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THE APPROXIMATION OF FUNCTIONS IN THE SENSE OF TCHEBYCHEV II

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This paper gives a certain generalization of the (classical) Haar condition and the corresponding theory of the approximation.

The detailed knowledge of all the theory, the notation and the terminology given in the paper [1] is necessary for understanding this paper.

1. THE HAAR DECOMPOSITION CONDITION

Assumption (for § 1.). Let B be a set, $n \in N$, $S = R$ or $S = C$, let card $B \geqq n$. Let $\mathscr{M}$ be a decomposition of the set B. Let $\omega \in \mathscr{M} \cup \{\emptyset\}$.

Definition 1. Let V be an n-dimensional subspace of S^B. We shall say that V satisfies *the Haar decomposition condition* (with respect to B, $\mathscr{M}$, ω) iff every non-trivial polynomial $Q \in V$ has at most $n - 1$ zeros in distinct classes of $\mathscr{M} - \{\omega\}$.

Remark. If card $(\mathscr{M} - \{\omega\}) \leqq n - 1$, then every n-dimensional subspace of S^B satisfies the Haar decomposition condition.

Theorem 1. Let card $(\mathscr{M} - \{\omega\}) > n$. Let V be a subspace of S^B generated by functions $Q_1, \ldots, Q_n \in S^B$. Then the following assertions are equivalent:

(1) $Q_1, \ldots, Q_n$ form a basis of V and V satisfies the Haar decomposition condition.

(2) If $x_1, \ldots, x_n \in B - \omega$ are in distinct classes of $\mathscr{M}$, then $\det Q_k(x_j) \neq 0$.

Proof. The proof of the assertion is simple.

Theorem 2. Let card $(\mathscr{M} - \{\omega\}) \geqq n$. Let V be a subspace of S^B, $\dim V \leqq n$. Then the following assertions are equivalent:

(1) $\dim V = n$ and V satisfies the Haar decomposition condition.

(2) If $x_1, \ldots, x_n \in B - \omega$ are in distinct classes of $\mathscr{M}$ and if $y_1, \ldots, y_n \in S$ are arbitrary, then there exists exactly one $P \in V$ such that $P(x_j) = y_j$ for $j = 1, \ldots, n$.

(3) If $1 \leqq m \leqq n$ and if $x_1, \ldots, x_m \in B - \omega$ are in distinct classes of $\mathcal{M}$, then $\dim_{\{x_1, \ldots, x_m\}} V = m$.

Proof. We shall prove that (1) implies (3); the rest of the proof is simple. Let (1) hold, let $x_1, \ldots, x_m \in B - \omega$ be in distinct classes. If $m = n$, then the assertion $\dim_{\{x_1, \ldots, x_n\}} V = n$ follows from Theorem 1(2) and from Theorem 23(2) of [1]. Let $m < n$; we can add such points $x_{m+1}, \ldots, x_n \in B - \omega$ that the points $x_1, \ldots, x_n$ are in distinct classes and hence $\dim_{\{x_1, \ldots, x_n\}} V = n$. By Theorems 23(4) and 23(1) of [1], we have $n = \dim_{\{x_1, \ldots, x_n\}} V \leqq \dim_{\{x_1, \ldots, x_m\}} V + (n - m) \leqq m + (n - m) = n$, hence $\dim_{\{x_1, \ldots, x_m\}} V = m$ and (3) is valid.

Remark. If $\mathcal{M} = \{\{x\}/x \in B\}$ and $\omega = \emptyset$, then the Haar decomposition condition is equivalent to the (classical) Haar condition (see [2], p. 25).

Theorem 3. Let $D \subset B$. Let us denote $\mathcal{N} = \{\alpha \cap D/\alpha \in \mathcal{M}\} - \{\emptyset\}$, $\varkappa = \omega \cap D$. Then $\mathcal{N}$ is a decomposition of D and $\varkappa \in \mathcal{N} \cup \{\emptyset\}$.

Let $\operatorname{card}(\mathcal{N} - \{\varkappa\}) \geqq n$. Let V be an n-dimensional subspace of S^B satisfying the Haar decomposition condition with respect to B, $\mathcal{M}$, ω; let us denote $W = \{Q_D/Q \in V\}$. Then W is an n-dimensional subspace of S^D satisfying the Haar decomposition condition with respect to $D, \mathcal{N}, \varkappa$.

Proof. The assertion is obvious.

2. THE QUOTIENT FUNCTION $p(x, y)$

Assumption (for § 2.). Let B be a set, $n \in N$, $S = R$ or $S = C$, let $\operatorname{card} B \geqq n$. Let $\mathcal{M}$ be a decomposition of B; let $\sim$ denote the equivalence on B corresponding to $\mathcal{M}$. Let $\omega \in \mathcal{M} \cup \{\emptyset\}$.

Let us suppose that for each $x, y \in B - \omega$ of the same class of $\mathcal{M}$ there is given a fixed non-zero number $p(x, y) \in S$. If $x, y, z \in B - \omega$ and $x \sim y$ and $y \sim z$, let the relation $p(x, z) = p(x, y) \cdot p(y, z)$ hold.

Let us denote $Y = Y(B, \mathcal{M}, \omega, p, S) = \{g \in S^B/g(x) = 0$ for all $x \in \omega$, $g(x) = p(x, y) \cdot g(y)$ for $x, y \in B - \omega$ and $x \sim y\}$. (In what follows we shall deal only with the functions of Y.)

Theorem 4. (1) We have $p(x, x) = 1$ for all $x \in B - \omega$.

(2) If $x, y \in B - \omega$ and $x \sim y$, then $p(x, y) \cdot p(y, x) = 1$.

(3) Y is a subspace of S^B.

(4) If $\mathcal{M} = \{\{x\}/x \in B\}$ and $\omega = \emptyset$, then $Y = S^B$.

(5) Let us choose for each class $\alpha \in \mathcal{M} - \{\omega\}$ a fixed point $x_\alpha \in \alpha$ and a number $c_\alpha \in S$. Then there exists one and only one $g \in Y$ such that $g(x_\alpha) = c_\alpha$ for all $\alpha \in \mathcal{M} - \{\omega\}$.

Proof. (1) $p(x, x) = p(x, x) \cdot p(x, x)$ and $p(x, x) \neq 0$, hence $p(x, x) = 1$.

(2) We have $p(x, y) . p(y, x) = p(x, x) = 1$.

(5) Let $g \in Y$ be such that $g(x_\alpha) = c_\alpha$ for all $\alpha \in \mathcal{M} - \{\omega\}$.

Then $g(x) = 0$ for all $x \in \omega$. If $x \in B - \omega$, then there exists one and only one $\alpha \in \mathcal{M} - \{\omega\}$ such that $x \in \alpha$; we have $g(x) = p(x, x_\alpha) \cdot g(x_\alpha) = p(x, x_\alpha) \cdot c_\alpha$. Hence there exists at most one $g \in Y$ such that $g(x_\alpha) = c_\alpha$ for all $\alpha \in \mathcal{M} - \{\omega\}$.

On the other hand, let us define $g \in S^B$ by the relations: $g(x) = 0$ for $x \in \omega$, $g(x) = p(x, x_\alpha) \cdot c_\alpha$ for $x \in \alpha$ where $\alpha \in \mathcal{M} - \{\omega\}$. Then $g \in Y$ and $g(x_\alpha) = c_\alpha$ for all $\alpha \in \mathcal{M} - \{\omega\}$.

Definition 2. A point $x \in B$ will be called *a significant point* iff $x \in B - \omega$ and $|p(y, x)| \leqq 1$ for all $y \in B - \omega$ such that $y \sim x$.

Theorem 5. Let V be an n-dimensional subspace of Y, $f \in Y$.

(1) We have $\operatorname{card}(\mathcal{M} - \{\omega\}) \geqq n$.

(2) If $x \in \omega$, then $Q(x) - f(x) = 0$ for all $Q \in V$.

(3) If $x, y \in B - \omega$ and $x \sim y$, then $Q(y) - f(y) = p(y, x) \cdot [Q(x) - f(x)]$ for all $Q \in V$.

(4) Let x be a significant point. If $y \sim x$, then $|Q(y) - f(y)| \leqq |Q(x) - f(x)|$ for all $Q \in V$.

(5) Let $P \in V$ and $0 < \|P - f\| < +\infty$. Let $x \in B$ be such a point that $|P(x) - f(x)| = \|P - f\|$ (such a point is called an extreme point of B). Then x is a significant point.

Proof. (1) Let $Q_1, \ldots, Q_n$ form a basis of V. By Theorem 21 or [1], there exist points $x_1, \ldots, x_n \in B$ such that $\det Q_k(x_j) \neq 0$. Evidently $x_j \notin \omega$ for $j = 1, \ldots, n$. Let us admit that $x_i \sim x_j$ and $i \neq j$. Then $Q_k(x_i) = p(x_i, x_j) \cdot Q_k(x_j)$ for $k = 1, \ldots, n$, hence $\det Q_k(x_j) = 0$, which is a contradiction. Therefore $x_1, \ldots, x_n$ are in distinct classes of $\mathcal{M} - \{\omega\}$, hence $\operatorname{card}(\mathcal{M} - \{\omega\}) \geqq n$.

(5) Necessarily $x \in B - \omega$. Let us admit that there exists $y \in B$ such that $y \sim x$ and $|p(y, x)| > 1$. Then by (3), $|P(y) - f(y)| = |p(y, x)| \cdot |P(x) - f(x)| > |P(x) - f(x)| = \|P - f\|$, which is a contradiction.

Theorem 6. Let V be an n-dimensional subspace of Y satisfying the Haar decomposition condition. Let $f \in Y$, let us denote $\mu = \min_{Q \in V} \|Q - f\|$.

(1) Let $M \neq \emptyset$ be a minimal set (i.e. $\mu > 0, f \notin V$). Then:

a) $M \cap \omega = \emptyset$;

b) the points of M are in distinct classes of $\mathcal{M} - \{\omega\}$;

c) if $x \in M$, then x is a significant point;

d) $\operatorname{card} M \geqq n + 1$ (and if $S = R$, then $\operatorname{card} M = n + 1$);

e) $\dim_M V = n$.

(2) Suppose that there exists a minimal set $M \neq \emptyset$. If $P \in V$ and $\|P - f\| = \mu$, then P has at least $n + 1$ extreme points in distinct classes of $\mathcal{M} - \{\omega\}$.

(3) Suppose that there exists a minimal set M. Then there exists one and only one $P \in V$ such that $\| P - f \| = \mu$.

Proof. (1) a) Let us admit that $x \in M \cap \omega$. We have $\| Q - f \|_{M-\{x\}} = \| Q - f \|_M$ for all $Q \in V$, hence $\mu(M - \{x\}) = \mu(M)$, which is a contradiction. Hence $M \cap \omega = \emptyset$.

b) Let us admit that $x, y \in M$ and $x \sim y$. Since $p(x, y) \cdot p(y, x) = 1$, we may assume $| p(x, y) | \leqq 1$. By Theorem 5(3), $| Q(x) - f(x) | \leqq | Q(y) - f(y) |$ for all $Q \in V$, which is in contradiction with Theorem 16(1) of [1].

c) Let $x \in M$, let $P \in V$ be such a polynomial that $\| P - f \| = \mu$. By Theorems 9(4) and 17 of [1], we have $| P(x) - f(x) = \mu = \| P - f \|$. Since $\mu > 0$, x is a significant point by Theorem 5(5).

d) Let us admit that card $M = m \leqq n$. By a), b) and Theorem 2, we have $\dim_M V = m$. By Theorem 24 of [1], we have $\mu = \mu(M) = 0$, which is a contradiction. Hence card $M \geqq n + 1$.

e) By a), b), d) and by Theorem 2, we have $\dim_D V = n$ even for each subset $D \subset M$ with at least n points. Hence $\dim_M V = n$, too.

(2) By Theorems 9(4) and 17 of [1], we have $| P(x) - f(x) | = \mu$ for all $x \in M$. The assertion follows now from (1a), (1b), (1d).

