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ACTIVITIES IN A ONE-DIMENSIONAL CONTINUOUS NEURAL NETWORK

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This paper is dedicated to the seventieth anniversary of Professor D. Mangeron

I. INTRODUCTION

A discrete neural model has been investigated from a mathematical, computational and physiological point of view in Refs [1]–[4]. In Ref [5] this model has been extended to a two-dimensional continuous neural network. In Ref [7] we studied the activity propagation in a special two-dimensional continuous neural model. In the present work we deal with the spatial and temporal propagation of nervous activities in a one-dimensional neural network with a special structure.

The neurons of the neural network considered in this paper are supposed to be distributed over the half-axis $\boldsymbol{R} = \{x \mid x > 0\}$ and are connected to each other in a specific way which will be explained below.

Let $u = u(t, x)$ be the normalized and smoothed rate of generation of nerve impulses of the neuron $\boldsymbol{P} = \boldsymbol{P}(x) \in \boldsymbol{R}$ at time t as described in Ref [5]. By this normalization we have $0 \leqq u \leqq 1$. Let the rate of influence, the rate of excitation or inhibition, of a cell $\boldsymbol{Q} \equiv \boldsymbol{Q}(x) \in \boldsymbol{R}$ on the activity of the cell $\boldsymbol{P}$ be denoted by $\tilde{\boldsymbol{K}}(\boldsymbol{P}; \boldsymbol{Q}) \equiv \boldsymbol{K}(x; \xi)$. We assume that $\tilde{\boldsymbol{K}}(\boldsymbol{P}; \boldsymbol{Q})$ is a Volterra kernel such that

$$\text{(I.1)} \qquad \boldsymbol{K}(x; \xi) = \begin{cases} \mu \boldsymbol{K}_0(x - \xi) & \text{if } \boldsymbol{P}, \boldsymbol{Q} \in \boldsymbol{R} \text{ and } 0 \leqq \xi \leqq x, \\ 0, & \text{otherwise} \end{cases}$$

where μ is a real constant and $\boldsymbol{K}_0$ is a given sufficiently smooth function defined on $\boldsymbol{R}$ with the property

$$\text{(I.2)} \qquad \varkappa = \int_0^\infty |\boldsymbol{K}_0(\xi)|^2 \, d\xi < \infty.$$

A neuron $\boldsymbol{Q}$ excites the neuron $\boldsymbol{P}$ if $\tilde{\boldsymbol{K}}(\boldsymbol{P}; \boldsymbol{Q}) > 0$, inhibits $\boldsymbol{P}$ if $\tilde{\boldsymbol{K}}(\boldsymbol{P}; \boldsymbol{Q}) < 0$, and it is not connected to $\boldsymbol{P}$ if $\tilde{\boldsymbol{K}}(\boldsymbol{P}; \boldsymbol{Q}) = 0$.

Let $\alpha = \alpha(x)$ be the rate of self-inhibition of the neuron P, and $f = f(t, x)$ be the external input at time t on P. We assume that

$$\alpha_0 \leqq \alpha(x) \leqq \alpha_1 \ \forall x \in R, \tag{I.3}$$

where α_0 and α_1 are certain positive numbers with $\alpha_0 > 1$. Further, let λ, $c_k's$ and $\gamma_k's$ be real numbers such that

$$0 < \gamma_1 < \gamma_2 < \dots < \gamma_m, \tag{I.4}$$

and consider the function

$$H(t) = \begin{cases} \lambda \sum_{k=1}^{m} c_k e^{-\gamma_k t} & \text{for } t > 0, \\ 0, & \text{for } t \leqq 0. \end{cases} \tag{I.5}$$

The function $H(t)$ will be taken as the self-regulation (adaptation) function of the neural network.

