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ON HOMEOMORPHIC TOPOLOGIES AND EQUIVALENT SET-SYSTEMS

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1. Introduction. Let P be a non-void set. Set-systems $\mathscr{S}_1, \mathscr{S}_2 \subset \exp P$ are said to be equivalent if there exists a permutation f of the set P (i.e. a one-to-one mapping of P onto itself) such that $\mathscr{S}_2 = \{f(X) : X \in \mathscr{S}_1\}$, (cf. [6] p. 323). To every topology u (in the sense of [2] or [5]) it can be assigned such a set-system $\mathscr{S}(u)$ so that topologies u, v are homeomorphic if and only if set-systems $\mathscr{S}(u)$, $\mathscr{S}(v)$ are equivalent in the above sense and $u \neq v$ implies $\mathscr{S}(u) \neq \mathscr{S}(v)$. The aim of this note is to give a constructive proof of the possibility of a non-trivial extension of the set-system valued mapping $\mathscr{S}$ onto a system of more general topologies which do not satisfy the so called U-axiom (the idempotency of closures).

2. Preliminaries. By a topological space we mean the so called Čech's topological space (see [1]), that is a pair (P, u), where P is a set and u a mapping of $\exp P$ into itself satisfying the following axioms:

$$1^0\ u\emptyset = \emptyset, \qquad 2^0\ X \subset uX \quad \text{for } X \subset P, \qquad 3^0\ X \subset Y \subset P \text{ implies } uX \subset uY.$$

If

$$4^0\ u(X \cup Y) = uX \cup uY, \qquad X \subset P, \qquad Y \subset P$$

holds then the topology u is called an A-topology and (P, u) an A-space (closure operations, closure spaces in the terminology of [2]). Topologies fulfiling axioms 1^0 through 3^0 and

$$5^0\ uuX = uX \qquad \text{for each} \qquad X \subset P \qquad \text{(U-axiom)}$$

are called U-topologies and corresponding spaces U-spaces. If axioms 1_0 through 5_0 are satisfied, we speak about AU-spaces, AU-topologies (topologies in the sense of [2] or [5]).

Denote by $\mathscr{C}(P)$ the lattice of all topologies on the set P (with respect to the ordering: $u, v \in \mathscr{C}(P)$, $u \leqq v$ if $uX \subset vY$ for each $X \subset P$, cf. [7] 1.2., 2.1.). For $u, v \in \mathscr{C}(P)$ there holds $(u \vee v)\, X = uX \cup vX$, $(u \wedge v)\, X = uX \cap vX$, $X \subset P$. Subsystems of $\mathscr{C}(P)$ of all A-topologies and U-topologies are denoted by $\mathscr{A}(P)$ and $\mathscr{U}(P)$ respectively

Let $\mathscr{C}^*(P)$ means the set of al totally irreducible elements in the lattice $(\mathscr{C}(P), \wedge, \vee)$, $\mathscr{C}^*_{\max}(P)$ the set of all maximal elements in $\mathscr{C}^*(P)$ and let $\mathscr{C}^*_0(P)$ be the system of all atoms in $(\mathscr{C}(P), \vee, \wedge)$. For $u, v \in \mathscr{C}(P)$, $u \cong v$ means that u, v are homeomorphic for $S_1 \subset \exp P$, $S_2 \subset \exp P$, $S_1 \sim S_2$ means that S_1, S_2 are equivalent in the above sense, i.e. there exists a permutation f of the set P such that $S_2 = \{f(X) : X \in S_1\}$. In this case we shall write $S_2 = f(S_1)$. Similarly, if $S_2 = \{f^{-1}(X) : X \in S_1\}$, then we write $S_2 = f^{-1}(S_1)$. The permutation group of the set P will be denoted by $\Phi(P)$. If X is a set, ϱ the equivalence relation on X then X/ϱ denotes a decomposition of X induced by ϱ. A system of topologies $\mathscr{X} \subset \mathscr{C}(P)$ is called topological if $u \in X$, $v \in \mathscr{C}(P)$, $v \cong u$ implies $v \in X$.

In [7], 2.3. is given a characterization of totally irreducible elements of $\mathscr{C}(P)$:

Proposition 1. *$u \in \mathscr{C}^*(P)$ iff there exists a non-void set $X_0 \subset P$ and a point $a \in P$ such that $uX = X \cup \{a\}$ for $X \subset P$, $X_0 \subset X$ and $uX = X$ otherwise.*

From here it follows immediately

Lemma 1. *The topology $u \in \mathscr{C}(P)$ belongs to $\mathscr{C}^*_{\max}(P)$ iff there exist points $a \in P$, $b \in P$ so that $uX = X \cup \{b\}$ if $a \in X$ and $uX = X$ otherwise.*

Evidently, $\mathscr{C}^*_{\max}(P)$ is a topological system.