(3) If $M = \emptyset$, then $f \in V$ and the assertion is evident. If $M \neq \emptyset$, then $\dim_M V = n$ by (1e) and the assertion follows from Theorem 20(3) of [1].

Remark. Theorem 6(3) is a generalization of the classical Haar theorem, namely of the assertion of the sufficiency (see Theorem 19 of [2]). We can generalize also the assertion of the necessity (see Theorem 20 of [2]); we need, however, stronger assumptions. Theorem 7 is not used in the following theory.

Theorem 7. Suppose that there exists a number $d > 0$ such that for each $\alpha \in \mathscr{M} - \{\omega\}$ there exists a point $z_\alpha \in \alpha$ such that $| p(x, z_\alpha) | \leqq d$ for all $x \in \alpha$.

Let D be such a subset of B that $p(x, y) = 1$ for $x, y \in D - \omega$ and $x \sim y$. Let $\mathscr{T}$ be a topology on D. Let us denote $\mathscr{N} = \{\alpha \cap D / \alpha \in \mathscr{M}\} - \{\emptyset\}$; then $\mathscr{N}$ is a decomposition of D. Let us denote $\mathscr{F} = \{\mathscr{A} \subset \mathscr{N} / U\mathscr{A} \in \mathscr{T}\}$; then $\mathscr{F}$ is a topology on $\mathscr{N}$. Suppose that $(\mathscr{N}, \mathscr{F})$ is a compact Hausdorff T-space.

Let V be an n-dimensional subspace of Y not satisfying the Haar decomposition condition (with respect to $B, \mathscr{M}, \omega$). Let P be a non-trivial polynomial of V having zeros $x_1, \ldots, x_n$ in distinct classes $\alpha_1, \ldots, \alpha_n \in \mathscr{M} - \{\omega\}$. Suppose that P is bounded in B and continuous in D with respect to the topology $\mathscr{T}$.

Then there exists a function $f \in Y$ continuous in D with respect to $\mathscr{T}$ which has infinitely many polynomials of the best approximation in V.

Proof. We give only the principle ideas:

1. We may assume $\| P \| = \frac{1}{d}$, $x_k = z_{\alpha_k}$ and $x_k \in D$ for $\alpha_k \cap D \neq \emptyset$.

2. There exist $b_1, \ldots, b_n \in S$ not all zero such that $\sum_{j=1} b_j Q(x_j) = 0$ for all $Q \in V$.

3. There exist a function $g \in S^D$ continuous in D with respect to $\mathscr{T}$ with the following properties: $g(x) = 0$ for all $x \in D \cap \omega$; $g(x) = g(y)$ for $x, y \in D - \omega$ and $x \sim y$; $g(x_k) = \operatorname{sign} b_k$ for $\alpha_k \cap D \neq \emptyset$; $|g(x)| \leqq 1$ for all $x \in D$.

4. Let us define $f \in S^B$ in this way: $f(x) = 0$ for $x \in \omega$ and for $x \in \alpha$, $\alpha \cap D = \emptyset$, $\alpha \notin \{\alpha_1, \ldots, \alpha_n\}$; $f(x) = p(x, z_\alpha) \cdot g(z_\alpha) \cdot (1 - |P(z_\alpha)|)$ for $x \in \alpha$, $\alpha \cap D \neq \emptyset$; $f(x) = p(x, x_k) \cdot (\operatorname{sign} b_k) \cdot (1 - |P(x_k)|)$ for $x \in \alpha_k$, $\alpha_k \cap D = \emptyset$. Then $\mu = \min_{Q \in V} \|Q - f\| = 1$ and $\|aP - f\| = 1$ for all $a \in S$ such that $|a| \leqq 1$.

Remark. In Theorem 20 of [2] there are the following assumptions: B is a compact Hausdorff T-space, V is an n-dimensional subspace of $C(B)$ not satisfying the (classical) Haar condition. We take $\mathscr{M} = \{\{x\}/x \in B\}$, $\omega = \emptyset$, $D = B$. Then $\mathscr{N} = \mathscr{M}$ and $(\mathscr{N}, \mathscr{F})$ is a compact Hausdorff T-space. If $x \sim y$, then $x = y$ and $p(x, y) = 1$. By Theorem 7, there exists $f \in C(B)$ having infinitely many polynomials of the best approximation in V.

3. THE APPROXIMATION

Assumption (for § 3.). Let $n \in N$, $S = R$. Let D be a set, $\mathscr{N}$ a decomposition of D ($\sim$ the corresponding equivalence on D), $\varkappa \in \mathscr{N} \cup \{\emptyset\}$. Let us suppose that for each $x, y \in D - \varkappa$ of the same class of $\mathscr{N}$ there is given a fixed non-zero number $q(x, y) \in R$. If $x, y, z \in D - \varkappa$ and $x \sim y$ and $y \sim z$, let the relation $q(x, z) = q(x, y) \cdot q(y, z)$ hold.

Let B be a subset of D. Let us denote $\mathscr{M} = \{\alpha \cap B/\alpha \in \mathscr{N}\} - \{\emptyset\}$, $\omega = \varkappa \cap B$. Let us suppose $\operatorname{card}(\mathscr{M} - \{\omega\}) \geqq n + 1$.

Let W be an n-dimensional subspace of $Y(D, \mathscr{N}, \varkappa, q, R)$ satisfying the Haar decomposition condition with respect to $D, \mathscr{N}, \varkappa$. Let $Q_1, \ldots, Q_n$ form a basis of W.

Suppose that there are given an interval $J \subset R^*$, a set $I \subset D - \varkappa$ and a one-one mapping ξ of J onto I. Let every $Q \in W$ have the following property: if $Q[\xi(s)]$ is non-zero in a subinterval $\langle c, d \rangle \subset J$, then $Q[\xi(c)] \cdot Q[\xi(d)] > 0$. (The same is true e.g. when $Q[\xi(s)]$ is continuous in J.)

Let $f \in R^B$ be such a function that $f(x) = 0$ for all $x \in \omega$ and $f(x) = q(x, y) . f(y)$ for $x, y \in B - \omega$, $x \sim y$.

Remark. (1) $\mathscr{M}$ is a decomposition of B, $\omega \in \mathscr{M} \cup \{\emptyset\}$.

(2) If $x, y \in B$ and $x \sim y$, then we define $p(x, y) = q(x, y)$. The function p satisfies the requirements of the Assumption for § 2 with respect to $B, \mathscr{M}, \omega$. We have $Y(B, \mathscr{M}, \omega, p, R) = \{g \in R^B/g(x) = 0$ for $x \in \omega, g(x) = q(x, y) \cdot g(y)$ for $x, y \in B - \omega$ and $x \sim y\}$, i.e. $f \in Y(B, \mathscr{M}, \omega, p, R)$.

(3) Let us denote $V = \{Q_B/Q \in W\}$. We can easily prove (by Theorem 3 etc.) that V is an n-dimensional subspace of $Y(B, \mathscr{M}, \omega, p, R)$ satisfying the Haar decomposition condition with respect to $B, \mathscr{M}, \omega$.

(4) Let us denote $\mu = \min_{Q \in V} \| Q - f \|$. If $Q \in W$, let us denote $\| Q - f \| = = \sup_{x \in B} | Q(x) - f(x) | = \| Q_B - f \|$. Then $\mu = \min_{Q \in W} \| Q - f \|$, too.

(5) The restrictions of the functions $Q_1, \ldots, Q_n$ to the set B form a basis of V. When we apply the theorems of [1] and of § 1 and § 2, we must realize that under the basis of V these restrictions must be understood. However, in the theorems and formulae we shall speak only about the polynomials of W.

(6) For $x, y \in I$ let us denote: $x \prec y$ iff $\xi^{-1}(x) < \xi^{-1}(y)$, $x \precsim y$ iff $x \prec y$ or $x = y$.

(7) If $B = D$, then $\mathcal{M} = \mathcal{N}$, $\omega = \varkappa$, $p = q$, $V = W$, too. If we consider such a case, we shall speak only about $B, \mathcal{M}, \omega, p, V$.

(8) If $I \subset (D - \varkappa) \cap R^*$ is an interval and if each polynomial $Q \in W$ is continuous in I, we take mostly $J = I$, $\xi(x) \equiv x$. Then $x \prec y$ iff $x < y$.

(9) All these assumptions and constructions are necessary for the applications; see § 4.

Theorem 8. Let $x_1 \prec \ldots \prec x_{n+1}$ be such points in I that $x_1 \precsim x \precsim x_{n+1}$ and $x \sim x_k$ implies $x = x_k$ (for each $x \in I$ and $k = 1, \ldots, n + 1$). For $k = 1, \ldots, n + 1$ let us denote

$$C_k = (-1)^{k-1} \cdot \begin{vmatrix} Q_1(x_1) \cdots Q_1(x_{k-1}) \; Q_1(x_{k+1}) \cdots Q_1(x_{n+1}) \\ \vdots \\ Q_n(x_1) \cdots Q_n(x_{k-1}) \; Q_n(x_{k+1}) \cdots Q_n(x_{n+1}) \end{vmatrix}.$$

Then the numbers $C_1, \ldots, C_{n+1}$ are non-zero and alternate their signs.

Proof. Let $k \in \{1, \ldots, n\}$. For all $x \in D$ let us put

$$Q(x) = \begin{vmatrix} Q_1(x_1) \cdots Q_1(x_{k-1}) \; Q_1(x) \; Q_1(x_{k+2}) \cdots Q_1(x_{n+1}) \\ \vdots \\ Q_n(x_1) \cdots Q_n(x_{k-1}) \; Q_n(x) \; Q_n(x_{k+2}) \cdots Q_n(x_{n+1}) \end{vmatrix}.$$

Then $Q \in W$. If $s \in \langle \xi^{-1}(x_k), \xi^{-1}(x_{k+1}) \rangle$, then the points $x_1, \ldots, x_{k-1}, \xi(s), x_{k+2}, \ldots, x_{n+1}$ are in distinct classes of $\mathcal{N} - \{\varkappa\}$, hence $Q[\xi(s)] \neq 0$ by Theorem 1. Hence (by the Assumption) $Q(x_k) \cdot Q(x_{k+1}) > 0$. We have $C_k = (-1)^{k-1} Q(x_{k+1})$ $C_{k+1} = (-1)^k Q(x_k)$, hence $C_k \cdot C_{k+1} < 0$.

Remark. If each class $\alpha \in \mathcal{N} - \{\varkappa\}$ has at most one point in the set $\{x \in I / x_1 \precsim x \precsim x_{n+1}\}$, then the assumption of Theorem 8 is fulfilled.

Theorem 9. Let $P \in W$ have the following property: there exist points $x_1 \prec \ldots \prec x_{n+1}$ in I such that $x_1 \precsim x \precsim x_{n+1}$ and $x \sim x_k$ implies $x = x_k (x \in I, k = 1, \ldots, n+1)$, points $t_1, \ldots, t_{n+1} \in B$ and a number $h \in \{-1, +1\}$ such that for $k = 1, \ldots, n + 1$ we have $t_k \sim x_k$ and

$$P(t_k) - f(t_k) = h \cdot \operatorname{sign} q(t_k, x_k) \cdot (-1)^k \cdot d_k, \qquad \text{where} \qquad d_k \geqq 0.$$

(1) For $k = 1, \ldots, n + 1$ let us denote

$$D_k = (-1)^{k-1} \cdot \begin{vmatrix} Q_1(t_1) \cdots Q_1(t_{k-1}) \ Q_1(t_{k+1}) \cdots Q_1(t_{n+1}) \\ \vdots \\ Q_n(t_1) \cdots Q_n(t_{k-1}) \ Q_n(t_{k+1}) \cdots Q_n(t_{n+1}) \end{vmatrix} .$$

Then $\mu \geqq \mu(\{t_1, \ldots, t_{n+1}\}) = \dfrac{\Sigma \mid D_k \mid \cdot \mid P(t_k) - f(t_k) \mid}{\Sigma \mid D_k \mid} \geqq \min\limits_{k=1,\ldots,n+1} \mid P(t_k) - f(t_k) \mid$.