Let $I = \{t \mid t > 0\}$ and $D = I \times R$. Consider the space U of all functions $u = u(t, x)$ defined on D which are absolutely L-integrable in $x \in R$ for fixed t ($0 \leqq t < < \infty$), and absolutely continuous in $t \in I$ for fixed $x \in R$, and are such that $0 \leqq u \leqq 1$ in D. Note that the function

$$S\{g\}(t, x) = \frac{1}{1 + \exp\{-g(t, x)\}}, \tag{I.6}$$

maps $L^\infty(D)$ into itself; it is monotonically increasing with g and $0 < S\{g\} < 1$ for any bounded element of $L^\infty(D)$.

It can be easily demonstrated as in Ref [5], with some minor modifications, that the nervous activities in the neural network under considerations are governed by the nonlinear integro-differential difference equation

$$\left(\frac{\partial}{\partial t} + \alpha(x)\right) u(t, x) = \tag{I.7}$$

$$= S\{f(t, x) + \lambda \int_0^t H(t - \tau)\, u(\tau, x)\, d\tau + \mu \int_0^x K_0(x - \xi)\, u(t - h, \xi)\, d\xi\}$$

for $(t, x) \in D$, subject to the initial condition

$$u(t, x) = \varphi_0(x) \qquad \text{for} \quad t \in I_0, \qquad x \in R, \tag{I.8}$$

where $\psi_0(x)$ is the initial firing rate of the neuron P in the initial time-interval $I_0 = = \{t \mid -h \leqq t \leqq 0\}$, and h is a small time-lag which occurs in the neural interactions. Here the initial firing rate is supposed to be independent of t for the sake of simplicity.

For small λ and μ Eq (I.7) can be approximated by the following linear inhomogeneous integro-differential difference equation:

(I.9) $$\left(\frac{\partial}{\partial t} + \alpha(x)\right) u(t, x) =$$
$$= f_0(t, x) + f_1(t, x)\, \lambda \int_0^t H(t - \tau)\, u(\tau, x)\, d\tau + \mu \int_0^x K_0(x - \xi)\, u(t - h, \xi)\, d\xi,$$

where

(I.10) $$f_0 = S\{f\}, \qquad f_1 = f_0 - f_0^2.$$

By making a few natural modifications in the proof given in § II of Ref [5] we can demonstrate that Eq (I.7) has a unique solution $u = u(t, x)$ in U which satisfies the initial condition (I.8). The same result is also valid for the initial value problem (I.8)–(I.9).

In the present paper we study the initial value problem (I.8)–(I.9) in the case $m = 1$ and

(I.11) $$K_0(x) = e^{-\beta x},$$

where β is a positive number. Further, to simplify our writing we shall denote $c_1 y$ by λ and γ_1 by γ, and assume that $f = f(x)$, that is the external force f is stationary with respect to time so that f_0 and f_1 are independent of time t: $f_0 = f_0(x)$, $f_1 = f_1(x)$.

Note that the neurons in the network always excite (inhibit) each other if $\mu > 0$ ($\mu < 0$), by virtue of Eqs (I.1) and (I.11). The structure of the neural network with the above restrictions is surely not very realistic and oversimplified. Nevertheless it provides some insights on the nervous activities in the idealized conditions.

II. ASSOCIATED PARTIAL DIFFERENTIAL DIFFERENCE EQUATION

Consider the linearized neural network described by Eq (I.9) with $f = f(x)$, $f_0 = f_0(x)$, $f_1 = f_1(x)$, $H(t) = e^{-\gamma t}$ and $K_0(x) = e^{-\beta x}$. Put

(II.1) $$v(t, x) = \int_0^x e^{-\gamma(x-\xi)}\, u(t, \xi)\, d\xi,$$
$$v_j(t, x) = \int_0^x e^{-\gamma(x-\xi)}\, \varphi_j(\xi)\, d\xi \qquad (j = 0, 1),$$

where $\varphi_0(x)$ is the given initial function in (I.8), and

(II.2) $$\varphi_1(x) = \mu f_1(x)\, v_0(x) - \alpha(x)\, \varphi_0(x) \left(= \left.\frac{\partial u}{\partial t}\right|_{t \in I_0} \right).$$