3. Auxiliary assertions. Let P be a set of the cardinality at least 5, $u \in \mathscr{C}^*(P)$. By T_u will be denoted the set $X_0 \subset P$ and by a_u the point a both considered in proposition 1. §2. Put $\mathscr{C}^*_1(P) = \{u \in \mathscr{C}^*(P) : \text{card } T_u \geqq 2, \text{card } (P - T_n) \geqq 3\}$. Evidently, $\mathscr{C}^*_{\max}(P) = \{u \in \mathscr{C}^*(P) : \text{card } T_u = 1\}$. Put $\mathscr{T}_C(P) = \{u \vee v : u \in \mathscr{C}^*_1(P), v \in \mathscr{C}^*_{\max}(P), T_v = \{a_u\}, a_v \notin T_u\}$. It is easy to see that a topology w belongs to $\mathscr{T}_C(P)$ iff there exist a set $X_1 \subset P$ with card $X_1 \geqq 2$, card $(P - X_1) \geqq 3$ and points $x_1, x_2 \in P - X_1$, $x_1 \neq x_2$ such that $X \subset P$, $X_1 \subset X$ implies $wX = X \cup \{x_1\}$, $X \subset P$, $x_1 \in X$ implies $wX = X \cup \{x_2\}$ and $wX = X$ otherwise. If we denote by u, v topologies from $\mathscr{C}^*_1(P)$, $\mathscr{C}^*_{\max}(P)$ respectively such that $w = u \vee v$, then $X_1 = T_u$, $\{x_1\} = \{a_u\} = T_v$, $x_2 = a_v$. If $w \in \mathscr{T}_C(P)$, then by $T(w)$ will be denoted the set X_1 (considered above), by λ_w and b_w the above considered point x_1 and x_2 respectively. Hence, there is defined a one-to-one mapping $\mathbf{T}$ of the system $\mathscr{T}_C(P)$ into the set $2^P \times P \times P$ by the rule: $\mathbf{T}(u) = \langle \mathrm{T}(u), a_u, b_u \rangle$, for $u \in \mathscr{T}_C(P)$.

Further, denote by $\mathscr{A}_1(P)$ a system of all A-topologies on P satisfying the following condition:

There exists a pair X_1, X_2 of non-void disjoint subsets of the set P with $X_1 \cup X_2 = P$ such that

(i) $uX_1 = X_1 \cup X_2$,

(ii) $uX = X \cup X_1$ if $X \subset P$, $X \cap X_1 \neq \emptyset$,

(iii) $uX = X$ if $X \subset P$, $X \cap X_1 = \emptyset$ or $X_1 \cup X_2 \subset X$.

Clearly, $\mathscr{A}_1(P) \neq \emptyset$. To every A-topology u from the system $\mathscr{A}_1(P)$ is assigned a pair of sets X_1, X_2 with above described properties. We shall denote these sets by $L_1(u)$,

$L_2(u)$. It is easy to see that $\mathscr{C}^*_{\max}(P) \subsetneqq \mathscr{A}_1(P)$. Put $\mathscr{T}_A(P) = \{u \in \mathscr{A}_1(P) : \text{card}\, L_1(u) \geqq \geqq 2,\ P \neq L_1(u) \cup L_2(u)\}$ and finally $\mathscr{T}(P) = \mathscr{T}_A(P) \cup \mathscr{T}_C(P) \cup \mathscr{U}(P)$.

Let f be a permutation of the set P. By $\bar{f}$ will be denoted a mapping of $\mathscr{C}(P)$ into itself induced by the permutation f in the way: $\bar{f}(u)\, X = f^{-1}uf(X)$ for $u \in \mathscr{C}(P)$, $X \subset P$, (i.e. $\bar{f}(u)$ is a topology projectively generated by the mapping $f : P \to (P, u)$). Notice that in [7] 1.4. is $\bar{f}(u)$ denoted by $f \circ u$, where is also examined that for $f \in \Phi(P)$ is $\bar{f}$ an automorphism of the lattice $(\mathscr{C}(P), \vee, \wedge)$. It is clear that a system $\mathscr{S}(P) \subset \subset \mathscr{C}(P)$ is topological iff it is $\bar{f}$-stable for every permutation of the set P, i.e. $\bar{f}(\mathscr{S}(P)) \subset \subset \mathscr{S}(P)$ for each $f \in \Phi(P)$. A union of an arbitrary collection of topological systems is evidently a topological system.