(2) Let us define the numbers $C_1, \ldots, C_{n+1}$ as in Theorem 8. Then $\mu(\{t_1, \ldots, t_{n+1}\}) = \dfrac{\Sigma \mid C_k \mid \cdot \mid q(x_k, t_k) \mid \cdot \mid P(t_k) - f(t_k) \mid}{\Sigma \mid C_k \mid \cdot \mid q(x_k, t_k) \mid}$.

(3) If $\mid P(t_k) - f(t_k) \mid = \| P - f \|$ for $k = 1, \ldots, n + 1$, then $\| P - f \| = \mu$.

Proof. Let us denote $w = q(t_1, x_1) \cdot \ldots \cdot q(t_{n+1}, x_{n+1})$. Let $k \in \{1, \ldots, n + 1\}$. Then we have $Q_i(t_k) = q(t_k, x_k) \cdot Q_i(x_k)$ for $i = 1, \ldots, n$, hence $D_k = q(t_1, x_1) \cdot \ldots \cdot q(t_{k-1}, x_{k-1}) \cdot q(t_{k+1}, x_{k+1}) \cdot \ldots \cdot q(t_{n+1}, x_{n+1}) \cdot C_k = \dfrac{w}{q(t_k, x_k)} \cdot C_k = w . q(x_k, t_k) \cdot C_k$. By Theorem 8, there exists $a \in \{-1, +1\}$ such that sign $C_k = a \cdot (-1)^k$ for $k = 1, \ldots, n + 1$, hence sign $D_k =$ sign $w \cdot$ sign $q(x_k, t_k) \cdot a \cdot (-1)^k$. Let us denote $b = a \cdot h \cdot$ sign w. Then for $k = 1, \ldots, n + 1$ we have $b \cdot D_k \cdot [P(t_k) - f(t_k)] = b \cdot \mid D_k \mid \times$ $\times$ sign $w \cdot$ sign $q(x_k, t_k) \cdot a \cdot (-1)^k \cdot h \cdot$ sign $q(t_k, x_k) \cdot (-1)^k \cdot d_k = \mid D_k \mid \cdot d_k \geqq 0$. Therefore (1) follows from Theorem 28(6) of [1] (we take t_k, D_k instead of x_k, C_k).

(2) follows from (1), if we substitute $\mid D_k \mid = \mid w \mid \cdot \mid q(x_k, t_k) \mid \cdot \mid C_k \mid$, (3) follows from (1).

Theorem 10. Let $P \in W$ have the property: there exist points $x_1 \prec \ldots \prec x_{n+1}$ in $I \cap B$ such that $x_1 \precsim x \precsim x_{n+1}$ and $x \sim x_k$ implies $x = x_k (x \in I, k = 1, n + 1)$ and a number $h \in \{-1, +1\}$ such that for $k = 1, \ldots, n + 1$ we have

$$P(x_k) - f(x_k) = h \cdot (-1)^k . d_k, \qquad \text{where} \qquad d_k \geqq 0.$$

(1) Let us define $C_1, \ldots, C_{n+1}$ as in Theorem 8. Then $\mu \geqq \mu(\{x_1, \ldots, x_{n+1}\}) = \dfrac{\Sigma \mid C_k \mid . \mid P(x_k) - f(x_k) \mid}{\Sigma \mid C_k \mid} \geqq \min\limits_{k=1,\ldots,n+1} \mid P(x_k) - f(x_k) \mid$.

(2) If $\mid P(x_k) - f(x_k) \mid = \| P - f \|$ for $k = 1, \ldots, n + 1$, then $\| P - f \| = \mu$.

Proof. Theorem 10 follows from Theorem 9. We take $t_k = x_k$, hence $q(t_k, x_k) = 1$, $C_k = D_k$.

Theorem 11. Let $M = \{t_1, \ldots, t_{n+1}\}$ be a minimal set (see Theorem 6(1)). Suppose that there exist such points $x_1 \prec \ldots \prec x_{n+1}$ in I that $t_k \sim x_k$ for $k = 1, \ldots, n + 1$. Let $P \in W$ and $\| P - f \| = \mu$.

(1) Let us define $C_1, \ldots, C_{n+1}$ as in Theorem 8. Then there exists $b \in \{-1, +1\}$ such that for $k = 1, \ldots, n + 1$

$$P(t_k) - f(t_k) = b \cdot \text{sign } q(t_k, x_k) \cdot \text{sign } C_k \cdot \| P - f \|.$$

(2) Let $x_1 \leqq x \leqq x_{n+1}$ and $x \sim x_k$ imply $x = x_k (x \in I, k = 1, \ldots, n + 1)$.

a) Then there exists a number $h \in \{-1, +1\}$ such that for $k = 1, \ldots, n + 1$ we have

$$P(t_k) - f(t_k) = h \cdot \operatorname{sign} q(t_k, x_k) \cdot (-1)^k \cdot \| P - f \| .$$

b) Let $u_1, \ldots, u_{n+1} \in B$ be such points that $u_k \sim t_k$ for $k = 1, \ldots, n + 1$. Then for $k = 1, \ldots, n + 1$ we have

$$P(u_k) - f(u_k) = h \cdot | q(u_k, t_k) | \cdot \operatorname{sign} q(u_k, x_k) \cdot (-1)^k \cdot \| P - f \| .$$

c) If $x_1, \ldots, x_{n+1} \in B$, then for $k = 1, \ldots, n + 1$ we have

$$P(x_k) - f(x_k) = h \cdot | q(x_k, t_k) | \cdot (-1)^k \cdot \| P - f \| .$$

Proof. Let us define $D_1, \ldots, D_{n+1}$ as in Theorem 9, let us denote $w = q(t_1, x_1) \cdot \ldots \cdot q(t_{n+1}, x_{n+1})$. Then $D_k = w \cdot q(x_k, t_k) \cdot C_k$ for $k = 1, \ldots, n + 1$ (see proof of Theorem 9). By Theorem 31(2) of [1] (where we take t_k, D_k instead of x_k, C_k), there exists $a \in \{-1, +1\}$ such that $P(t_k) - f(t_k) = a \cdot \operatorname{sign} D_k \cdot \| P - f \| = a \cdot \operatorname{sign} w \cdot \operatorname{sign} q(x_k, t_k) . \operatorname{sign} C_k \cdot \| P - f \|$ for $k = 1, \ldots, n + 1$. Let us put $b = a \cdot \operatorname{sign} w$; since $\operatorname{sign} q(x_k, t_k) = \operatorname{sign} q(t_k, x_k)$, our assertion is valid.

(2a) By Theorem 8, there exists $c \in \{-1, +1\}$ such that $\operatorname{sign} C_k = c \cdot (-1)^k$ for $k = 1, \ldots, n + 1$. Let us denote $h = b \cdot c$; the assertion follows now from (1).

(2b) $P(u_k) - f(u_k) = q(u_k, t_k) \cdot [P(t_k) - f(t_k)] = | q(u_k, t_k) | \cdot \operatorname{sign} q(u_k, t_k) \cdot h \cdot \operatorname{sign} q(t_k, x_k) \cdot (-1)^k \cdot \| P - f \| = h \cdot | q(u_k, t_k) | \cdot \operatorname{sing} q(u_k, x_k) \cdot (-1)^k \cdot \| P - f \|$.

(2c) follows from (2b) for $u_k = x_k$.

Theorem 12. (1) Suppose that $\alpha \cap B \neq \emptyset$ implies $\alpha \cap I \neq \emptyset$ for each $\alpha \in \mathcal{N} - \{\varkappa\}$. Let $M \neq \emptyset$ be a minimal set. Then there exist (significant) points $t_1, \ldots, t_{n+1} \in B$ (in distinct classes of $\mathcal{N} - \{\varkappa\}$) and points $x_1 \prec \ldots \prec x_{n+1}$ in I such that $M = \{t_1, \ldots, t_{n+1}\}$ and $t_k \sim x_k$ for $k = 1, \ldots, n + 1$.

(2) Suppose that $\alpha \cap B \neq \emptyset$ implies $\operatorname{card} (\alpha \cap I) \leqq 1$ for each $\alpha \in \mathcal{N} - \{\varkappa\}$. If $x_1 \prec \ldots \prec x_{n+1}$ are arbitrary points in I and if there exist points $t_1, \ldots, t_{n+1} \in B$ such that $t_k \sim x_k$ for $k = 1, \ldots, n + 1$, then $x_1 \leqq x \leqq x_{n+1}$ and $x \sim x_k$ implies $x = x_k$.

Proof. (1) By Theorem 6(1), M has exactly $n + 1$ points which are significant and are in distinct classes of $\mathcal{M} - \{\omega\}$; let us denote them by $t_1, \ldots, t_{n+1}$. For $k = 1, \ldots, n + 1$ let $\alpha_k \in \mathcal{N}$ be the class containing t_k. Then $\alpha_k \neq \varkappa$, $\alpha_k \cap B \neq \emptyset$, hence $\alpha_k \cap I \neq \emptyset$. Let us choose $x_k \alpha\, \infty_k \cap I$ arbitrarily. The points $x_1, \ldots, x_{n+1}$ are distinct; we may assume that the points $t_1, \ldots, t_{n+1}$ are denoted so that $x_1 \prec \ldots \prec x_{n+1}$.

(2) Let $k \in \{1, \ldots, n + 1\}$. Let $\alpha_k \in \mathcal{N}$ be the class containing x_k. Then $\alpha_k \neq \varkappa$ and $\alpha_k \cap B \neq \emptyset$, hence $\alpha_k \cap I = \{x_k\}$ and the validity of the assertion is proved.

Theorem 13. Suppose that $\alpha \cap B \neq \emptyset$ implies card $(\alpha \cap I) = 1$ for each $\alpha \in \mathcal{N} - \{\varkappa\}$. Suppose that there exists a minimal set, let $P \in W$.

Then $\| P - f \| = \mu$ iff there exist points $t_1, \ldots, t_{n+1} \in B$ (in distinct classes of $\mathcal{N} - \{\varkappa\}$), points $x_1 \prec \ldots \prec x_{n+1}$ in I and a number $h \in \{-1, +1\}$ such that for $k = 1, \ldots, n+1$ we have $t_k \sim x_k$ and

$$P(t_k) - f(t_k) = h \cdot \operatorname{sign} q(t_k, x_k) \cdot (-1)^k \cdot \| P - f \|.$$

Proof. Let the latter condition be fulfilled. Then we have $\| P - f \| = \mu$ by Theorems 12(2) and 9(3).

Let $\| P - f \| = \mu = 0$. Since card $(\mathcal{M} - \{\omega\}) \geqq n + 1$, there exist distinct classes $\alpha_1, \ldots, \alpha_{n+1} \in \mathcal{N} - \{\varkappa\}$ such that $\alpha_k \cap B \neq \emptyset$ for $k = 1, \ldots, n+1$. Let $\{x_k\} = \alpha_k \cap I$, $t_k \in \alpha_k \cap B$. By a renumeration we can attain that $x_1 \prec \ldots \prec x_{n+1}$ and the assertion holds.

Let $\| P - f \| = \mu > 0$. Then the assertion follows from Theorems 12(1), 12(2) and 11(2a).

Theorem 14. Suppose that there exists a minimal set. Then there exists one and only one $P \in W$ such that $\| P - f \| = \mu$.

Proof. By Theorem 6(3) there exists exactly one $Q \in V$ such that $\| Q - f \| = \mu$. Since card $(\mathcal{M} - \{\omega\} \geqq n + 1$, two distinct polynomials of W cannot coincide in B (see Theorem 2). If $P \in W$ is the only polynomial for which $P_B = Q$, then P is the only polynomial of W such that $\| P - f \| = \mu$.

Theorem 15. Let a subset $A \subset B$ be compact with respect to some topology, let the function $| Q - f |$ be continuous in A for any $Q \in W$. Suppose that if $\alpha \in \mathcal{N} - \{\varkappa\}$ and $\alpha \cap B \neq \emptyset$, then there exists a significant point $x \in \alpha \cap A$. Then A is a representative subset (and there exists a minimal set).