Hence

(II.3) $$v\,|_{t \in I_0} = v_0(x), \qquad \left.\frac{\partial u}{\partial t}\right|_{t \in I_0} = v_1(x)\; (v_j\,|_{x=0} = 0)$$

and

(II.4) $$v\,|_{x=0} = 0.$$

Further

(II.5) $$\left(\frac{\partial}{\partial x} + \beta\right) v(t, x) = u(t, x).$$

Elimination of $u(t, x)$ between Eqs (I.9) and (II.5) yields the integrodifferential difference equation

(II.6) $$\left(\frac{\partial}{\partial t} + \alpha(x)\right)\left(\frac{\partial}{\partial x} + \beta\right) v(t, x) = f_0(x) + \mu f_1(x)\, v(t - h, x) + \\ + \lambda f_1(x) \int\limits_0^t e^{-\gamma(t-\tau)} \left(\frac{\partial}{\partial x} + \beta\right) v(\tau, x)\, d\tau,$$

and the differentiation of the two sides of Eq (II.6) with respect to t yields

(II.7) $$\left(\frac{\partial^2}{\partial t^2} + \alpha(x)\frac{\partial}{\partial t} - \lambda f_1(x)\right)\left(\frac{\partial}{\partial x} + \beta\right) v(t, x) = \\ = \mu f_1(x)\frac{\partial v(t - h, x)}{\partial t} - \lambda\gamma f_1(x) \int\limits_0^t e^{-\gamma(t-\tau)} \left(\frac{\partial}{\partial x} + \beta\right) v(\tau, x)\, d\tau.$$

Now we eliminate the terms that contain integrals, obtaining the following linear inhomogeneous third order partial differential difference equation in y:

(II.8) $$\frac{\partial^3 v(t, x)}{\partial t^2\, \partial x} + a_1(x)\frac{\partial^2 v(t, x)}{\partial t^2} + a_2(x)\frac{\partial^2 v(t, x)}{\partial t\, \partial x} + \\ + a_3(x)\frac{\partial v(t, x)}{\partial t} + a_4(x)\frac{\partial v(t, x)}{\partial x} + a_5(x)\, v(t, x) = \\ = \gamma f_0(x) + \gamma\mu f_1(x)\, v(t - h, x) + \mu f_1(x)\frac{\partial v(t - h, x)}{\partial t},$$

where

(II.9) $$a_1(x) = \beta, \qquad a_2(x) = \alpha(x) + \gamma, \qquad a_3(x) = \beta[\alpha(x) + \gamma], \\ a_4(x) = \gamma\alpha(x) - \lambda f_1(x), \qquad a_5(x) = \beta[\gamma\alpha(x) - \lambda f_1(x)].$$

The functions u and v determine each other uniquely by virtue of Eqs (II.1) and (II.5), respectively. In the next section the solution $v(t, x)$ of Eq (II.8) satisfying the conditions (II.3) and (II.4) will be constructed by the methods of integral equations.

III. ASSOCIATED INTEGRAL EQUATION

To establish the integral equation which is satisfied by the function $v(t, x)$, we integrate successively both sides of Eq (II.8) twice with respect to t between 0 and t, once with respect to x between 0 and x. Then, taking into account the conditions (II.3) and (II.4), and putting

$$\text{(III.1)}\qquad \begin{aligned} A(t, x) &= -[a_2(x) + ta_4(x)], \\ B(t, x) &= -\beta, \\ C(t, x) &= a_2'(x) - a_3(x) + t[a_4(x) - a_5'(x)], \\ D(t, x) &= (1 + \gamma t) f_1(x) \end{aligned}$$