Lemma 2. *Let P be a set. Systems $\mathscr{T}_A(P)$, $\mathscr{T}_C(P)$, $\mathscr{T}(P)$ are topological.*

Proof. Let $u \in \mathscr{T}_A(P)$, $f \in \Phi(P)$. It holds $f^{-1}(L_1(u)) \cap f^{-1}(L_2 u)) = \emptyset$. Let $X \subset P$ be such a set that $X \cap f^{-1}(L_1(u)) \neq \emptyset$. Since $\emptyset \neq f(X \cap f^{-1}(L_1(u))) = f(X) \cap L_1(u)$ and $u \in \mathscr{A}_1(P)$ we have $\bar{f}(u)\, X = f^{-1}uf(X) = f^{-1}(f(X) \cup L_1(u)) = f^{-1}uL_1(u) = = f^{-1}(L_1(u) \cup L_2(u)) = f^{-1}(L_1(u)) \cup f^{-1}(L_2(u))$. From $X \cap f^{-1}(L_1(u)) = \emptyset$ there follows $f(X) \cap L_1(u) = \emptyset$, i.e. $f^{-1}uf(X) = X$. We get that $L_i(\bar{f}(u)) = f^{-1}(L_i(u))$ $(i = 1, 2)$ thus it holds $\bar{f}(u) \in \mathscr{T}_A(P)$, i.e. the system $\mathscr{T}_A(P)$ is topological. It can be proved in a similar way that the system $\mathscr{T}_C(P)$ is topological hence the system $\mathscr{T}(P)$, which is a union of $\mathscr{T}_A(P)$, $\mathscr{T}_C(P)$ and $\mathscr{U}(P)$, is a topological system, too.

Lemma 3. *Let P be an infinite set. It holds* card $[(\mathscr{T}(P) \cap \mathscr{A}(P)) - \mathscr{U}(P)] = 2^{\text{card}\, P}$, card $[((\mathscr{T}(P) \cap \mathscr{A}(P)) - \mathscr{U}(P))/\cong] = \text{card}\, P$.

Proof. Let $u \in \mathscr{T}_A(P)$, $x \in L_1(u)$. Then $u\{x\} = L_1(u) \neq \{x\}$, $u^2\{x\} = uL_1(u) = = L_1(u) \cup L_2(u) \neq L_1(u)$, thus $u^2 \neq u$ and we have that $\mathscr{T}_A(P) \cap \mathscr{U}(P) = \emptyset$. Further, for arbitrary $u \in \mathscr{T}_C(P)$ and arbitrary $x \in T(u)$ there holds $u[\{x\} \cup (T(u) - \{x\})] = = uT(u) = T(u) \cup \{a_u\} \neq T(u) = \{x\} \cup (T(u) - \{x\}) = u\{x\} \cup u(T(u) - \{x\})$, thus $\mathscr{T}_A(P) \cap \mathscr{T}_C(P) = \emptyset$. It holds $(\mathscr{T}(P) \cap \mathscr{A}(P)) - \mathscr{U}(P) = \mathscr{T}_A(P)$. If we put $\mathscr{S} = = \{\langle X, Y\rangle \in \exp' P \times \exp' P : \text{card}\, X \geqq 2$ and $X \cap Y = \emptyset\}$, where $\exp' P = = \exp P - \{\emptyset\}$, then we have $\text{card}\, \mathscr{S} = 2^{\text{card}\, P} . 2^{\text{card}\, P} = 2^{\text{card}\, P}$ for $\text{card}\, P \geqq \aleph_0$. The mapping $\mathbf{L} : \mathscr{T}_A(P) \to \mathscr{S}$ defined by the rule $\mathbf{L}(u) = \langle L_1(u), L_2(u)\rangle$, for $u \in \mathscr{T}_A(P)$, is bijective, hence $\text{card}\, \mathscr{T}_A(P) = 2^{\text{card}\, P}$. Assign to every A-topology $u \in \mathscr{T}_A(P)$ a triad of cardinal numbers $\langle \mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3\rangle_u$, where $\mathfrak{m}_i = \text{card}\, L_i(u)$ for $i = 1, 2$ and $\mathfrak{m}_3 = = \text{card}(P - (L_1(u)) \cup L_2(u))$. Evidently, if $u, v \in \mathscr{T}_A(P)$ are nonhomeomorphic topologies then $\langle \mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3\rangle_u \neq \langle \mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3\rangle_v$, hence card $[\mathscr{T}_A(P)/\cong] \leqq \text{card} \times \times \{\langle \mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3\rangle : \mathfrak{m}_i \leqq \text{card}\, P,\ i = 1, 2, 3\} = \text{card}\, P$. On the other hand, if a, b are arbitrary points in P, $\mathscr{L} \subset 2^P$ is a set-system of the cardinality card P such that $X \in \mathscr{L}$ implies $a \notin X$ and $X \in \mathscr{L}$, $Y \in \mathscr{L}$, $X \neq Y$ implies $\text{card}\, X \neq \text{card}\, Y$ then $\text{card}\, P = \text{card}\, \mathscr{L} = \text{card}\, \{u \in \mathscr{T}_A(P) : L_1(u) \in \mathscr{L},\ L_2(u) = \{a\}\} \leqq \text{card}\, [\mathscr{T}_A(P)/\cong]$. Therefore it holds card $[((\mathscr{T}(P) \cap \mathscr{A}(P)) - \mathscr{U}(P))/\cong] = \text{card}\, P$.

