Archivum Mathematicum

Jitka Kühnová
Maillet's determinant $D_{p^{n+1}}$

Archivum Mathematicum, Vol. 15 (1979), No. 4, 209--212

Persistent URL: http://dml.cz/dmlcz/107044

ARCH. MATH. 4, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XV: 209—212, 1979

MAILLET'S DETERMINANT $D_{p^{n+1}}$

JITKA KÜHNOVÁ, Brno

(Received January 19, 1978)

1. INTRODUCTION

In Carlitz and Olson's paper [1] there is defined the so called Maillet's determinant D_p/p is a prime $\geqq 3$, $(r, p) = 1$, $r \,.\, r' \equiv 1 \pmod{p}$, the symbol $R(r)$ denotes the least positive residue of $r \pmod{p}$, $D_p = \det(R(r \,.\, s'))$, $r, s = 1, 2, \ldots, (p - 1)$ [2] and there is proved the relation

$$D_p = \pm p^{(p-3)/2} \,.\, h_0^-$$

where h_0^- denotes the first factor of the class number of the p$^{\text{th}}$ cyclotomic field.

The purpose of our paper is to prove an analogical relation for determinant $D_{p^{n+1}}$

$$D_{p^{n+1}} = \pm p^{(n+1)(N-1)} \,.\, h_n^- \tag{1}$$

p is an odd prime, $n \geqq 0$, $N = p^n(p - 1)$ [2]. The method of proving this relation differs from that presented in Carlitz and Olson's paper [1]. It reduces a certain matrix B, for which relation

$$h_n^- = |\det B|$$

is valid, where h_n^- denotes the first factor of the class number of the $p^{n+1\text{th}}$ cyclotomic field.

For $n = 0$ this relation was proved by Newman in [2] and by application of this result to a non-negative integer number n we get the above mentioned general relation (Skula [3]).

2. NOTATION

In the present paper the following symbols will be used:

p	an odd prime
n	a non-negative integer
Z	the ring of integers

$N = p^n(p-1)/2$

r a primitive root with respect to the modulus p^{n+1}

r_j the integer ($j \in \mathbf{Z}$, $0 < r_j < p^{n+1}$

$r_j \equiv r^j \pmod{p^{n+1}}$ for $j \geqq 0$

$r_j \,.\, r^{-j} \equiv 1 \pmod{p^{n+1}}$ for $j < 0$

h_n^- the first factor of the class number of the p^{n+1} cyclotomic field generated by $p^{n+1\text{th}}$ roots of unity over the rational field

$B = (b_{ij})_{0 \leqq i, j \leqq N-1}$ a matrix of order N, where

$$b_{00} = p^{n+1} - 2,$$
$$b_{0j} = 1 - r_j,\ 1 \leqq j \leqq N - 1,$$
$$b_{i0} = 1 - r_i,\ 1 \leqq i \leqq N - 1,$$
$$b_{ij} = 1/p^{n+1}(r_i r_j - r_{i+j}),\ 1 \leqq i, j \leqq N - 1.$$

(2) By [3] $h_n^- = |\det B|$

$Y^* = \{y/y = 1, 2, \ldots, p^{n+1} - 1, (y, p) = 1\}$, hence Y^* is a reduced set of residues with respect to the modulus p^{n+1} and card $Y = \varphi(p^{n+1}) = 2N$

y' the integer, where

$y \,.\, y' \equiv \pmod{p^{n+1}}$ for $y \in Y^*$

$R(y)$ the least positive residue $y \pmod{p^{n+1}}$

$$Y = \{y/y = 1, 2, \ldots, (p^{n+1} - 1)/2, (y, p) = 1\}$$

(3) $D_{p^{n+1}} = \det\,(R(x \,.\, y'))_{x, y \in Y}$

(our definition of Maillet's determinant $D_{p^{n+1}}$)

3. MATRIX B REDUCTIONS

Let I, $J \subseteq \mathbf{Z}$, $I \cup \{0\}$, $J \cup \{0\}$ is a complete set of residues with respect to the modulus N and card I = card $J = N - 1$. We denote

$$(4) \qquad \Delta(I, J) = \left| \det \begin{pmatrix} p^{n+1} - 2 & \ldots & 1 - r_j & \ldots \\ \vdots & & \vdots & \\ 1 - r_i & \ldots & (r_i r_j - r_{i+j})/p^{n+1} & \ldots \\ \vdots & & \vdots & \end{pmatrix}_{i, j \in I, J} \right| .$$

We can suppose that I, $J \subseteq \{1, 2, \ldots, 2N\} - \{N\}$.