and

$$\text{(III.2)}\qquad g(t, x) = v_0(x) + tv_1(x) + \\ + \int_0^x \left\{ \frac{\gamma}{2} t^2 f_0(\xi) + [\beta + t(a_3(\xi) - \mu f_0(\xi))]\, v_0(\xi) + [\beta + a_2(\xi)]\, tv_1(\xi) \right\} \mathrm{d}\xi,$$

we obtain the following integro-difference equation

$$\text{(III.3)}\qquad v(t, x) = g(t, x) + \mu \int_{-h}^{t-h} \int_0^x D(t - \tau - h, \xi)\, v(\tau, \xi)\, \mathrm{d}\tau\, \mathrm{d}\xi + (Tv)(t, x),$$

where

$$(Tv)(t, x) = \int_0^t A(t - \tau, x)\, v(\tau, x)\, \mathrm{d}\tau + \int_0^x B(t, \xi)\, v(t, \xi)\, \mathrm{d}\xi + \int_0^t \int_0^x C(t - \tau, \xi)\, v(\tau, \xi)\, \mathrm{d}\tau\, \mathrm{d}\xi.$$

(III.4)

Note that the double integral on the right-hand side of Eq (III.3) involve only the values of the function $v(\tau, \xi)$ for $-h \leqq \tau \leqq t - h$. Put

$$\text{(III.5)}\qquad I_n = \{t \mid (n - 1)\, h \leqq t \leqq nh\}, \qquad n = 0, 1, 2, \dots$$

and

$$\text{(III.6)}\qquad w_n(t, x) = v(t, x) \qquad \text{for } t \in I_n.$$

Then we have

$$\text{(III.7)}\qquad w_n(t, x) = g_n(t, x) + (Tw_n)(t, x) \qquad \text{for } t \in I_n,$$

where

$$\text{(III.8)}\qquad g_n(t, x) = g(t, x) + \mu \sum_{k=1}^{n-2} \int_{(k-1)h}^{kh} \int_0^x D(t - \tau - h, \xi)\, w_k(\tau, \xi)\, \mathrm{d}\tau\, \mathrm{d}\xi + \\ + \mu \int_{(n-2)h}^{t-h} \int_0^x D(t - \tau - h, \xi)\, w_{n-1}(\tau, \xi)\, \mathrm{d}\tau\, \mathrm{d}\xi.$$

Thus, if the functions $w_0(t, x)$, $w_1(t, x)$, ..., $w_n(t, x)$ are known, the function $g_n(t, x)$ is completely determined on $D_n = I_n \times D$, and Eq (III.8) is a pure integral equation.

Although the operator T involves different simple and double integral operators, Eq (III.7) can be solved uniquely and the solution $w_n(t)$ is smooth on D_n since the functions A, B, C, D and g are sufficiently smooth by virtue of the hypotheses of § I. We omit the details here.

Integral equations involving different simple and double integral operators first investigated by M. Picone in his great work on partial differential equations (cf. Ref [6]). We recently dealt with such equations in several occasions (cf. Ref [7] – [9]).

Before closing this section we note that $w_0(t, x) = v_0(x)$ by virtue of Eq (II.3), and

$$g_1(t,x) = g(t,x) + \left(t + \frac{\gamma}{2} t^2\right) \int_0^x v_0(\xi)\, d\xi, \qquad (t,x) \in \boldsymbol{D}_1. \tag{III.8}$$

Hence $w_1(t, x)$ is well determined on $\boldsymbol{D}_1$. Using w_0 and w_1 we can construct w_2 on $\boldsymbol{D}_2$, and so on. This stepwise continuation of the solution in the forward direction of time is well known in the theory of differential difference equations (cf. [10] – [11]).