Proof. Let $x \in B - \omega$, let $\alpha \in \mathcal{N}$ be the class containing x. Then $\alpha \neq \varkappa$, $\alpha \cap B \neq \emptyset$, hence there exists a significant point $y \in \alpha \cap B$. As $| q(x, y) | \leqq 1$, we have $| Q(x) - f(x) | \leqq | Q(y) - f(y) |$ for all $Q \in W$.

Let $x \in \omega$. As $A \neq \emptyset$, we can choose arbitrary $y \in A$ and then $| Q(x) - f(x) | = 0 \leqq | Q(y) - f(y) |$ for all $Q \in W$.

Lemma. Let $x, y \in D$ be such points that $| Q(x) - f(x) \leqq | Q(y) - f(y) |$ for all $Q \in W$. Then there exists a number $d \in R$ such that $| d | \leqq 1$, $f(x) = d \cdot f(y)$ and $Q(x) = d \cdot Q(y)$ for all $Q \in W$. (The proof is not difficult and we do not give it here.)

Theorem 16. Let $A \subset B$ be a representative subset.

(1) If $x \in B - \omega$, then there exists $y \in A$ such that $x \sim y$ and $| q(x, y) | \leqq 1$.

(2) Let the class $\alpha \in \mathcal{N} - \{\varkappa\}$ contain at least one significant point (of course with respect to p). Then there is a significant point in $\alpha \cap A$, too.

Proof. (1) Let $x \in B - \omega$; let $y \in A$ be such a point that $|Q(x) - f(x)| \leqq \leqq |Q(y) - f(y)|$ for all $Q \in W$. By Lemma, there exists $d \in R$ such taht $|d| \leqq 1$ and $Q(x) = d \cdot Q(y)$ for all $Q \in W$. Then $\dim_{\{x,y\}} W \leqq 1$ and hence $x \sim y$ by Theorem 2. Then $q(x, y) = d$ and $|q(x, y)| \leqq 1$.

(2) Let $x \in \alpha$ be a significant point, let y be the point mentioned in (1). Then $|q(x, y)| = 1$. If $z \in \alpha \cap B$, then $|p(z, y)| = |p(z, x)| \cdot |p(x, y)| = |p(z, x)| \leqq 1$, hence y is a significant point, too.

4. APPLICATIONS

A. The (Classical) Haar Condition

Assumption. Let $S = R$, $n \in N$, $a, b \in R^*$, $a < b$. Let W be an n-dimensional subspace of $C\langle a, b\rangle$, let every non-trivial polynomial $Q \in W$ have at most $n - 1$ zeros in $\langle a, b\rangle$ (the Haar condition). Let $B \subset \langle a, b\rangle$ be compact, card $B \geqq n + 1$, let $f \in C(B)$. Let us denote $\mu = \min_{Q \in W} \| Q - f \|$.

Remark. We take $D = \langle a, b\rangle$, $\mathcal{N} = \{\{x\}/x \in \langle a, b\rangle\}$, $\varkappa = \emptyset$. We have, card $(\mathcal{M} - \{\omega\}) =$ card $B \geqq n + 1$. W is an n-dimensional subspace of $Y(D, \mathcal{N}, \varkappa, q, R) = R^{\langle a,b\rangle}$ satisfying the Haar decomposition condition with respect to $D, \mathcal{N}, \varkappa$. We take $I = J = \langle a, b\rangle$, $\xi(s) \equiv s$; then card $(\alpha \cap I) = 1$ for all $\alpha \in \mathcal{N}$. Since B is a representative subset, there exists a minimal set.

Remark. As $x \sim y$ implies $x = y$, it is not necessary to define q explicitely; we always have $q(x, y) = 1$. The situation will be similar in the other applications; moreover, if $x \sim y$ and $x \neq y$, it is sufficient to define $q(x, y)$; we have $q(y, x) = = \dfrac{1}{q(x,y)}$.

Theorem 17. (1) Let $P \in W$ have the property: there exist points $x_1 < \ldots < x_{n+1}$ in B such that the numbers $P(x_k) - f(x_k)$ $(k = 1, \ldots, n + 1)$ alternate their signs. Then $\mu \geqq \mu(\{x_1, \ldots, x_{n+1}\}) \geqq \min_{k=1,\ldots,n+1} |P(x_k) - f(x_k)|$.

(2) Let $P \in W$. Then $\|P - f\| = \mu$ iff there exist points $x_1 < \ldots < x_{n+1}$ in B and $h \in \{-1, +1\}$ such that $P(x_k) - f(x_k) = h \cdot (-1)^k \cdot \|P - f\|$ for $k = 1, \ldots, n + 1$.

(3) There exists one and only one $P \in W$ such that $\|P - f\| = \mu$.

Proof. (1) follows from Theorems 12(2) and 10(1); (2) follows from Theorem 13 (where $t_k \sim x_k$ implies $t_k = x_k$); (3) follows from Theorem 14.

Remark. If we introduce a basis $Q_1, \ldots, Q_n$ of W, we can get a better estimation in (1) from Theorems 9 and 10. The same will be true of the other applications.

B. Functions with Zero Values at the End Points

Assumption. Let $S = R$, $n \in N$, $a, b \in R^*$, $a < b$. Let W be an n-dimensional subspace of $C\langle a, b\rangle$, let $Q(a) = 0$ for all $Q \in W$ and let every non-trivial polynomial $Q \in W$ have at most $n - 1$ zeros in $(a, b\rangle$. Let $B \subset \langle a, b\rangle$ be compact, card $(B - \{a\}) \geqq \geqq n + 1$. Let $f \in C(B)$ and $f(a) = 0$ in case $a \in B$. Let us denote $\mu = \min_{Q \in W} \| Q - f \|$.

Remark. We take $D = \langle a, b\rangle$, $\mathcal{N} = \{\{x\}/x \in \langle a, b\rangle\}$, $\varkappa = \{a\}$; q is defined implicitely. We have card $(\mathcal{M} - \{\omega\}) =$ card $(B - \{a\}) \geqq n + 1$. W is an n-dimensional subspace of $Y(D, \mathcal{N}, \varkappa, q, R) = \{g \in R^{\langle a, b\rangle}/g(a) = 0\}$ satisfying the Haar decomposition condition with respect to $D, \mathcal{N}, \varkappa$. If $x \in \omega$, then $x = a$ and $x \in B$, hence $f(x) = = 0$. We take $I = J = (a, b\rangle$, $\xi(s) \equiv s$. Then $\varkappa \cap I = \emptyset$, card $(\alpha \cap I) = 1$ for all $\alpha \in \mathcal{N} - \{\varkappa\}$. As B is a representative subset, there exists a minimal set.

Theorem 18. (1) Let $P \in W$ have this property: there exist points $x_1 < \ldots < x_{n+1}$ in $B - \{a\}$ such that the numbers $P(x_k) - f(x_k)$ $(k = 1, \ldots, n + 1)$ alternate their signs. Then $\mu \geqq \mu(\{x_1, \ldots, x_{n+1}\}) \geqq \min_{k=1, \ldots, n+1} | P(x_k) - f(x_k) |$.

(2) Let $P \in W$. Then $\| P - f \| = \mu$ iff there exist points $x_1 < \ldots < x_{n+1}$ in $B - \{a\}$ and $h \in \{-1, +1\}$ such that $P(x_k) - f(x_k) = h \cdot (-1)^k \cdot \| P - f \|$ for $k = 1, \ldots, n + 1$.

(3) There exists one and only one $P \in W$ such that $\| P - f \| = \mu$.

Proof. (1) follows from Theorems 12(2) and 10(1); (2) follows from Theorems 10(2) and 13 (we have $t_k = x_k$); (3) follows from Theorem 14.

Remark. (1) If we examine the functions being of zero value at b, we get similar results.

(2) We can also examine the functions having zero values at both a and b. We assume that $Q(a) = Q(b) = 0$ for all $Q \in W$, every non-trivial polynomial $Q \in W$ has at most $n - 1$ zeros in (a, b), card $(B - \{a, b\}) \geqq n + 1$, $f(a) = 0$ in the case $a \in B$ and $f(b) = 0$ in the case $b \in B$. We take $\varkappa = \{a, b\}$, $I = J = (a, b)$ etc. Theorem 17 will hold also in this case, only the points $x_1 < \ldots < x_{n+1}$ will be in $B - \{a, b\} = B \cap (a, b)$.

C. Functions with Proportional Values at the End Points

Assumption. Let $S = R$, $n \in N$, $a, b \in R^*$, $a < b$, $d \in R$, $d \neq 0$. Let W be an n-dimensional subspace of $C\langle a, b\rangle$, let $Q(a) = d \cdot Q(b)$ for all $Q \in W$ and let each non-trivial polynomial $Q \in W$ have at most $n - 1$ zeros in $\langle a, b)$. Let $B \subset \langle a, b\rangle$ be compact, let card $B \geqq n + 2$ in the case $a, b \in B$ and card $B \geqq n + 1$ in the other cases. Let $f \in C(B)$ and $f(a) = d \cdot f(b)$ in the case $a, b \in B$. Let us denote $\mu = = \min_{Q \in W} \| Q - f \|$.

Remark. We take $D = \langle a, b\rangle, \mathcal{N} = \{\{x\}/x \in (a, b)\} \cup \{a, b\}, \varkappa = \emptyset, q(a, b) = d$. We have $\operatorname{card}(\mathcal{M} - \{\omega\}) \geqq n + 1$. W is an n-dimensional subspace of $Y(D, \mathcal{N}, \varkappa, q, R) = \{g \in R^{\langle a, b\rangle}/g(a) = d \cdot g(b)\}$ satisfying the Haar decomposition condition with respect to $D, \mathcal{N}, \varkappa$ (as $Q(b) = 0$ iff $Q(a) = 0$). The function f satisfies the requirements. Let us put $I = J = \langle a, b\rangle$, $\xi(s) \equiv s$. If $x_1 < \ldots < x_{n+1}$ are such points in $\langle a, b\rangle$ that $x_1 > a$ or $x_{n+1} < b$, then $x_1 \leqq x \leqq x_{n+1}$ and $x \sim x_k$ implies $x = x_k$. If $\alpha \in \mathcal{N}$, then $\alpha \cap I \neq \emptyset$. Since B is a representative subset, there exists a minimal set.

Theorem 19. (1) Let $P \in W$ have the following property: there exist points $x_1 < \ldots < x_{n+1}$ in B such that either $x_1 > a$ or $x_{n+1} < b$ and the numbers $P(x_k) - f(x_k)$ $(k = 1, \ldots, n + 1)$ alternate their signs. Then $\mu \geqq \mu(\{x_1, \ldots, x_{n+1}\}) \geqq \min_{k=1,\ldots,n+1} |P(x_k) - f(x_k)|$.

(2) Let $P \in W$. Then $\|P - f\| = \mu$ iff there exist points $x_1 < \ldots < x_{n+1}$ in B and a number $h \in \{-1, +1\}$ such that either $x_1 > a$ or $x_{n+1} < b$ and $P(x_k) - f(x_k) = h \cdot (-1)^k \cdot \|P - f\|$ for $k = 1, \ldots, n + 1$.

(3) There exists one and only one $P \in W$ such that $\|P - f\| = \mu$.

Proof. (1) follows from Theorem 10(1); (3) follows from Theorem 14. As for (2): Let $\|P - f\| = \mu > 0$. Let $x_1 < \ldots < x_{n+1}$ be the points in B which form a minimal set. Then either $x_1 > a$ or $x_{n+1} < b$ (else $x_1 \sim x_{n+1}$) and the assertion follows from Theorem 11(2c) (we take $t_k = x_k$).

Remark. Let $a, b \in B$. Let $P \in W$, $\|P - f\| = \mu > 0$; then the points $x_1 < \ldots < x_{n+1}$ of Theorem 19(2) are significant by Theorem 5(5). Hence, if $|d| < 1$, then $x_1 > a$; if $|d| > 1$, then $x_{n+1} < b$.

Theorem 20. We have $\operatorname{sign} d = (-1)^{n-1}$.

