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ON SOME OPERATOR DEFINED ON EQUATIONAL CLASSES OF ALGEBRAS

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§ 0.

Let $\mathbf{K}_E$ be an equational class of algebras of the type τ defined by the set of axioms E. We denote by $C(E)$ the set of all consequences of E. Let φ, Ψ be terms in $\mathbf{K}_E$. An equality $\varphi = \Psi$ is called to be non-trivializing (see [2]) iff it is of the form $x = x$ or none of the terms φ, Ψ is a single variable. Denote by $N(E)$ the set of all non-trivializing consequences of E. Obviously $C(N(E)) = N(E)$.

It was shown in [2] that if there exists in $\mathbf{K}_E$ a unary term $q(x)$ not being a single variable such that the equality $q(\mathbf{x}) = x$ is satisfied in $\mathbf{K}_E$, then an algebra $\mathfrak{A}$ belongs to $\mathbf{K}_{N(E)}$ iff $\mathfrak{A}$ is isomorphic to subdirect product of algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ where $\mathfrak{A}_1 \in \mathbf{K}_E$ and in $\mathfrak{A}_2$ all fundamental operations are equal to one constant c.

In this paper we give another representation of algebras from $\mathbf{K}_{N(E)}$ without the assumption of existence of the term $q(x)$.

§ 1.

First we prove some properties.

(i) If $\mathfrak{A} = (X; \mathbf{F})$ is an algebra and $r : X \to X$ is mapping satisfying the condition

$$r(r(x)) = r(x) \tag{1}$$

then for any $a \in r(X)$ we have $a = r(a)$.

Proof: If any $a \in r(X)$, then there exists $b \in X$ such that

$$a = r(b). \tag{2}$$

Hence

$$r(r(b)) = r(a)$$
$$r(r(b)) = r(b).$$

Lemma. *If* $\mathfrak{A} = (X; \mathbf{F})$ *is an algebra and* $r : X \to X$ *satisfies* (1) *and*

(3) $\mathfrak{B} = (r(X); \mathbf{F})$ *is a subalgebra of* $\mathfrak{A} = (X)\mathbf{F}$;

(4) $a_1, \ldots, a_n \in X$ *and* $f(x_1, \ldots, x_n) \in \mathbf{F}$ *implies* $f(a_1, \ldots, a_n) = f(r(a_1), \ldots, r(a_n))$

then r *is an endomorphism of* $\mathfrak{A} = (X; \mathbf{F})$.

Proof: Obviously $r(a_1), \ldots, r(a_n) \in r(X)$. By (3) $\mathfrak{B}$ is a subalgebra so for any $f(x_1, \ldots, x_n) \in \mathbf{F}$ we have $f(r(a_1), \ldots, r(a_n)) \in r(X)$. Hence by (i)

$$r(f(r(a_1), \ldots, r(a_n))) = f(r(a_1), \ldots, r(a_n)).$$

From the last equality we get by (4)

$$r(f(a_1, \ldots, a_n)) = r(f(r(a_1), \ldots, r(a_n))) = f(r(a_1), \ldots, r(a_n)). \text{ q. e. d.}$$

Theorem. *If an algebra* $\mathfrak{A} = (X; \mathbf{F})$ *is of the type* τ, *then this algebra belongs to the class* $\mathbf{K}_{N(E)}$ *iff there exists a mapping* $r : X \to X$ *such that*

(5) $$r(r(x)) = r(x)$$

(6) $$\mathfrak{B} = (r(X); \mathbf{F}) \in \mathbf{K}_E$$

(7) *if* $f(x_1, \ldots, x_n) \in \mathbf{F}$ *and* $a_1, \ldots, a_n \in X$, *then* $f(a_1, \ldots, a_n) = f(r(a_1), \ldots, r(a_n))$.

Proof: If $N(E) = C(E)$ it is enough to put $r(x) = x$. We must prove the theorem if the set $N(E)$ is a proper subset of $C(E)$. We have three possible cases:

1° $\mathbf{F} = \emptyset$

2° $\mathbf{F} \neq \emptyset$ and $(x = y) \in C(E)$

3° $\mathbf{F} \neq \emptyset$ and $(x = y) \notin C(E)$.