Lemma 4. *Let P be an infinite set. It holds* card $[\mathscr{T}(P) - (\mathscr{A}(P) \cup \mathscr{U}(P))] = = 2^{\text{card}\, P}$, card $[(\mathscr{T}(P) - (\mathscr{A}(P) \cup \mathscr{U}(P)))/\cong] = \text{card}\, P$.

Proof. If $u \in \mathcal{T}_C(P)$ then $u \notin \mathcal{A}(P) \cup \mathcal{U}(P)$. Hence $\mathcal{T}_C(P) = \mathcal{T}(P) - (\mathcal{A}(P) \cup \cup \mathcal{U}(P))$. Now, similarly as in the proof of lemma 3, we put $\mathcal{S} = \{\langle X, x, y\rangle : X \subset P, x \in P, y \in P, \text{card } X \geqq 2, \text{card}(P - X) \geqq 3, x \notin X, y \notin X, x \neq y\}$. The mapping $\mathbf{T} : \mathcal{T}_C(P) \to \mathcal{S}$, defined by $\mathbf{T}(u) = \langle T(u), a_u, b_u\rangle$ for $u \in \mathcal{T}_C(P)$ is evidently bijective hence $\text{card } \mathcal{T}_C(P) = \text{card } \mathcal{S} = 2^{\text{card } P} . \text{card } P . \text{card } P = 2^{\text{card } P}$. Denote by $\mathcal{M}$ the set of pairs of cardinal numbers $\{\langle \mathfrak{m}_1, \mathfrak{m}_2\rangle : \mathfrak{m}_i \leqq \text{card } P \text{ for } i = 1, 2\}$. We have (similarly as in the proof of lemma 3) that $\text{card } (\mathcal{T}_C(P)/\cong) \leqq \text{card } \mathcal{M} = \text{card } P$. On the other hand, let $a \in P$, $b \in P$, $a \neq b$ be arbitrary but fixed points, $\mathcal{L}$ be a system of subsets $X \subset P$ such that $\text{card } \mathcal{L} = \text{card } P$, $X \in \mathcal{L}$ implies $\text{card } X \geqq 2$, $a \notin X$, $b \notin X$ and such that $X, Y \in \mathcal{L}$, $X \neq Y$ implies $\text{card } X \neq \text{card } Y$. Then we have $\text{card } P = = \text{card } \mathcal{L} \leqq \text{card } \{u \in \mathcal{T}_C(P) : u \text{ non} \cong v\} = \text{card } [\mathcal{T}_C(P)/\cong]$.

Lemma 5. *Let P be an infinite set, $\mathcal{X} \in [\mathcal{T}(P) - \mathcal{U}(P)]/\cong$. Then it holds* $\text{card } \mathcal{X} \geqq \geqq \text{card } P$.

Proof. $\mathcal{T}(P) - \mathcal{U}(P) = \mathcal{T}_A(P) \cup \mathcal{T}_C(P)$ with disjoint summands. Let $u \in \mathcal{T}_A(P)$. Since $\text{card } P \geqq \aleph_0$ there exists a set $X \in \{L_1(u), L_2(u), P - (L_1(u) \cup L_2(u))\}$ such that $\text{card } X = \text{card } P$. Suppose that $X = L_1(u)$. Let $a \in L_2(u)$. Put $T_u = \{v_x \in \mathcal{T}_A(P) : : L_1(v_x) = (L_1(u) - \{x\}) \cup \{a\}, L_2(v_x) = (L_2(u) - \{a\}) \cup \{x\}, x \in L_1(u)$. Then $\text{card } T_u = \text{card } L_1(u) = \text{card } P$ and every A-topology belonging to the system T_u is homeomorphic to the A-topology u. (If $v_{x_0} \in T_u$ then the permutation $f \in \Phi(P)$ defined by $f(x) = x$ for $x \in P$, $x_0 \neq x \neq a$ and $f(x_0) = a$, $f(a) = x_0$ is a homeomorphism of the space (P, u) onto the space (P, v_{x_0}). Thus $\mathcal{X} \in \mathcal{T}_A(P)/\cong$ implies $\text{card } \mathcal{X} \geqq \text{card } P$. In the same way we get that $\mathcal{X} \in \mathcal{T}_C(P)/\cong$ implies $\text{card } \mathcal{X} \geqq \text{card } P$, as well.

Let $u \in T(u)$. Denote by $D(u)$ the system of all subsets of P closures of which are proper subsets of P dense in the space (P, u), i.e. $D(u) = \{X \subset P : uX \neq P, u^2X = P\}$. Further, for $u \in \mathcal{T}(P)$ we put $F(u) = D(u) \cup C(u)$, where $C(u)$ is the system of all closed sets in the space (P, u). It is clear that $u \in u \in \mathcal{U}(P)$ iff $D(u) = \emptyset$, hence $F(u) = = C(u)$ for each $u \in \mathcal{U}(P)$. In futher development we shall deal with properties of the mapping $F : \mathcal{T}(P) \to \exp \exp P$. Cardinality of the set P is supposed at least 5.