Proof (we give only the principle ideas). Let $Q_1, \ldots, Q_n$ form a basis of W, let us choose points $x_1, \ldots, x_{n-1}$ such that $a_1 < x_1 < \ldots < x_{n-1} < b$. For all $x \in \langle a, b\rangle$ let us put

$$Q(x) = \begin{vmatrix} Q_1(x) & Q_1(x_1) & \ldots & Q_1(x_{n-1}) \\ \vdots & & & \\ Q_n(x) & Q_n(x_1) & \ldots & Q_n(x_{n-1}) \end{vmatrix}.$$

Then $Q \in W$, $Q(x) \neq \emptyset$ for $x \in \langle a, b\rangle - \{x_1, \ldots, x_{n+1}\}$. We can prove that Q changes the sign at each point x_k: Let e.g. $Q(x) > 0$ for $0 < |x - x_k| \leqq u$. Let $T \in W$ be such that $T(x_k) = 1$ and $T(x_j) = 0$ for $j \neq k$. Then there exists $c > 0$ such that $Q - cT$ has two zeros in $(x_k - u, x_k) \cap (x_k, x_k + u)$: of course $x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n-1}$ are zeros of $Q - cT$, too, which is a contradiction. Hence $\operatorname{sign} Q(b) = (-1)^{n-1} \cdot \operatorname{sign} Q(a) = (-1)^{n-1} \cdot \operatorname{sign} d \cdot \operatorname{sign} Q(b)$, i.e. $\operatorname{sign} d = (-1)^{n-1}$.

D. Functions with Proportional values at m Points

Assumption. Let $S = R$, $n \in N$, $a, b \in R^*$, $a < b$, $m \in N$. Let $B \subset \langle a, b\rangle$ be compact, card $B \geqq n + 1$. Let us consider distinct points $z_1, \dots, z_m \in \langle a, b\rangle - B$ and non-zero numbers $d_2, \dots, d_m \in R$. Let W be an n-dimensional subspace of $C\langle a, b\rangle$, let $Q(z_k) = d_k \cdot Q(z_1)$ for $k = 2, \dots, m$ and for all $Q \in W$, let each non-trivial polynomial $Q \in W$ have at most $n - 1$ zeros in $\langle a, b\rangle - \{z_2, \dots, z_m\}$. Let $f \in C(B)$; let us denote $\mu = \min_{Q \in W} \| Q - f \|$.

Remark. We take $D = \langle a, b\rangle, \mathcal{N} = \{\{x\}/x \in \langle a, b\rangle - \{z_1, \dots, z_m\}\} \cup \{z_1, \dots, z_m\}$, $\varkappa = \emptyset$. Let us denote $d_1 = 1$ and $q(x_k, x_j) = d_k/d_j$ for $k, j = 1, \dots, m$. W is an n-dimensional subspace of $Y(D, \mathcal{N}, \varkappa, q, R) = \{g \in R^{\langle a, b\rangle}/g(z_k) = d_k \cdot g(z_1)$ for $k = 2, \dots, m\}$ satisfying the Haar decomposition condition with respect to $D, \mathcal{N}, \varkappa$ (as $Q(z_k) = 0$ implies $Q(z_1) = 0$). We have card $(\mathcal{M} - \{\omega\}) =$ card $B \geqq n + 1$. If $x, y \in B$ and $x \sim y$, then $x = y$, hence there is no condition for f. Let us put $I = J = \langle a, b\rangle$, $\xi(s) \equiv s$. If $\alpha \in \mathcal{N}$ and $\alpha \cap B \neq \emptyset$, then $\alpha \neq \{z_1, \dots, z_m\}$ and card $(\alpha \cap I) = 1$. As B is a representative subset, there exists a minimal set.

Theorem 21. All the three assertions hold also in this case, they are the same as in Theorem 17.

E. Generalized Even and Odd Functions

Assumption. Let $S = R, n \in N, 0 < a \leqq +\infty, d \in R, d \neq 0$. Let W be an n-dimensional subspace of $C\langle -a, a\rangle$, let $Q(-x) = d \cdot Q(x)$ for all $x \in (0, a\rangle$ and $Q(0) = 0$ for all $Q \in W$. Let every non-trivial polynomial $Q \in W$ have at most $n - 1$ zeros in $(0, a\rangle$. Let $B \subset \langle -a, a\rangle$ be compact, let card $(\{|x|/x \in B, x \neq 0\}) \geqq n + 1$. Let $f \in C(B)$ be such that $f(0) = 0$ in case $0 \in B$ and $f(-x) = d \cdot f(x)$ in case $x > 0$, $x \in B$, $-x \in B$. Let us denote $\mu = \min_{Q \in W} \| Q - f \|$.

Remark. We take $D = \langle -a, a\rangle, \mathcal{N} = \{\{-x, x\}/x \in \langle 0, a\rangle\}$, $\varkappa = \{0\}$, $q(-x, x) = d$ for $0 < x \leqq a$. We have card $(\mathcal{M} - \{\omega\}) =$ card $(\{|x|/x \in B, x \neq 0\}) \geqq n + 1$. W is an n-dimensional subspace of $Y(D, \mathcal{N}, \varkappa, q, R) = \{g \in R^{\langle -a, a\rangle}/g(0) = 0$, $g(-x) = d \cdot g(x)$ for all $x \in (0, a\rangle\}$ satisfying the Haar decomposition condition with respect to $D, \mathcal{N}, \varkappa$. The function f satisfies the requirements. We can take either $I = J = (0, a\rangle$ or $I = J = \langle -a, 0)$, $\xi(s) \equiv s$. Then $\alpha \cap I = \emptyset$ and card $(\alpha \cap I) = 1$ for all $\alpha \in \mathcal{N} - \{\varkappa\}$. As B is a representative subset, there exists a minimal set.

Theorem 22. (1) Let $P \in W$ have this property: there exist points $x_1 < \dots < x_{n+1}$ in I, points $t_1, \dots, t_{n+1} \in B$ and $h \in \{-1, +1\}$ such that for $k = 1, \dots, n + 1$ we have either $t_k = x_k$ and $P(t_k) - f(t_k) = h \cdot (-1)^k \cdot d_k$, or $t_k = -x_k$ and $P(t_k) - f(t_k) = h \cdot (\operatorname{sign} d) \cdot (-1)^k \cdot d_k$, where $d_k \geqq 0$. Then $\mu \geqq \mu(\{x_1, \dots, x_{n+1}\}) \geqq \min_{k=1,\dots,n+1} d_k$.

(2) Let $P \in W$. Then $\| P - f \| = \mu$ iff there exist points $x_1 < \ldots < x_{n+1}$ in I, points $t_1, \ldots, t_{n+1} \in B$ and $h \in \{-1, +1\}$ such that for $k = 1, \ldots, n+1$ we have either $t_k = x_k$ and $P(t_k) - f(t_k) = h \,.\, (-1)^k \cdot \| P - f \|$, or $t_k = -x_k$ and $P(t_k) - f(t_k) = h \cdot (\operatorname{sign} d) \cdot (-1)^k \cdot \| P - f \|$.

(3) There exists one and only one $P \in W$ such that $\| P - f \| = \mu$.

Proof. (1) follows from Theorems 12(2) and 9(1); (2) follows from Theorem 13; (3) follows from Theorem 14.

Remark. Let $P \in W$, let $x \in \langle -a, a \rangle$ be such a point that $x \in B$, $-x \in B$ and $| P(x) - f(x) | = \| P - f \| > 0$. If $| d | < 1$, then $x > 0$; if $| d | > 1$, then $x < 0$.

Remark. (1) If $d = -1$, then the functions are odd.

(2) Let $d = 1$. We may change the assumptions in this way: we omit the assumptions $Q(0) = 0$ and $f(0) = 0$ and assume that every non-trivial polynomial $Q \in W$ has at most $n - 1$ zeros in $\langle 0, a \rangle$. Then we take $\varkappa = \emptyset$, $I = \langle 0, a \rangle$ or $I = \langle -a, 0 \rangle$ etc. Then the functions are even and all the three assertions of Theorem 22 hold also in this case. We can substitute sign $d = 1$ and simplify the assertions (1) and (2).

F. The Approximation on a Generalized Arc

Assumption. Let $S = R$, $n \in N$, $a, b \in R^*$, $a < b$. Let $\xi(s)$ be a one-one mapping of $\langle a, b \rangle$ onto some set I. Let W be an n-dimensional subspace of R^I, let every non-trivial polynomial $Q \in W$ have at most $n - 1$ zeros in I and for every $Q \in W$ let the function $Q[\xi(s)]$ be continuous in $\langle a, b \rangle$. Let $B \subset I$ be such a subset that $\xi^{-1}(B)$ is a compact subset of $\langle a, b \rangle$, let card $B \geqq n + 1$. Let $f \in R^B$ be such a function that $f[\xi(s)]$ is continuous in $\xi^{-1}(B)$. Let us denote $\mu = \min\limits_{Q \in W} \| Q - f \|$.

Remark. We take $D = I$, $\mathcal{N} = \{\{x\}/x \in I\}$, $\varkappa = \emptyset$; q is defined implicitely. We have card $(\mathcal{M} - \{\omega\}) = \operatorname{card} B \geqq n + 1$. W is an n-dimensional subspace of $Y(D, \mathcal{N}, \varkappa, q, R) = R^I$ satisfying the Haar decomposition condition with respect to $D, \mathcal{N}, \varkappa$. We take $J = \langle a, b \rangle$, we have card $(\alpha \cap I) = 1$ for all $\alpha \in \mathcal{N}$.

We transfer the topology from $\langle a, b \rangle$ onto I by means of the mapping ξ. Then each $Q \in W$ is continuous in I, B is compact and f is continuous in B. B is a representative subset and consequently there exists a minimal set.

Theorem 23. (1) Let $P \in W$ have this property: there exist points $x_1, \ldots, x_{n+1} \in B$ such that $\xi^{-1}(x_1) < \ldots < \xi^{-1}(x_{n+1})$ and the numbers $P(x_k) - f(x_k)$ $(k = 1, \ldots, n+1)$ alternate their signs. Then $\mu \geqq \mu(\{x_1, \ldots, x_{n+1}\}) \geqq \min\limits_{k=1,\ldots,n+1} | P(x_k) - f(x_k) |$.

(2) Let $P \in W$. Then $\| P - f \| = \mu$ iff there exist points $x_1, \ldots, x_{n+1} \in B$ and $h \in \{-1, +1\}$ such that $\xi^{-1}(x_1) < \ldots < \xi^{-1}(x_{n+1})$ and $P(x_k) - f(x_k) = h \cdot (-1)^k \times \times \| P - f \|$ for $k = 1, \ldots, n+1$.

(3) There exists one and only one $P \in W$ such that $\|P - f\| = \mu$.

Proof is the same as that of Theorem 17.

Remark. Any theory formulated for an interval can be transferred in this way onto a generalized arc.

G. Trigonometric Polynomials

Theorem 24. (1) Let $a_0, \ldots, a_m, b_1, \ldots, b_m \in R$ be not all zero. Then the trigonometric polynomial $Q(x) = a_0 + \sum_{k=1}^{m} (a_k \cdot \cos kx + b_k \cdot \sin kx)$ has at most $2m$ zeros in $\langle 0,2\pi)$.

(2) Let $a_0, \ldots, a_m \in R$ be not all zero. Then the even trigonometric polynomial $Q(x) = \sum_{k=0}^{m} a_k \cdot \cos kx$ has at most m zeros in $\langle 0, \pi\rangle$.

(3) Let $b_1, \ldots, b_m \in R$ be not all zero. Then the odd trigonometric polynomial $Q(x) = \sum_{k=1}^{m} b_k \cdot \sin kx$ has at most $m - 1$ zeros in $(0, \pi)$.

Proof. Theorem 24 is well-known and can be proved e.g. by expressing $Q(x)$ by means of algebraic polynomials; we have $Q(x) = e^{-imx} \cdot \sum_{k=0}^{2m} c_k \cdot (e^{ix})^k$ for (1), $Q(x) = \sum_{k=0}^{m} c_k \cdot (\cos x)^k$ for (2), $Q(x) = (\sin x) \cdot \sum_{k=0}^{m-1} c_k \cdot (\cos x)^k$ for (3).