IV. A SPECIAL CASE

In this section we shall briefly investigate the specil case $\alpha(x, y) \equiv \alpha$, $f(t, x) \equiv f$ and $h = 0$ where α and f are constants, $\alpha > 1$. In this case f_0, f_1 and $a_k's$ ($k = 1, 2, 3, 4, 5$) are also constant, and Eq (II.8) is a pure partial differential equation of the form

$$\frac{\partial^3 v}{\partial t^2\, \partial x} + b_1 \frac{\partial^2 v}{\partial t^2} + b_2 \frac{\partial^2 v}{\partial t\, \partial x} + b_3 \frac{\partial v}{\partial t} + b_4 \frac{\partial v}{\partial x} + b_5 v = \gamma f_0 \tag{IV.1}$$

subjected to the conditions (II.3) and (II.4), where

$$\begin{aligned} b_1 &= \beta, \qquad b_2 = \alpha + \gamma, \qquad b_3 = \beta(\alpha + \gamma) - \mu f_1, \\ b_4 &= \alpha\gamma - \lambda f_1, \qquad b_5 = \beta(\alpha\gamma - \lambda f_1 - \mu f_1). \end{aligned} \tag{IV.2}$$

Although $v(t, x)$ can be constructed by the integral equation method described in § III, we shall try to construct it analytically by means of Laplace transformation. For this purpose, put

$$V = V(t; s) = \int_0^\infty e^{-sx} v(t, x)\, dx \tag{IV.3}$$

and

$$\chi_k(s) = \int_0^\infty e^{-sx} v_k(x)\, dx \qquad (k = 0,1). \tag{IV.4}$$

Then multiplying both sides of Eqs (IV.1), (II.3) and (II.4) by e^{-sx} and integrating with respect to x over the interval $(0, \infty)$, we find the ordinary differential equation

$$(s + b_1) \frac{\partial^2 V}{\partial t^2} + (b_2 s + b_3) \frac{\partial V}{\partial t} + (b_4 s + b_5) V = \frac{\gamma f_0}{s} \tag{IV.5}$$

subjected to the initial conditions

$$V \bigg|_{t=0} = \chi_0(s), \quad \frac{\partial V}{\partial t} \bigg|_{t=0} = \chi_1(s). \tag{IV.6}$$

The associated characteristic equation is

$$(s + b_1) r^2 + (b_2 s + b_3) r + (b_4 s + b_5) = 0, \tag{IV.7}$$

and its roots are

$$r_{1,2}(s) = \frac{-(b_2 s + b_3) \mp \sqrt{\Delta(s)}}{2(s + b_1)} \tag{IV.8}$$

where

$$\Delta(s) = (b_2^2 - 4b_4) s^2 + 2(b_2 b_3 - 2b_1 b_4 - b_5) s + (b_3^2 - 4b_1 b_5). \tag{IV.9}$$

Hence, if $\Delta(s) \not\equiv 0$ we have $r_1(x) \not\equiv r_2(s)$ and

$$V(t; s) = \frac{\gamma f_0}{s(b_4 s + b_5)} + C_1(s) e^{r_1(s)t} + C_2(s) e^{r_2(s)t} \tag{IV.10}$$

where

$$\begin{cases} C_1(s) = \dfrac{r_2(s)\chi_0(s) - \chi_1(s)}{r_2(s) - r_1(s)} - \dfrac{\gamma f_0 r_2(s)}{s(b_4 s + b_5)(r_2(s) - r_1(s))}, \\ C_2(s) = \dfrac{\chi_1(s) - r_1(s)\chi_0(s)}{r_2(s) - r_1(s)} + \dfrac{\gamma f_0 r_1(s)}{s(b_4 s + b_5)(r_2(s) - r_1(s))}. \end{cases} \tag{IV.11}$$

Accordingly, we have

$$v(t, x) = \frac{\gamma f_0}{b_5}(1 - e^{-\Theta x}) + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \{C_1(s) e^{r_1(s)t} + C_2(s) e^{r_2(s)t}\} e^{sx}\, dx, \tag{IV.12}$$

where $\Theta = b_5/b_4$ and c is a suitably chosen positive number.