Lemma 6. *Let $u \in \mathcal{T}_A(P)$, $v \in \mathcal{T}_C(P)$. Then $F(u)$ non $\sim$ $F(v)$.*

Proof. Admit that there exists such a permutation f of the set P that $f(F(u)) = = F(v)$. Let $x_0 \in L_1(u)$, $x_1 \in L_1(u)$ be arbitrary points, $x_0 \neq x_1$. Such points exist because of $\text{card } L_1(u) \geqq 2$. Since $\{x_0\} \notin F(u)$, $\{x_1\} \notin F(u)$ and f is a permutation of P, we have $\{f(x_0)\} \notin F(v)$, $\{f(x_1)\} \notin F(v)$. However, the only singleton which does not belong to the system $F(v)$ is $\{a_v\}$. This is a contradiction, hence systems $F(u)$, $F(v)$ are not equivalent.

Corollary. *Let $u \in \mathcal{T}_A(P)$, $v \in \mathcal{T}_C(P)$. Then $F(u) \neq F(v)$.*

Lemma 7. *Let $u \in \mathcal{U}(P)$, $v \in \mathcal{T}_A(P)$. Then $F(u)$ non $\sim$ $F(v)$.*

Proof. Admit that there exists a permutation $f \in \Phi(P)$ such that $f(F(u)) = F(v)$.

Let $x_0 \in L_1(v)$. Since $[P - (L_1(v) \cup L_2(v))] \cup \{x_0\} \in D(v) \subset F(v)$, there exists a set $X \in F(u) = C(u)$ such that $f(X) = [P - (L_1(v) \cup L_2(v))] \cup \{x_0\}$. Since $L_1(v) \cup \cup L_2(v) \in C(v) \subset F(v)$ there exists a set $Y \in C(u)$ with the property $f(Y) = L_1(v) \cup \cup L_2(v)$. There is $X \cap Y \in C(u)$ (an intersection of an arbitrary system of closed sets in a U-space is a closed set), thus $f(X \cap Y) \in F(v)$. From $\{x_0\} = f(X) \cap f(Y) = f(X \cap Y)$ and $\{x_0\} \notin C(v)$, $\{x_0\} \notin D(v)$ (because of $L_1(v) \cup L_2(v) \neq P$) we get a contradiction. Hence $F(u)$ non $\sim F(v)$.

Corollary. *Let $u \in \mathscr{U}(P)$, $v \in \mathscr{T}_A(P)$. Then $F(u) \neq F(v)$.*

Lemma 8. *Let $u \in \mathscr{U}(P)$, $v \in \mathscr{T}_C(P)$. Then $F(u)$ non $\sim F(v)$.*

Proof. Suppose similarly as above that $f(F(v)) = F(u)$ for some $f \in \Phi(P)$. Since $v[P - \{a_v, b_v\}] = P - \{b_v\}$; $v^2[P - \{a_v, b_v\}] = v[P - \{b_v\}] = P$, thus $P - \{a_v, b_v\} \in \in F(v)$, we have that $P - \{f(a_v), f(b_v)\} = f[P - \{a_v, b_v\}] \in F(u) = C(u)$. Further, $v(T(v) \cup \{a_v, b_v\}) = T(v) \cup \{a_v, b_v\}$ hence the set $f(T(v)) \cup \{f(a_v), f(b_v)\}$ is closed in the space (P, u). Then $f(T(v)) = [P - \{f(a_v), f(b_v)\}] \cap [f(T(v))) \cup \{f(a_v), f(b_v)\}]$ is a closed set in (P, u). From here $T(v) = f^{-1}f(T(v)) \in F(v)$. Since card $(P - T(v)) \geqq \geqq 3$, thus $T(v) \notin D(v)$ we have $T(v) \in C(v)$, i.e. $v(T(v)) = T(v)$, which is a contradiction. Hence $F(u)$ non $\sim F(v)$.

Corollary. *Let $u \in \mathscr{U}(P)$, $v \in \mathscr{T}_C(P)$. Then $F(u) \neq F(v)$.*

Lemma 9. *Let $u \in \mathscr{T}_A(P)$, $v \in \mathscr{T}_A(P)$, $u \neq v$. Then $F(u) \neq F(v)$.*