Definition 3. Let the symbol $C_{2\pi}$ denote the system of all the continuous functions in R which are periodic with the period 2π.

Remark. Let W mean the system of all the trigonometric polynomials of at most the m-th degree, let $f \in C_{2\pi}$. We shall aproximate f by the polynomials $Q \in W$ in R. As $\max_{x \in R} |Q(x) - f(x)| = \max_{x \in \langle 0, 2\pi\rangle} |Q(x) - f(x)|$ for all $Q \in W$, we may investigate the problem only in $\langle 0, 2\pi\rangle$. This problem can be solved according to 0 § 4.C, if we take $a = 0$, $b = 2\pi$, $d = 1$, $B = \langle 0, 2\pi\rangle$, $n = 2m + 1 = \dim W$.

Theorem 25. (1) Let $P \in W$ have this property: there exist points $x_1 < \ldots < x_{2m+2}$ in $\langle 0, 2\pi)$ such that the numbers $P(x_k) - f(x_k)$ (k $= 1, \ldots, 2m + 2$) alternate their signs. Then $\mu \geqq \mu(\{x_1, \ldots, x_{2m+2}\}) \geqq \min_{k=1,\ldots,2m+2} |P(x_k) - f(x_k)|$.

(2) Let $P \in W$. Then $\|P - f\| = \mu$ iff there exist points $x_1 < \ldots < x_{2m+2}$ in $\langle 0, 2\pi)$ and $h \in \{-1, +1\}$ such that $P(x_k) - f(x_k) = h \cdot (-1)^k \cdot \|P - f\|$ for $k = 1, \ldots, 2m + 2$.

(3) There exists one and only one $P \in W$ such that $\|P - f\| = \mu$.

Proof. See Theorem 19. To the assertion (2): Theorem 19 admits also the case $x_1 > 0$, $x_{2m+2} = 2\pi$. Then we can put $x_0 = 0$; we have $P(x_0) - f(x_0) = P(x_{2m+2}) - f(x_{2m+2}) = h \cdot (-1)^{2m+2} \cdot \| P - f \| = h \cdot (-1)^0 \cdot \| P - f \|$. We can take $x_0, \dots, x_{2m+1}$ and renumerate them.

Remark. Let now W represent the system of all the even trigonometric polynomials of at most the m-th degree, let $f \in C_{2\pi}$ be even. We shall approximate f by the polynomials $Q \in W$ in R. As $\max_{x \in R} | Q(x) - f(x) | = \max_{x \in \langle 0, \pi \rangle} | Q(x) - f(x) |$ for all $Q \in W$, we may investigate the problem only on $\langle 0, \pi \rangle$. This problem can be solved according to § 4.A, if we take $a = 0$, $b = \pi$, $B = \langle 0, \pi \rangle$, $n = m + 1 = \dim W$. We shall not formulate the theorem since it would be the same as Theorem 17, if we substitute $B = \langle 0, \pi \rangle$, $n = m + 1$.

Remark. Let now W mean the system of all the odd trigonometric polynomials of at most the m-th degree, let $f \in C_{2\pi}$ be odd. We shall aproximate f by the polynomials $Q \in W$ in R. Since $\max_{x \in R} | Q(x) - f(x) | = \max_{x \in \langle 0, \pi \rangle} | Q(x) - f(x) |$ for all $Q \in W$, we can investigate the problem only in $\langle 0, \pi \rangle$. This problem was mentioned in Remark (2) of 0 § 4.B. We take $a = 0$, $b = \pi$, $B = \langle 0, \pi \rangle$, $n = m = \dim W$. We can formulate

Theorem 26. (1) Let $P \in W$ have this property: there exist points $x_1 < \dots < x_{m+1}$ in $(0, \pi)$ such that the numbers $P(x_k) - f(x_k)$ $(k = 1, \dots, m + 1)$ alternate their signs. Then $\mu \geqq \mu(\{x_1, \dots, x_{m+1}\}) \geqq \min_{k=1,\dots,m+1} | P(x_k) - f(x_k) |$.

(2) Let $P \in W$. Then $\| P - f \| = \mu$ iff there exist points $x_1 < \dots < x_{m+1}$ in $(0, \pi)$ and $h \in \{-1, +1\}$ such that $P(x_k) - f(x_k) = h \cdot (-1)^k \cdot \| P - f \|$ for $k = 1, \dots, m + 1$.

(3) There exists one and only one $P \in W$ such that $\| P - f \| = \mu$.

H. Another Approach to the Trigonometric Polynomials

Remark. Let W be the system of all the trigonometric polynomials of at most the m-th degree, let $f \in C_{2\pi}$. We shall approximate f by the polynomials $Q \in W$ in R, let $\mu = \min_{Q \in W} \| Q - f \|$.

Let us denote $n = 2m + 1$, $S = R$, $D = B = R$. Let us give a decomposition $\mathcal{N}$ of R by means of the equivalence on R: $x \sim y$ iff $\dfrac{x - y}{2\pi}$ is integer. Let $\varkappa = \emptyset$, $q(x, y) = 1$ for $x \sim y$.

W is an n-dimensional subspace of $Y(D, \mathcal{N}, \varkappa, q, R) = \{g \in R^R / g(x)$ is 2π-periodic in $R\}$ satisfying the Haar decomposition condition with respect to $D, \mathcal{N}, \varkappa$. The function f satisfies the requirements of the Assumption for § 3.

Let $I = J = \langle 0, 2\pi)$, $\xi(s) \equiv s$. We have card $(\alpha \cap I) = 1$ for all $\alpha \in \mathcal{N}$. The set

$A = \langle 0, 2\pi\rangle$ is a representative subset (e.g. by Theorem 15), hence there exists a minimal set.

We can now derive Theorem 25 once again; (1) follows from Theorems 12(2) and 10(1); (2) follows from (1) and from Theorem 11(2c) (since we may assume $M \subset \subset \langle 0, 2\pi)$ by Theorem 15 of [1]); (3) follows from Theorem 14.

Remark. In the same way we can investigate also the even trigonometric polynomials (we take $x \sim y$ iff either $\frac{x-y}{2\pi}$ or $\frac{x+y}{2\pi}$ is integer, $\varkappa = \emptyset$, $q(x, y) = 1$ for $x \sim y$, $n = m + 1$, $I = \langle 0, \pi\rangle$, $A = \langle 0, \pi\rangle$) and the odd trigonometric polynomials (we take $\varkappa = \{k\pi / k \text{ integer}\}$, $x \sim y$ iff either $x, y \in \varkappa$ or $x, y \in R - \varkappa$ and one of the numbers $\frac{x-y}{2\pi}$, $\frac{x+y}{2\pi}$ is integer; if $x, y \in R - \varkappa$ and $\frac{x-y}{2\pi}$ is integer, we take $q(x, y) = 1$; if $x, y \in R - \varkappa$ and $\frac{x+y}{2\pi}$ is integer, we take $q(x, y) = -1$; $n = m$, $I = (0, \pi)$, $A = \langle 0, \pi\rangle$).

Remark. We can investigate also the approximation on a subset, i.e. $B \subset R$, f is defined only on B. We can solve the problem if B has a representative subset A. The compactness of A may be investigated with respect to the usual topology on R, but we may introduce also another topology on R and investigate the compactness of A with respect to it.

Remark. Let U be the system of all the trigonometric polynomials of at most the m-th degree, $g \in C_{2\pi}$. Let $h(x)$ be a continuous positive real funciton in R. We can approximate the function $f = hg$ by the polynomials of $\{hQ / Q \in U\}$ if we are able to prove the existence of a representative subset (e.g. for $h(x) = e^{-x}$).

5. THE HAAR NODE CONDITION

Remark. In what follows we shall consider functions having common zeros (or values) at several points. We distinguish two types of the zeros according to the behaviour of the function in a neighbourhood of the zero point. We consider only real functions.

Definition 4. Let g be a real function defined in some set $I \subset R^*$, let $z \in I$ be a point.

(1) The point z will be called *a cross zero* of the function g iff there exists a number $u > 0$ such that $\langle z - u, z + u\rangle \subset I$, $g(z) = 0$ and either $g(x) < 0$ for $x \in \langle z - u, z)$ and $g(x) > 0$ for $x \in (z, z + u\rangle$, or $g(x) > 0$ for $x \in \langle z - u, z)$ and $g(x) < 0$ for $x \in (z, z + u\rangle$.

(2) The point z is called *a touch zero* of the function g iff there exists a number $u > 0$ such that $\langle z - u, z + u\rangle \subset I$, $g(z) = 0$ and either $g(x) > 0$ for $0 < |x - z| \leqq \leqq u$ or $g(x) < 0$ for $0 < |x - z| \leqq u$.

Remark. If z is a cross zero or a touch zero of g, then z is inside I and g has no other zeros in some neighbourhood of z.

Theorem 27. (1) Let g be defined (at least) in $\langle a, b\rangle$, let $a < z < b$. Let $g(z) = 0$ and $g(x) \neq 0$ for all $x \in \langle a, z) \cup (z, b\rangle$. Suppose that if either $a \leqq c \leqq d < z$ or $z < c \leqq d \leqq b$, then $g(c) \cdot g(d) > 0$. Then z is either a cross zero or a touch zero of g, moreover, $g(x)$ has a constant sign in $\langle a, z)$ and a constant sign in $(z, b\rangle$.

(2) Let g be continuous in $\langle a, b\rangle$, let $a < z < b$. Let $g(z) = 0$ and $g(x) \neq 0$ for all $x \in \langle a, z) \cup (z, b\rangle$. Then the assertions of (1) hold.

(3) Let g have derivatives up to the r-th order at a point z ($r \in N$). Let $g(z) = g'(z) = \ldots = g^{(r-1)}(z) = 0$, $g^{(r)}(z) \neq 0$. If r is odd (even), then z is a cross (touch) zero of g.

Proof. Assertions (1) and (2) are obvious, (3) follows immediately from a well-known theorem.

Assumption (for § 5.). Let $n \in N$, $m \in N_0$, let $I \subset R^*$ be an interval. Suppose that there are given points $z_1 < \ldots > z_m$ in I (called nodes) and numbers $t_1, \ldots, t_m \in \{1, 2\}$. Let us denote $I' = I - \{z_1, \ldots, z_m\}$.

Remark. Let us denote $A(I) = \{g \in R^I / \text{if } \langle c, d\rangle \subset I \text{ and } g(x) \neq 0 \text{ for all } x \in \langle c, d\rangle, \text{ then } g(c) \cdot g(d) > 0\}$, $X(I) = \{g \in A(I) / g(z_1) = \ldots = g(z_m) = 0\}$.

(1) We have $C(I) \subset A(I)$.

(2) Let $g \in A(I)$. Then $g(x)$ keeps the sign in each subinterval of I, in which $g(x) \neq 0$.

(3) Let $g \in A(I)$ and let z be an isolated zero of g (inside I). Then z is either a cross zero or a touch zero of g.

Definition 5. Let $g \in X(I)$. A point $x \in I$ will be called *an additional zero* of g iff either

(1) $x \in I'$ and $g(x) = 0$; or

(2) $x = z_k$, $t_k = 1$ and z_k is a touch zero of g (inside I); or

(3) $x = z_k$, $t_k = 2$ and z_k is a cross zero of g (inside I).

Remark. If $c, d \in I'$ and $c \leqq d$, then $t(c, d)$ will denote the sum of all t_k for such k that $c < z_k < d$.

Remark. If $x_1 \leqq \ldots \leqq x_r$ are points in I' ($r \geqq 2$), then $t(x_1, x_r) = t(x_1, x_2) + \ldots + t(x_{r-1}, x_r)$.

Theorem 28. Let $g \in X(I)$, let $c \leqq d$ be such points in I' that the function g has no additional zero in $\langle c, d\rangle$. Then $\operatorname{sign} g(d) = (-1)^{t(c,d)} \cdot \operatorname{sign} g(c) \neq 0$.