Further, if $\Delta(s) = 0$, then $r_1(s) = r_2(s) = \dfrac{b_2 s + b_3}{2(s + b_1)} = r(s)$ and

$$V(t; s) = \frac{f_0}{s(b_4 s + b_5)} + [C_1(s) t + C_2(s)] e^{r(s)t} \tag{IV.13}$$

where

$$\begin{cases} C_1(s) = \chi_1(s) + \left[\dfrac{b_2}{2} + \dfrac{b_3 - b_1 b_2}{2(s + b_1)}\right] \chi_0(s) - \dfrac{\gamma f_0 (b_2 s + b_3)}{2s(s + b_1)(b_4 s + b_5)}, \\ C_2(s) = \chi_0(s) - \dfrac{\gamma f_0}{s(b_4 s + b_5)}. \end{cases} \tag{IV.14}$$

Accordingly, we find

$$v(t, x) = \frac{\gamma f_0}{b_5}(1 - e^{-\Theta x}) + [Y(x) t + Z(x)] R(t, x) \tag{IV.15}$$

where

$$\begin{cases} Y(x) = v_1(x) + \dfrac{b_2}{2} v_0(x) + \dfrac{b_3 - b_1 b_2}{2} \displaystyle\int_0^x e^{-b_1(x-\xi)} v_0(\xi)\, d\xi - \\ \qquad - \dfrac{\gamma f_0}{2}\left[\dfrac{b_3}{b_1 b_5} + \dfrac{b_3 - b_1 b_2}{b_1(b_1 b_4 - b_5)} e^{-b_1 x} + \dfrac{b_3 b_4 - b_2 b_5}{b_5(b_5 - b_1 b_4)} e^{-\Theta x}\right] \\ Z(x) = v_0(x) - \dfrac{\gamma f_0}{b_5}(1 - e^{-\Theta x}) \end{cases} \tag{IV.16}$$

and

$$\text{(IV.17)}\; R(t,x) = e^{-(b_2/2)t}\left\{1 + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[e^{-b_1 x}\sqrt{\frac{2x}{(b_3 - b_1 b_2)t}}\, J_1\left(\sqrt{2(b_3 - b_1 b_2)tx}\right)\right]\right\}$$

where $J_1(z)$ is the Bessel function of the first kind of order 1.

Finally we note that $\Delta(s) \equiv 0$ if and only if

$$\text{(IV.18)} \quad b_2^2 - 4b_4 = 0, \qquad b_2 b_3 - 2b_1 b_4 - b_5 = 0, \qquad b_3^2 - 4b_1 b_5 = 0.$$

The first and second equations in (IV.18) yield respectively the relationships

$$\text{(IV.19)} \qquad \lambda = -\frac{(\alpha - \gamma)^2}{4f_1}, \qquad \mu = \frac{\beta(\alpha + \gamma)^2}{4\alpha f_1}$$

and the third equation yields the relationships

$$\text{(IV.20)} \qquad \gamma = \alpha[-2 + \sqrt{4\alpha - 21}], \qquad \alpha > \frac{25}{4}$$

since $\gamma > 0$. Note that $0 < f_1 \leqq \frac{1}{4}$. Hence if the structural parameters $\alpha, \beta, \gamma, f_1, \lambda$ and μ satisfy the conditions (IV.19) and (IV.20), then the activities in the neural model described by Eqs (II.5) and (IV.1) show oscillations according to the formulas (II.5) and (IV.15)–(IV.17).