Proof. Let $u \in \mathscr{T}_A(P)$, $v \in \mathscr{T}_A(P)$ be different A-topologies. Then either $L_1(u) \neq \neq L_1(v)$ or $L_1(u) = L_1(v)$ and $L_2(u) \neq L_2(v)$. Suppose that $L_1(u) - L_1(v) \neq \emptyset$. Let $a \in L_1(u) - L_1(v)$. Then $v\{a\} = \{a\}$, thus $\{a\} \in C(v) \subset F(v)$. On the other hand, $u\{a\} = L_1(u) \neq \{a\}$, $u^2\{a\} = L_1(u) \cup L_2(u) \neq P$, thus $\{a\} \notin C(u) \cup D(u) = F(u)$. Hence $F(u) \neq F(v)$ in this case. The same result we get under the assumption $L_1(v) - - L_1(u) \neq \emptyset$. Now, let $L_1(u) = L_1(v)$, $L_2(u) \neq L_2(v)$. If $L_2(u) - L_2(v) \neq \emptyset$ we choose a point $a \in L_2(u) - L_2(v)$ and a point $b \in L_1(u)$. Put $X = P - \{a, b\}$. Then $uX = = L_1(u) \cup X = P - \{a\}$ and $u^2X = u(P - \{a\}) = P$, thus $X \in D(u) \subset F(u)$. Similarly $vX = P - \{a\}$. However $v^2X = v(P - \{a\}) = P - \{a\}$, thus $X \notin C(v) \cup D(v) = = F(v)$. Hence $F(u) \neq F(v)$ again. If $L_2(v) - L_2(u) \neq \emptyset$ then we get $F(u) \neq F(v)$ in a similar way as above.

Lemma 10. *Let $u \in \mathscr{T}_C(P)$, $v \in \mathscr{T}_C(P)$, $u \neq v$.* Then $F(u) \neq F(v)$.

Proof. Topologies $u \in \mathscr{T}_C(P)$, $v \in \mathscr{T}_C(P)$ are different iff exactly one of the following cases occurs:

(1,1) $\quad T(u) = T(v), \quad a_u \neq a_v, \quad b_u = b_v,$

(1,2) $\quad T(u) = T(v), \quad a_u \neq a_v, \quad b_u \neq b_v,$

(1,3) $\quad T(u) = T(v), \quad a_y = a_v, \quad b_u \neq b_v,$

(2,1) $\quad T(u) \neq T(v), \quad a_u = a_v, \quad b_u = b_v,$

(2,2) $\qquad T(u) \neq T(v), \quad a_u \neq a_v, \quad b_u = b_v,$

(2,3) $\qquad T(u) \neq T(v), \quad a_u \neq a_v, \quad b_u \neq b_v,$

(2,4) $\qquad T(u) \neq T(v), \quad a_u = a_v, \quad b_u \neq b_v.$

In cases (1,1), (1,2) there holds $u\{a_u\} = \{a_u, b_u\}$, $u^2\{a_u\} = \{a_u, b_u\} \neq P$, thus $\{a_u\} \notin F(u)$, $\{a_u\} \in F(v)$. If (1,3) occurs we have $u\{a_u, b_u\} = \{a_u, b_u\}$, $v\{a_u, b_u\} = \{a_u, b_u, b_v\}$. Since $v^2\{a_u, b_u\} = \{a_u, b_u, b_v\} \neq P$, it holds $\{a_u, b_u\} \in F(u)$, $\{a_u, b_v\} \notin F(v)$. Now, consider these possibilities: $T(u) \subsetneqq T(v)$, $T(v) \subsetneqq T(u)$, $T(u) \parallel T(v)$. If $T(u) \subsetneqq T(v)$ then in cases (2,1)–(2,4) there is $uT(u) = T(u) \cup \{a_u\} \neq T(u) = vT(u)$ and $u^2T(u) = T(u) \cup \{a_u, b_u\} \neq P$ for $\operatorname{card}(P - T(u)) \geqq 3$. Thus $T(u) \notin F(u)$, $T(u) \in F(v)$. If $T(v) \subsetneqq T(u)$ then similarly as above $uT(v) = T(v)$, $vT(v) = T(v) \cup \{a_v\} \neq T(v)$, $v^2T(v) = T(v) \cup \{a_v, b_v\} \neq P$, hence $T(v) \in F(u)$, $T(v) \notin F(v)$. Let $T(u) \parallel T(v)$. In cases (2,1), (2,4) it holds $vT(u) = T(u)$ for $a_v \notin T(u)$. However, $uT(u) \neq T(u)$, $u^2T(u) \neq P$, thus $T(u) \notin F(u)$, $T(u) \in F(v)$. In cases (2,2), (2,3) are a_u, a_v different. It can be shown, similarly as in cases (1,1), (1,2) that set-systems $F(u)$, $F(v)$ are also different. Therefore we get that set-systems $F(u)$, $F(v)$ are distinguished in all possible cases (1,1)–(2,4) by a suitable subset of P, q.e.d.