Proof. Suppose that there are exactly $z_p < \ldots < z_q$ in $\langle c, d\rangle$. The function g keeps the sign in the intervals $\langle c, z_p)$, (z_p, z_{p+1}), $\ldots$, (z_{q-1}, z_q), $(z_q, d\rangle$. Let $k \in$

$\in \{p, \dots, q\}$; if $t_k = 1$, then z_k is a cross zero of g; if $t_k = 2$, then z_k is a touch zero of g. If the number of such $k \in \{p, \dots, q\}$ for which $t_k = 1$ is odd (even), then $g(c) \cdot g(d) < 0$ $(g(c) \cdot g(d) > 0)$ and $t(c, d) = t_p + \dots + t_q$ is odd (even), hence the assertion holds.

Remark. The numbers t_k are of the following meaning. Suppose that $g \in X(I)$ and z_k is an isolated zero of g (inside I). The number t_k determines the behaviour of g in some neighbourhood of z_k which is necessary for z_k to be an „allowed" zero of g (i.e. which is not additional). For $t_k = 1$ we allow a cross zero, for $t_k = 2$ we allow a touch zero; if z_k is a zero of the other type, then z_k is called an additional zero of g. If z_k is an end point of the interval I, then t_k has no meaning.

Definition 6. Let W be an n-dimensional subspace of $X(I)$. We shal say that W satisfies the Haar node condition (with respect to I, z_k, t_k) iff every non-trivial polynomial $Q \in W$ has at most $n - 1$ additional zeros in I.

Remark. If $m = 0$, then we have the classical Haar condition.

Theorem 29. Let W be an n-dimensional subspace of $X(I)$ satisfying the Haar node condition. Let $Q_1, \dots, Q_n$ form a basis of W.

(1) If $a_1, \dots, a_n \in R$ are not all zero, then $\sum_{k=1}^{n} a_k Q_k$ has at most $n - 1$ additional zeros in I.

(2) If $x_1, \dots, x_n \in I'$ are distinct, then $\det Q_k(x_j) \neq 0$ and $\dim_{\{x_1, \dots, x_n\}} W = n$.

(3) If $x_1, \dots, x_n \in I'$ are distinct and numbers $y_1, \dots, y_n \in R$ are arbitrary, then there exists one and only one $P \in W$ such that $P(x_k) = y_k$ for $k = 1, \dots, n$.

Proof. All the assertions are obvious.

Theorem 30. Let W be an n-dimensional subspace of $X(I)$ satisfying the Haar node condition, let $Q_1, \dots, Q_n$ form a basis of W. Let $x_1 < \dots < x_{n+1}$ be points in I'. For $k = 1, \dots, n + 1$ let us denote

$$C_k = (-1)^{k-1} \cdot \begin{vmatrix} Q_1(x_1) \dots Q_1(x_{k-1}) \; Q_1(x_{k+1}) \dots Q_1(x_{n+1}) \\ \vdots \\ Q_n(x_1) \dots Q_n(x_{k-1}) \; Q_n(x_{k+1}) \dots Q_n(x_{n+1}) \end{vmatrix}$$

The sign $C_k = (-1)^{t(x_1, x_k)+k-1}$. sign $C_1 \neq 0$ for $k = 1, \dots, n + 1$.

Proof. Let $k \in \{1, \dots, n\}$. For all $x \in I$ let us put

$$Q(x) = \begin{vmatrix} Q_1(x_1) \dots Q_1(x_{k-1}) \; Q_1(x) \; Q_1(x_{k+2}) \dots Q_1(x_{n+1}) \\ \vdots \\ Q_n(x_1) \dots Q_n(x_{k-1}) \; Q_n(x) \; Q_n(x_{k+2}) \dots Q_n(x_{n+1}) \end{vmatrix}$$

We have $Q \in W$ and $Q \not\equiv 0$. Since Q has additional zeros $x_1, \ldots, x_{k-1}, x_{k+2}, \ldots, x_{n+1}$, consequently Q has no other additional zero, namely Q has no additional zero in $\langle x_k, x_{k+1} \rangle$. By Theorem 28, we have sign $Q(x_{k+1}) = (-1)^{t(x_k, x_{k+1})} \cdot$ sign $Q(x_k) \neq 0$. As $C_k = (-1)^{k-1} Q(x_{k+1})$ and $C_{k+1} = (-1)^k Q(x_k)$, we have sign $C_{k+1} = (-1)^{t(x_k, x_{k+1})+1} \cdot$ sign C_k. Hence sign $C_k = (-1)^{t(x_{k-1}, x_k)+1} \cdot \ldots \cdot (-1)^{t(x_1, x_2)+1} \times$ $\times$ sign $C_1 = (-1)^{t(x_1, x_k)+k-1} \cdot$ sign C_1 for $k = 1, \ldots, n+1$.

6. THE APPROXIMATION

Assumption (for § 6.). Let $n \in N$, $m \in N_0$, let $I \subset R^*$ be an interval. Suppose that there are given points $z_1 < \ldots < z_m$ in I and numbers $t_1, \ldots, t_m \in \{1, 2\}$. Let $I' = I - \{z_1, \ldots, z_m\}$.

Let W be an n-dimensional subspace of $X(I)$ satisfying the Haar node condition. Let $Q_1, \ldots, Q_n$ form a basis of W.

Let $B \neq \emptyset$ be a subset of I, let us denote $B' = B - \{z_1, \ldots, z_m\}$. Let $f \in R^B$ be such a function that if $z_k \in B$, then $f(z_k) = 0$.

Remark. Let us denote $V = \{Q_B / Q \in W\}$. Then V is a subspace of R^B, $\dim V = \dim_B W \leqq n$. We shal approximate f by the polynomials $Q \in V$ on the set B; let us denote $\mu = \min_{Q \in W} \| Q - f \|$. If $Q \in W$, we denote $\| Q - f \| = \sup_{x \in B} | Q(x) - f(x) | = \| Q_B - f \|$; we have $\mu = \min_{Q \in W} \| Q - f \|$.

Theorem 31. (1) If card $B' \leqq n$, then $\mu = 0$.

(2) If card $B' > n$, then $\dim V = n$ and the restrictions of $Q_1, \ldots, Q_n$ to the set B form a basis of V.

Proof. (1) follows from Theorem 29(3), (2) follows from Theorem 29(2).

Theorem 32. Let $P \in W$ have this property: there exist points $x_1 < \ldots < x_{n+1}$ in B' and a number $h \in \{-1, +1\}$ such that for $k = 1, \ldots, n+1$ we have

$$P(x_k) - f(x_k) = h \,.\, (-1)^{t(x_1, x_k)+k} \cdot d_k, \qquad \text{where} \qquad d_k \geqq 0.$$

(1) Let us define $C_1, \ldots, C_{n+1}$ as in Theorem 30. Then $\mu \geqq \mu(\{x_1, \ldots, x_{n+1}\}) = \dfrac{\Sigma | C_k | \cdot | P(x_k) - f(x_k) |}{\Sigma | C_k |} \geqq \min_{k=1,\ldots,n+1} | P(x_k) - f(x_k) |$.

(2) If $| P(x_k) - f(x_k) | = \| P - f \|$ for $k = 1, \ldots, n+1$, then $\| P - f \| = \mu$.

Proof. (1) We have $\dim_{\{x_1, \ldots, x_{n+1}\}} V = \dim_{\{x_1, \ldots, x_{n+1}\}} W = n$ by Theorem 29(2). For $k = 1, \ldots, n+1$ we have $(-h \cdot \text{sign } C_1) \cdot C_k \cdot [P(x_k) - f(x_k)] = -h \cdot \text{sign } C_1 \cdot | C_k | \cdot (-1)^{t(x_1, x_k)+k-1} \cdot \text{sign } C_1 \cdot h \cdot (-1)^{t(x_1, x_k)+k} \cdot d_k = | C_k | \cdot d_k \geqq 0$ by Theorem 30. Now the assertion follows from Theorem 28(6) of [1].

(2) follows from (1).

Remark. If B is compact and if all the polynomials $Q \in W$ and the function f are continuous on B, then B is a representative subset and there exists a minimal set $M \subset \subset B$. If $M \neq \emptyset$, then $\mu > 0$ and necessarily card $B' \geqq n + 1$ by Theorem 31(1).

Theorem 33. (1) Let $M \neq \emptyset$ be a minimal set. Then $M \subset B'$, card $M = n + 1$ and $\dim_M V = \dim_M W = n$.

(2) Suppose that there exists a minimal set M and card $B' \geqq n$. Then there exists one and only one $P \in W$ such that $\|P - f\| = \mu$.

Proof. (1) Let us admit that $z_k \in M$. Then $\|Q - f\|_{M - \{z_k\}} = \|Q - f\|_M$ for all $Q \in V$, hence $\mu(M - \{z_k\}) = \mu(M)$, which is a contradiction; hence $M \subset B'$. Let us admit card $M \leqq n$, then we have $\mu = \mu(M) = 0$ by Theorem 29(3), which is a contradiction; hence card $M = n + 1$. By Theorem 29(2), we have $\dim_M V = = \dim_M W = n$.

(2) By Theorem 29(3), two distinct polynomials of W cannot coincide on B'. If $M = \emptyset$, then $\mu = 0, f \in V$ and the assertion is evident. If $M \neq \emptyset$, then $\dim_M V =$ by (1) and the assertion follows from Theorem 20(3) of [1].

Theorem 34. Let $M = \{x_1, \ldots, x_{n+1}\}$ be a minimal set, we can assume $x_1 < \ldots \ldots < x_{n+1}$. Let $P \in W$ be such a polynomial that $\|P - f\| = \mu$. Then there exists a number $h \in \{-1, +1\}$ such that $P(x_k) - f(x_k) = h \cdot (-1)^{t(x_1, x_k) + k} \cdot \|P - f\|$ for $k = 1, \ldots, n + 1$.

Proof. By Theorem 31(2) of [1], there exists $a \in \{-1, +1\}$ such that for $k = = 1, \ldots, n + 1$ we have $P(x_k) - f(x_k) = a \cdot \operatorname{sign} C_k \cdot \|P - f\| = a \cdot (-1)^{t(x_1, x_k) + k - 1} \times \times \operatorname{sign} C_1 \cdot \|P - f\|$; we take $h = -a \cdot \operatorname{sign} C_1$.

Theorem 35. Let card $B' \geqq n + 1$. Suppose that there exists a minimal set, let $P \in W$. Then $\|P - f\| = \mu$ iff there exist points $x_1 < \ldots < x_{n+1}$ in B' and a number $h \in \{-1, +1\}$ such that $P(x_k) - f(x_k) = h \cdot (-1)^{t(x_1, x_k) + k} \cdot \|P - f\|$ for $k = = 1, \ldots, n + 1$.

Proof. If the latter condition is fulfilled, then we have $\|P - f\| = \mu$ by Theorem 32(2).

Let $\|P - f\| = \mu$. If $\mu = 0$, then the assertion is trivial. If $\mu > 0$, then the assertion follows from Theorem 34.

Remark. The theory given in § 5 and § 6 corresponds to that of § 2 and § 3. The most important common fact is that we can find some relations between the signs of the numbers $C_1, \ldots, C_{n+1}$. If we consider any other properties of the polynomials $Q \in W$ which enable us to find some similar relations, we can derive all the theory analogous to these two theories. E.g., it is possible to construct a theory which is a common generalization of these two theories (such a theory is given in [4]).

7. THE CONNECTION WITH THE CLASSICAL HAAR CONDITION

Assumption (for § 7.). Let $n \in N$, $m \in N_0$, let $I \subset R^*$ be an interval. Let $z_1 < \ldots < z_m$ be points in I, let $I' = I - \{z_1, \ldots, z_m\}$.

Let Z be an $(n + m)$-dimensional subspace of $A(I)$ satisfying the classical Haar condition on I. Let $B \neq \emptyset$ be a subset of I, let us denote $B' = B - \{z_1, \ldots, z_m\}$. Let $w_1, \ldots, w_m \in R$ be fixed numbers.

Let $f \in R^B$ be such a function that if $z_k \in B$, then $f(z_k) = w_k$ (for $k = 1, \ldots, m$).

Remark. We take $t_1 = \ldots = t_m = 1$. A point $x \in I$ is an additional zero of $g \in X(I)$ iff either $x \in I'$ and $g(x) = 0$ or $x = z_k$ and z_k is a touch zero of g.

