exists $c \in A^1$ with $c \geqq b$. Distinguish two cases: 1) $b \notin A^0$, 2) $b \in A^0$. If 1) occurs, there exists $a \in A$ with $a < b$. Since $a \in \{x \in A : x \ngeqq b\}$, we can define a topology w as follows: $D_w(a) = \{O \subset A : O \supset \{x \in A : x \ngeqq b\}\}$, $D_w(z) = \{A\}$ for every $z \in A$, $z \neq a$. Obviously w is not the greatest topology and $w \in \beta(A, \leqq)$, hence there exists $c \in A$ with $w \leqq v(a, c)$, where a, c are noncomparable elements or $c \in A^0 \cup A^1$. The inequality $w \leqq v(a, c)$ implies $A - \{c\} \in D_w(a)$, i.e. $A - \{c\} \supset \{x \in A : x \ngeqq b\}$. Hence $c \geqq b$. As $b > a$, the elements a, c are comparable. Consequently $c \in A^0 \cup A^1$. It follows from $c > a$ that $c \in A^1$. If 2) occurs and $b \notin A^1$, there exists $b' \in A$, $b' > b$. Using what was proved above, there exists $c \in A^1$, $c \geqq b'$. Then also $c \geqq b$. Analogously we can prove that if $b \in A$, then $b \geqq d$ for some $d \in A^0$.

Finally, the condition (3) implies the condition (1). Take an arbitrary topology $w \in \alpha(A, \leqq)$ different from the greatest one. Then there exist $a, b \in A$ with $A - \{b\} \in \in D_w(a)$. Since w is convexly compatible with the ordering $\leqq$, there exists a convex set $W \in D_w(a)$ not containing b. It is $W \subset \{x \in A : x \ngeqq b\}$ or $W \subset \{x \in A : x \nleqq b\}$. Analyse, e.g., the first case. Let $c \in A^1$, $c \geqq b$. As obviously $a \neq c$, it is $v(a, c) \in \in \alpha_1(A, \leqq)$ and $w \leqq v(a, c)$, for otherwise $c \in W \subset \{x \in A : x \ngeqq b\}$, a contradiction. The proof of 6.10 is complete.

With respect to 1.4 and 6.9 it is not hard to prove the following lemmas.

6.11. Lemma. *Let w be a topology on A with the property (α), different from the greatest one. Then w is a meet of a nonempty set of dual atoms of the lattice $\alpha(A, \leqq)$ if and only if the following condition is fulfilled for every $a \in A$:*

If $O \in D_w(a)$, $O \neq A$, then there exists $b \in A^0 \cup A^1$ with $O = A - \{b\}$.

6.12. Lemma. *Let w be a topology on A with the property (β),* different from the greatest one. Then w is a meet of a nonempty set of dual atoms of the lattice $\beta(A, \leqq)$ if and only if the following condition is fulfilled for every $a \in A$:

If $O \in D_w(a)$, $O \neq A$, then there exists $b \in A$ such that $O = A - \{b\}$ and either b is noncomparable with a or $b \in A^0 \cup A^1$.

6.13. Theorem. *The lattice $\alpha(A, \leqq)$ is dually atomic if and only if* card $A = 2$.

Proof. If the lattice $\alpha(A, \leqq)$ is dually atomic, then the least topology is $\wedge \{v(a, b) : a \in A, b \in A^0 \cup A^1\}$. Applying 6.11, we get, that for every $a \in A$ there exists $b \in A^0 \cup A^1$ with $\{a\} = A - \{b\}$. Hence card $A = 2$. The sufficiency is obvious.

Analogously there can be proved the last theorem.

6.14. Theorem. *The lattice $\beta(A, \leqq)$ is dually atomic if and only if* card $A = 2$.

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