Now let $k \in I$, $k = i + N$, where $1 \leqq i \leqq N - 1$. Matrix elements from (4) in the row, corresponding to index k, are determined by means of relation $r_i + r_{i+N} = p^{n+1}$:

$$1 - r_k = 1 - r_{i+N} = 1 - p^{n+1} + r_i,$$

$$(r_k r_j - r_{k+j})/p^{n+1} = (r_{i+N} r_j - r_{i+j+N})/p^{n+1} =$$
$$= (r_j(p^{n+1} - r_i) - p^{n+1} + r_{i+j})/p^{n+1} = r_j - 1 - (r_i r_j - r_{i+j})/p^{n+1}.$$

Now the first row is added to that of matrix from (4), corresponding to index k. We get:

$$p^{n+1} - 2 + 1 - p^{n+1} + r_i = -1 + r_i = -(1 - r_i),$$
$$1 - r_j - 1 + r_j - (r_i r_j - r_{i+j})/p^{n+1} = -(r_i r_j - r_{i+j})/p^{n+1}.$$

Matrix B is symmetric and therefore analogously the same results are obtained also for columns. So if we change index sets I, J then there is changed only the sign of $\det B$, and thus

$$\Delta(I, J) = |\det B|. \tag{5}$$

4. COMPUTATION OF $D_{p^{n+1}}$

It is obvious that the order of Maillet's determinant $D_{p^{n+1}}$ is card $Y = N$ and that

$$D_{p^{n+1}} = |\det (r_{i-j})_{r_i, r_j \in Y}| = |\det (r_{i+j})_{r_i, r_{-j} \in Y}|. \tag{6}$$

Let $I^* = \{1 \leqq i \leqq 2N/2 \leqq r_i \leqq (p^{n+1} - 1)/2\}$

$$J^* = \{1 \leqq 2N/2 \leqq r_{-j} \leqq (p^{n+1} - 1)/2\}.$$

Then I^*, j^* contain $N - 1$ elements and $I^* \cup \{0\}$, $J^* \cup \{0\}$ is a complete set of residues with respect to the modulus N.

From (4) there follows

$$\Delta(I^*, J^*) = \left| \det \begin{pmatrix} p^{n+1} - 2 & \dots & 1 - r_j & \dots \\ \vdots & & \vdots & \\ 1 - r_i & \dots & (r_i r_j - r_{i+j})/p^{n+1} & \dots \\ \vdots & & \vdots & \end{pmatrix}_{i, j \in I^*, J^*} \right|.$$

If we multiply all the rows except the first one by number p^{n+1} we get

$$\Delta(I^*, J^*) = p^{-(n+1)(N-1)} \left| \det \begin{pmatrix} p^{n+1} - 2 & \dots & 1 - r_j & \dots \\ \vdots & & \vdots & \\ (1 - r_i)\, p^{n+1} & \dots & r_i r_j - r_{i+j} & \dots \\ \vdots & & \vdots & \end{pmatrix}_{i, j \in I^*, J^*} \right|.$$

Let $1 \leqq u \leqq 2N$ and $r_u = p^{n+1} - 2$. Then $r_{-u} = (p^{n+1} - 1)/2$ and thus $u \in J$.

The form of the column in the matrix presented in the last expression, corresponding to index u, is determined:

$$1 - r_u = 1 - p^{n+1} + 2 = 3 : p^{n+1};$$

since $r_{i+u} + 2r_i = p^{n+1}$, we have $r_i r_u - r_{i+u} = (p^{n+1} - 2)\, r_i - p^{n+1} + 2r_i = = p^{n+1}(r_i - 1)$.

Thus this column is of the form

$$\begin{pmatrix} 3 - p^{n+1} \\ \vdots \\ p^{n+1}(r_i - 1) \end{pmatrix}.$$

If the column, corresponding to index u, is added to the first column, then we get

$$\Delta(I^*, J^*) = p^{-(n+1)(N-1)} \left| \det \begin{pmatrix} 1 & \dots & 1 - r_j & \dots \\ \vdots & & \vdots & \\ 0 & \dots & r_i r_j - r_{i+j} & \dots \\ \vdots & & \vdots & \\ 0 & & \vdots & \end{pmatrix}_{i, j \in I^*, J^*} \right|$$

If the first column is added to (-1) multiple of the other columns, then

$$\Delta(\mathrm{I}^*, J^*) = p^{-(n+1)(N-1)} \left| \det \begin{pmatrix} 1 & \dots & r_j & \dots \\ \vdots & & \vdots & \\ 0 & \dots & -r_i r_j + r_{i+j} & \dots \\ \vdots & & \vdots & \\ 0 & & \vdots & \end{pmatrix}_{i, j \in I^*, J^*} \right|$$

If r_i multiple of the first row is added to the row corresponding to index i, then

$$(7) \qquad \Delta(I^*, J^*) = p^{-(n+1)(N-1)} \left| \det \begin{pmatrix} 1 & \dots & r_j & \dots \\ \vdots & & \vdots & \\ r_i & \dots & r_{i+j} & \dots \\ \vdots & & \vdots & \end{pmatrix}_{i, j \in I^*, J^*} \right|$$

From (6), (7), (5) and (2) we obtain (1):

$$D_{p^{n+1}} = \pm p^{(n+1)(N-1)} \cdot h_n^- .$$

REFERENCES

[1] **Carlitz, L., Olson, F. R.:** *Maillet's Determinant*, Proceedings of the American Mathematical Society, Vol. 6, No. 2, 1955, pp. 265—269.

[2] **Newman, M.:** *A Table of the First Factor for Prime Cyclotomic Fields*, Mathematics of Computation, Vol. 24, No. 109, 1970, pp 215—219.

[3] **Skula, L.:** *Another Proof of Iwasawa's Class Number Formula*, to appear.

J. Kühnová
662 95 Brno, Janáčkovo nám. 2a
Czechoslovakia