If $c, d \in I'$ and $c \leqq d$, then $t(c, d)$ is equal to the number of z_k in (c, d).

Remark. Let us denote $U = \{Q \in Z / Q(z_1) = \ldots = Q(z_m) = 0\}$, $W = \{Q \in Z / Q(z_k) = w_k$ for $k = 1, \ldots, m\}$.

Theorem 36. *U is an n-dimensicnal subspace of $X(I)$ satisfying the Haar node condition.*

Proof. U is a subspace of $X(I)$. Let us choose arbitrary distinct points $x_1, \ldots, x_n \in I'$. By the Haar condition (see Lemma (4) in § 2.4. of [1] where we take $n + m$ instead of n), there exist $Q_1, \ldots, Q_n \in Z$ such that for $k = 1, \ldots, n$ we have $Q_k(x_k) = 1$, $Q_k(x_j) = 0$ for $j = 1, \ldots, k - 1, k + 1, \ldots, n$ and $Q_k(z_j) = 0$ for $j = 1, \ldots, m$. Then $Q_1, \ldots, Q_n$ are independent polynomials of U.

On the other hand, if $Q \in U$, then the polynomials Q and $\sum_{k=1}^{n} Q(x_k) \cdot Q_k$ have the same values at $m + n$ points $x_1, \ldots, x_n, z_1, \ldots, z_m$, hence $Q = \sum_{k=1}^{n} Q(x_k) \cdot Q_k$ (see Lemma (4) in § 2.4. of [1]). Therefore $Q_1, \ldots, Q_n$ form a basis of U, hence dim $U = n$.

Let $P \in U$, $P \not\equiv 0$. Let P have n additional zeros in I, let p of them (denoted by $u_1, \ldots, u_p$) be in $\{z_1, \ldots, z_m\}$ and $n - p$ of them (denoted by $v_1, \ldots, v_{n-p}$) be in I'. If $p = 0$, then P has $n + m$ zeros $v_1, \ldots, v_n, z_1, \ldots, z_m$, which is a contradiction. Hence $p \geqq 1$. Let $k \in \{1, \ldots, p\}$; then u_k is a touch zero of P (inside I). There exist points $a_k, b_k \in I$ with these properties:

(1) $a_k\ u_k\ b_k$ for $k = 1 \ldots, p$;

(2) P has a constant sign in $\langle a_k, b_k \rangle - \{u_k\}$ for $k = 1, \ldots, p$;

(3) if $j \neq k$, then $b_k < a_j$ or $b_j < a_k$.

There exists a polynomial $F \in Z$ such that $F(u_k) = \text{sign}\, P(a_k)$ for $k = 1, \ldots, p$, $F(v_k) = 0$ for $k = 1, \ldots, n - p$ and $F(z_k) = 0$ for $z_k \notin \{u_1, \ldots, u_p\}$. We can choose such $c > 0$ that $c \cdot |F(a_k)| < |P(a_k)|$ and $c \cdot |F(b_k)| < |P(b_k)|$ for $k = 1, \ldots, p$. Let us put $Q = P - cF$; we have $Q \in Z$, $Q \not\equiv 0$. We have sign $Q(a_k) =$ sign $Q(b_k) =$ sign $P(a_k)$, sign $Q(u_k) = -$sign $P(a_k)$ for $k = 1, \ldots, p$; hence Q has a zero in

(a_k, u_k) and a zero in (u_k, b_k). Moreover, $Q(v_k) = 0$ for $k = 1, \ldots, n - p$ and $Q(z_k) = 0$ for $z_k \notin \{u_1, \ldots, u_p\}$; all these zeros are distinct. Hence Q has $2p + (n - p) + (m - p) = m + n$ zeros in I, which is a contradiction; U satisfies the Haar node condition.

Remark. We shall approximate the function f by the polynomials $Q \in W$ in the set B. Let us denote $\mu = \inf_{Q \in W} \| Q - f \|$.

Theorem 37. Let us choose arbitrary fixed $T \in W$, let us denote $g = f - T_B$. Then we have:

(1) $g \in R^B$; if $z_k \in B$, then $g(z_k) = 0$ $(k = 1, \ldots, m)$.

(2) $W = \{Q + T / Q \in U\}$.

(3) Let $P \in W$ and $Q \in U$ be such that $P = Q + T$. Then $P(x) - f(x) = Q(x) - g(x)$ for all $x \in B$, hence $\| P - f \| = \| Q - g \|$.

(4) $\mu = \min_{Q \in U} \| Q - g \|$; hence there exists $P \in W$ such that $\| P - f \| = \mu$ and it may be written $\mu = \min_{Q \in W} \| Q - f \|$.

Corollary. All the assertions of § 6. hold if we write U and g instead of W and f. However, by Theorem 37(3), they hold also if we write W and f again (i.e. in the original formulation).

Remark. The meaning of the theory given in § 7. is the following: We approximate the function f in the set B only by the polynomials of Z which have the fixed given values $w_1, \ldots, w_m$ at the points $z_1, \ldots, z_m$. The numbers $w_1, \ldots, w_m$ must be given so that $f(z_k) = w_k$ in case $z_k \in B$.

§ 7. gives this theory only for the case when Z satisfies the Haar condition. It is possible to give such a theory also for the case when Z satisfies the Haar decomposition condition (see [4]).

A special case of the theory of § 7. was solved e.g. in [5].

8. THE APPROXIMATION WITH GIVEN DERIVATIVES

Assumption (for § 8.). Let $n \in N$, $m \in N_0$, let $I \subset R^*$ be an interval. Suppose that $z_1 < \ldots < z_m$ are points in I, let $I' = I - \{z_1, \ldots, z_m\}$.

Suppose that $r_1, \ldots, r_m \in N_0$ are such numbers that $r_k = 0$ if z_k is at the end of I. Let us denote $t_k = 1$ if r_k is even and $t_k = 2$ if r_k is odd. Let us denote $r = \sum_{k=1}^{m} (r_k + 1)$.

Let Z be an $(r + n)$-dimensional subspace of $A(I)$ with the following properties:

(1) If z_k is inside I, then every $Q \in Z$ has derivatives up to the order $r_k + 1$ at z_k.

(2) If we give

(a) $q \in N_0$ and points $u_1, \ldots, u_q \in I'$;

(b) numbers $s_1, \ldots, s_m \in N_0$ such that

(b1) if z_k is inside I, then $r_k \leqq s_k \leqq r_k + 1$;

(b2) if z_k is at the end of I, then $s_k = 0$;

(b3) $\sum_{k=1}^{m} (s_k + 1) + q = r + n$;

(c) numbers $w_1, \ldots, w_q, v_1^{(0)}, \ldots, v_1^{(s_1)}, \ldots, v_m^{(0)}, \ldots, v_m^{(s_m)} \in R$,

then there exists one and only one $P \in Z$ such that $P(u_k) = w_k$ for $k = 1, \ldots, q$ and $P^{(i)}(z_k) = v_k^{(i)}$ for $k = 1, \ldots, m$ and $i = 0, \ldots, s_k$.

Let us denote $U = \{Q \in Z / Q^{(i)}(z_k) = 0$ for $k = 1, \ldots, m$ and $i = 0, \ldots, r_k\}$.

Let $y_k^{(i)}$ ($k = 1, \ldots, m$ and $i = 0, \ldots, r_k$) be fixed real numbers; let us denote $W = \{Q \in Z / Q^{(i)}(z_k) = y_k^{(i)}$ for $k = 1, \ldots, m$ and $i = 0, \ldots, r_k\}$.

Let $B \neq \emptyset$ be a subset of I, let us denote $B' = B - \{z_1, \ldots, z_m\}$. Let $f \in R^B$ be such a function that if $z_k \in B$, then $f(z_k) = y_k^{(0)}$.

Theorem 38. U is an n-dimensional supspace of $X(I)$ satisfying the Haar node condition.

Proof. U is a subspace of $X(I)$. Let us choose arbitrary distinct points $x_1, \ldots, x_n \in I'$. Let us take $q = n$, $u_k = x_k$ for $k = 1, \ldots, n$ and $s_k = r_k$ for $k = 1, \ldots, m$; by (2), there exist $Q_1, \ldots, Q_n \in Z$ such that for $k = 1, \ldots, n$ we have $Q_k(x_k) = 1$, $Q_k(x_j) = 0$ for $j = 1, \ldots, k-1, k+1, \ldots, n$ and $Q_k^{(i)}(z_j) = 0$ for $j = 1, \ldots, m$ and $i = 0, \ldots, r_j$. Then $Q_1, \ldots, Q_n$ are independent polynomials of U.

On the other hand, if $Q \in U$, then the polynomials Q and $\sum_{k=1}^{n} Q(x_k) \,.\, Q_k$ have the same values at the points $x_1, \ldots, x_n$ and zero derivatives at each z_j up to the order r_j ($j = 1, \ldots, m$). By (2), we have $Q = \sum_{k=1}^{n} Q(x_k) \cdot Q_k$. Hence $Q_1, \ldots, Q_n$ form a basis of U and $\dim U = n$.

Let $P \in U$, $P \not\equiv 0$. Let P have n additional zeros in I and let p of them be in $\{z_1, \ldots, z_m\}$. Let us consider one of these z_k; it is inside I. Let us admit that $P^{(r_k+1)}(z_k) \neq 0$. Then for r_k odd (even) z_k is a touch (cross) zero of P (see Theorem 27(3)) and z_k is not an additional zero of P. Hence $P^{(r_k+1)}(z_k) = 0$.

We shall apply (2). If z_k is an additional zero of P, we put $s_k = r_k + 1$, otherwise $s_k = r_k$. Let $u_1, \ldots, u_{n-p}$ be the additional zeros of P in I'; we put $q = n - p$. We have $\sum_{k=1}^{m} (s_k + 1) + q = (r + p) + (n - p) = r + n$. We have $P(u_k) = 0$ for $k = 1, \ldots, q$ and $P^{(i)}(z_k) = 0$ for $k = 1, \ldots, m$ and $i = 1, \ldots, s_k$. By (2), there exists one and only one polynomial of Z with these properties. Hence $P \equiv 0$, which is a contradiction. U satisfies the Haar node condition.

Remark. We shall approximate the function f by the polynomials $Q \in W$ in the set B. Let us denote $\mu = \inf_{Q \in W} \| Q - f \|$.

Theorem 39. Since $W \neq \emptyset$ by (2), let us choose arbitrary fixed $T \in W$ and let us denote $g = f - T_B$. Then we have:

(1) $g \in R^B$; if $z_k \in B$, then $g(z_k) = 0$ $(k = 1, \ldots, m)$.

(2) $W = \{Q + T / Q \in U\}$.

(3) Let $P \in W$ and $Q \in U$ be such that $P = Q + T$. Then $P(x) - f(x) = Q(x) - f(x)$ for all $x \in B$, hence $\| P - f \| = \| Q - g \|$.

(4) $\mu = \min_{Q \in U} \| Q - g \|$; hence there exists $P \in W$ such that $\| P - f \| = \mu$ and it may be written $\mu = \min_{Q \in W} \| Q - f \|$.

Corollary. All the assertions of § 6. hold if we write U and g instead of W and f. However, by Theorem 39(3), they hold also if we write W and f again (i.e. in the original formulation).

Theorem 40. Let $I = R$. Let Z be the system of all the algebraic polynomials of at most the order $r + n - 1$. Then Z satisfies the Assumption for § 8.

Proof. (1) is evident, (2) follows from the well-known theorem of the interpolation theory.

REFERENCES

[1] J. Šmerk: *The Approximation of Functions in the Sense of Tchebychev I.* Archivum Mathematicum, XI, 1975.

[2] G. Meinardus: *Aproximace funkcí.* Prague 1968.

[3] R. E. Edwards: *Functional Analysis.* Moscow 1969 (Russian).

[4] J. Šmerk: *Rovnoměrná aproximace funkcí.* Thesis, Faculty of Science of JEP University, Brno 1972.

[5] S. Paszkowski: *On approximation with nodes.* Rozprawy matematyczne, XIV, Warszawa 1957.

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