Further, since $0 \leqq u(t, x) \leqq 1$ always in $\boldsymbol{D}$, the functions $v_0(x)$ and $v_1(x)$ are bounded on $\boldsymbol{R}$. We can easily verify that

$$\text{(IV.21)} \qquad \begin{cases} \lim\limits_{t\to+\infty} v(t,x) = \lim\limits_{t\to+\infty} \dfrac{\partial v(t,x)}{\partial t} = 0 & \text{for fixed } x \in R, \\ \lim\limits_{x\to+\infty} v(t,x) = \lim\limits_{x\to+\infty} \dfrac{\partial v(t,x)}{\partial t} = 0 & \text{for fixed } t \in \boldsymbol{I}, \end{cases}$$

whenever the conditions (IV.19)–(IV.20) are satisfied. Hence, under these conditions, we have the limits

$$\text{(IV.22)} \qquad \begin{cases} \lim\limits_{t\to+\infty} u(t,x) = 0 & \text{for fixed } x \in R, \\ \lim\limits_{x\to+\infty} u(t,x) = 0 & \text{for fixed } t \in \boldsymbol{I} \end{cases}$$

by virtue of Eq (II.5). The physiological significance of these limits are obvious.

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REFERENCES

[1] M. N. Oğuztöreli: *On the neural equations of Cowan and Stein* (1). Utilitas Mathematica 2, 305—317 (1972).

[2] K. V. Leung, D. Mangeron, M. N. Oǧuztöreli, R. B. Stein: a) *On a class of nonlinear integro-differential equations*. I. Rend. Accad. Naz Lincei 54, 522—528 (1973). b) *On a class of nonlinear integrodifferential equations*. II. Rend. Accad. Naz. Lincei 54, 699—705 (1973). c) *On a class of nonlinear integrodifferential equations*. III. Bull. Acad. Roy. Sci. Begique. Serie 5, 59, 492—499 (1973). d) *On the stability and numerical solutions of two neural models.* Utilitas Mathematica 5, 167—212 (1974).

[3] R. B. Stein, K. V. Leung, M. N. Oǧuztöreli, D. W. Williams: *Properties of small neural networks*. Kybernetik 14, 223—230 (1974).

[4] R. B. Stein, K. V. Leung, D. Mangeron, M. N. Oǧuztöreli: *Improved neuronal models for studying neural networks*. Kybernetik 15, 1—9 (1974).

[5] M. N. Oǧuztöreli: *On the activities in a continuous neural network*. Biol. Cybernetics 18, 41—48 (1975).

[6] M. Picone: a) *Sopra un problema dei valori al contorno nelle equazioni iperboliche alle derivate parziali delsecond'ordine e sopra una classe di equazioni integrali che a quello si riconnettono,* "Rend. Circolo Mat. Palermo", 31, 133—190 (1911); b) *Su alcuni problemi di determinazione delle soluzioni di equazioni lineari a derivate parziali del second'ordine di tipo iperbolico,* "Atti Accad. Istituto di Bologna, Cl. sci. fis.", Anno 258, ser XII, 8, 60—71 (1910); c) *Duodecim Doctorum Virorum Vitae et Operum Notitia,* "Pontificia Academia Scientiarum", A.D. MCMLXX, 117—146.

[7] M. N. Oǧuztöreli: *Activity propagation in a continuous neural network*. (in press).

[8] M. N. Oǧuztöreli: *A Goursat problem for the fourth moment equation*. J. Math. Phys. 16, 1244—1246 (1975).

[9] M. N. Oǧuztöreli: *On a pseudoparabolic differential equation*. Utilitas Mathematica 7, 121—133 (1975).

[10] L. E. El'sgol'ts, S. B. Norkin: *Introduction to the Theory and Application of Differential Equations with Deviating Arguments*. Academic Press. New York and London 1973.

[11] M. N. Oǧuztöreli: *Time-lag control system*. Academic Press. New York and London 1966.

[12] D. V. Widder: *The Laplace Transform*. Princeton University Press. Princeton 1946.

[13] A. Erdelyi, W. Magnus, F. Oberhettinger, F. G. Tricomi: *Tables of Integral Transforms*. Vol. II. McGraw-Hill. New York, Toronto, London 1954.

[14] G. E. Roberts, H. Kaufman: *Table of Laplace Transforms*. W. B. Saunders, Philadelphia and London, 1966.

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