For the sake of completeness we formulate here the following well-known theorem:

Lemma 11. *Let u, v be U-topologies. Then $u \neq v$ implies $F(u) \neq F(v)$ and u, v are homeomorphic iff $F(u) \sim F(v)$.*

Lemma 12. *Let $u \in \mathscr{T}(P)$, $v \in \mathscr{T}(P)$* be homeomorphic topologies. *Then $F(u) \sim F(v)$.*

Proof. Let f be a homeomorphic mapping of the space (P, u) onto the space (P, v). Then $C(u) = f^{-1}(C(v))$, (it follows e.g. from [2] 16 C.2. and 16 C.4.). Let $u \in \mathscr{T}(P) - \mathscr{U}(P)$. Then $D(u) \neq \emptyset$. Let $X \in D(u)$ be an arbitrary set, $Y = f(X)$. Then $Y \in D(v)$ for $vY = vf(X) = f(uX) \neq f(P) = P$ and $v^2Y = v^2f(X) = f(u^2X) = f(P) = P$, thus $v \in \mathscr{T}(P) - \mathscr{U}(P)$. Since $X \in f^{-1}(Y)$ we have $D(u) \subset f^{-1}(D(v))$. Let $X \in f^{-1}D(v))$. There exists $Y \in D(v)$ such that $X = f^{-1}(Y)$. Since $uX = P$ implies $P = f(uX) = vf(X) = vY$, uX is a proper subset in P. Further, $u^2X = f^{-1}f(u^2X) = f^{-1}(v^2Y) = f^{-1}(P) = P$, thus $X \in D(u)$. Therefore $D(u) = f^{-1}(D(v))$ and we get $F(u) = f^{-1}(F(v))$, i.e. $F(u) \sim F(v)$.

Lemma 13. *Let $u \in \mathscr{T}_A(P)$, $v \in \mathscr{T}_A(P)$ be A-topologies with the property $F(u) \sim F(v)$. Then u, v are homeomorphic.*

Proof. Let $u \in \mathscr{T}_A(P)$, $v \in \mathscr{T}_A(P)$ be such A-topologies that $F(u) \sim F(v)$. Let $f \in \Phi(P)$ be a permutation with $F(u) = f(F(v))$, $x \in L_1(v)$. Since $\{x\} \notin C(v)$, $\{x\} \notin D(v)$, i.e. $\{x\} \notin F(v)$, it holds $\{f(x)\} \notin F(u)$. Since every point $a \in P$ with the property $u\{a\} \neq \{a\}$ belongs to $L_1(u)$, there is $f(x) \in L_1(u)$, hence $L_1(v) \subset f^{-1}(L_1(u))$. Let $y \in f^{-1}(L_1(u))$. If $x \in P$ is a point with $x = f(y)$, then $x \in L_1(u)$, thus $\{x\} \notin F(u)$. Then $\{y\} = f^{-1}\{f(y)\} = f^{-1}\{x\} \notin F(v)$ hence $y \in L_1(v)$. Therefore we get the equality $L_1(v) = f^{-1}(L_1(u))$. Now let $x_0 \in L_1(v)$. Put $M = [P - (L_1(u) \cup f(L_2(v)))] \cup \{f(x_0)\}$.

Since $[P - (L_1(v) \cup L_2(v))] \cup \{x_0\} \in D(v)$, we have that $M = f[P - (L_1(v) \cup \cup L_2(v)] \cup f\{x_0\} \in F(u)$. Since $f(x_0) \in L_1(u)$ it holds $uM \neq M$, $u^2M = P$ thus $f(L_2(v)) \subset L_2(u)$. Admit that $f(L_2(v)) \neq L_2(u)$. There is $L_1(u) \cup f(L_2(v)) = f(L_1(v) \cup \cup L_2(v)) \in F(u)$. On the other hand, $u(L_1(u) \cup f(L_2(v)) = L_1(u) \cup L_2(u) \neq P$, $u^2(L_1(u) \cup f(L_2(v))) = u(L_1(u) \cup L_2(u)) = L_1(u) \cup L_2(u) = u(L_1(u) \cup f(L_2(v))$, thus $L_1(u) \cup f(L_2(v)) \notin C(u) \cup D(u) = F(u)$, which is a contradiction. Therefore $L_2(u) = = f(L_2(v))$, i.e. $L_2(v) = f^{-1}(L_2(u))$. From equalities $L_i(v) = f^{-1}(L_i(u))$, $i = 1, 2$, it follows immediately that A-topologies u, v are homeomorphic.

Lemma 14. *Let $u \in \mathcal{T}_C(P)$, $v \in \mathcal{T}_C(P)$ be such topologies that $F(u) \sim F(v)$. Then u, v are homeomorphic.*