If the case 1° holds, then any trivializing equality in $\mathbf{K}_E$ is of the form $x = y$. It means that $\mathbf{K}_E$ is a trivial class. Then it is enough to choose an element $d \in X$ and to put $r(x) = d$ for any $x \in X$ and the theorem holds. In the case 2° the values of all fundamental operations in $\mathfrak{A}$ are equal to one constant c. We put $r(x) = c$ for any $x \in X$. Observe that the constructions in cases 1° i 2° show also sufficiency of the condition. In the case 3° observe first that a trivializing equality, which exists by assumption in $C(E)$, must be of the form $g(x_1, \ldots, x_m) = x_i$ where $i \in \{1, \ldots, m\}$. We get $g(x, \ldots, x) = x$. Denote $g(x, \ldots, x) = r(x)$. From the last two equalities for any $(x_1, \ldots, x_n) \in \mathbf{F}$ it follows:

(8) $$r(f(x_1, \ldots, x_n)) = f(x_1, \ldots, x_n)$$

(9) $$r(r(x)) = r(x)$$

(10) $$f(x_1, \ldots, x_n) = f(r(x_1), \ldots, r(x_n)).$$

First we prove the necessity. Assume that $\mathfrak{A} \in \mathbf{K}_{N(E)}$. The equalities (8), (9), (10) are non-trivializing and therefore are satisfied in $\mathfrak{A}$ and obviously r maps X into X.

So (5) and (7) follows from (9) and (10). By (8) for any $f(x_1, \ldots, x_n) \in \mathbf{F}$ and $a_1, \ldots, \ldots, a_n \in r(X)$ we have $f(a_1, \ldots, a_n) \in r(X)$. Thus $\mathfrak{B} = (r(X); \mathbf{F})$ is a subalgebra of $\mathfrak{A}$.

Obviously $\mathfrak{B}$ satisfies any equality from $N(E)$. We prove that $\mathfrak{B}$ satisfies any trivializing equality $h(x_1, \ldots, x_s) = x_i$ belonging to $C(E)$. Let $x_1, \ldots, x_s \in r(X)$. By(i) and (9) we get $h(x_1, \ldots, x_s) = h(r(x_1), \ldots, r(x_s))$. The equality $h(r(x_1), \ldots, r(x_s)) = r(x_i)$ is non-trivializing and holds in $\mathfrak{A}$. Thus we have $h(x_1, \ldots, x_s) = r(x_i)$. Applying i we get $h(x_1, \ldots, x_s) = x_i$. So we proved the condition 6 what finishes the proof of necessity.

Proof of sufficiency: It is enough to show that $\mathfrak{A}$ satisfies any equality belonging to $N(E)$. From the assumption and lemma 1 it follows that r is an endomorphism of $\mathfrak{A}$. So (7) holds not only for the fundamental operations but also for any term different from single variable. Thus if

$$\varphi(x_{i_1}, \ldots, x_{i_m}) = \Psi(x_{j_1}, \ldots, x_{j_g}) \tag{11}$$

is not of the form $x = x$ and is non-trivializing consequence of E which is satisfied in $\mathfrak{B}$, then for any $a_{i_1}, \ldots, a_{i_m}, a_{j_1}, \ldots, a_{j_g} \in X$ we have

$$\varphi(r(a_{i_1}), \ldots, r(a_{i_m})) = \Psi(r(a_{j_1}), \ldots, r(a_{j_g})).$$

So we have

$$\varphi(a_{i_1}, \ldots, a_{i_m}) = \Psi(a_{j_1}, \ldots, a_{j_g}).$$

Thus (11) holds in $\mathfrak{A}$.

q.e.d.

Corrolary 1. *Any algebra* $\mathfrak{A} = (X; \mathbf{F})$ *is completely described by a pair* $(\mathfrak{A}_0, r(x))$, *where* $\mathfrak{A}_0 = (r(X); \mathbf{F}) \in \mathbf{K}_E$, $r(r(x)) = r(x)$ *and* r *satisfies* (7).

Corrolary 2. The proof of our theorem gives a method of writting down the axiomatics $N(E)$, when E is given. In particular if E is finite than we can find a finite axiomatics for $\mathbf{K}_{N(E)}$.

For example we give an axiomatics for $\mathbf{K}_{N(E)}$ if $\mathbf{K}_E$ is the class of lattices $(X; x + y, xy)$.

A1.	$xy = yx$	A1′.	$x + y = y + x$
A2.	$(xy)\,z = x(yz)$	A2′.	$(x + y) + z = x + (y + z)$
A3.	$(xx)\,y = xy$	A3′.	$xx + y = x + y$
A4.	$x + xy = xx$	A4′.	$x(x + y) = x + x$
A5.	$xx = x + x$		

The reader can check that it is enough to put $r(x) = xx$.

REFERENCES

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[2] J. Płonka: *On the subdirect Product of some Equational classes of Algebras,* Matematische Nachrichten, 1974, pp. 1—3.

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