Proof. Let $u \in \mathcal{T}_C(P)$, $v \in \mathcal{T}_C(P)$ be topologies with the required property. There exists a permutation $f \in \Phi(P)$ such that $f(F(u)) = F(v)$. Since $x \in P$, $x \neq a_v$ implies $\{x\} \in F(v)$, it holds $f(a_u) = a_v$. From $\{a_u, b_u\} \in C(u) \subset F(u)$ it follows $\{a_v, f(b_u)\} = = f\{a_u, b_u\} = F(v)$. Since $X \in D(v)$ implies card $X \geqq 3$ for card $(P - T(v)) \geqq 3$, there holds $\{a_v, f(b_u)\} \in C(u)$. From here, with respect to $v\{a_v\} = \{a_v, b_v\}$, we get $\{a_v, b_v\} \subset v\{a_v, f(b_v)\} = \{a_v, f(b_u)\}$, hence $f(b_u) = b_v$. We are going to show that $f(T(u)) = T(v)$. Put $X = f(T(u))$, let $a \in T(u)$. There is $f(T(u) - \{a\}) = f(T(u)) - - \{f(a)\} = X - \{f(a)\}$, where $f(a) \in X$. Further, $u(T(u) - \{a\}) = T(u) - \{a\}$, i.e. $T(u) - \{a\} \in F(u)$, hence $X - \{f(a)\} \in F(v)$. Since the system $D(v)$ contains the only set $P - \{a_v, b_v\}$ and $a_u \neq a \neq b_u$, i.e. $a_v \neq f(a) \neq b_v$, it holds $P - \{a_v, b_v\} \neq \neq X - \{f(a)\}$, hence $X - \{f(a)\} \in C(v)$. It means that $v(X - \{f(a)\}) = X - \{f(a)\}$. Since $T(u) \notin F(u)$ it holds that $X \notin F(v)$ thus $vX \neq X$ and we have $T(v) \subset X = f(T(u))$ for $a_v \notin X$. Futher, $T(v)$ is not a subset of $X - \{f(a)\}$, hence $f(a) \in T(v)$. Since a was an arbitrary point from $T(u)$ we have $f(T(u)) \subset T(v)$, thus $f(T(u)) = T(v)$. Therefore u, v are homeomorphic.

4. Main theorem. Now, we summarize results obtained in the preceding paragraph. Let A, B be sets, ϱ be a binary relation on A, σ a binary relation on B. We say that the mapping $\varphi : A \to B$ is an embedding of a monorelational system (A, ϱ) into a monorelational system (B, σ) if φ is injective and for every pair of elements $a \in A$, $b \in A$, there holds $a \varrho b$ iff $f(a) \sigma f(b)$.

Theorem. *Let P be an infinite set. There exists a topological system $\mathcal{T}(P) \subset \mathcal{C}(P)$ with the property $\mathcal{U}(P) \subset \mathcal{T}(P)$ and a mapping $F : \mathcal{T}(P) \to \exp\exp P$ such that it holds:*

1° $\operatorname{card}[(\mathcal{T}(P) \cap \mathcal{A}(P)) - \mathcal{U}(P)] = \operatorname{card}[\mathcal{T}(P) - (\mathcal{A}(P) \cup \mathcal{U}(P))] = = 2^{\operatorname{card} P}$, $\operatorname{card}[((\mathcal{T}(P) \cap \mathcal{A}(P)) - \mathcal{U}(P))/\cong] = = \operatorname{card}[(\mathcal{T}(P) - (\mathcal{A}(P) \cup \mathcal{U}(P)))/\cong] = \operatorname{card} P$ *and* $\mathcal{X} \in [\mathcal{T}(P) - \mathcal{U}(P)]/\cong$ *implies* $\operatorname{card} \mathcal{X} \geqq \operatorname{card} P$.

2° *$F : \mathcal{T}(P) \to \exp\exp P$ is an embedding of the monorelational system $(\mathcal{T}(P), \cong)$, into the monorelational system* $(\exp\exp P, \sim)$.

3° *If u is a U-topology on P, then $F(u)$ is the system of all closed subsets of the space (P, u).*

Proof. Let P be an infinite set. Let symbols $\mathscr{T}(P)$ and $F : \mathscr{T}(P) \to \exp \exp P$ have the same meaning as in the beginning of the preceding paragraph. By lemma 2, the system $\mathscr{T}(P)$ is topological. Assertion 3° is cointained in lemmas 3, 4, 5. Assertion 2° follows from lemmas 6 to 14 and corollaries of lemmas 6, 7, 8. Assertion 3° is an immediate consequence of the definition of the mapping F, q.e.d.

Let (P, u) be a topological space which is not a U-space, v a U-modification of the topology u (see [1], 6.1). Then the system $C(u)$ of all closed sets of the space (P, u) coincides with the system $C(v)$ of all closed sets of the U-space (P, v) hence using subsets of P, closures of which are proper dense subsets of (P, u), we get different set-systems for u and v with above described properties.

Let us mention in this connection a problem formulated in [6] p. 328: Is it possible to assign to any Čech's space (P, u) the system $\mathscr{S}(u) \subset \exp P$ so that $u \neq v$ implies $\mathscr{S}(u) \neq \mathscr{S}(v)$ and u, v are homeomorphic iff $\mathscr{S}(u) \sim \mathscr{S}(v)$?

From results of paper [3] there follows the negative answer for card $\mathscr{C}(P) >$ card $\exp \exp P$ if card $P = 4$ or 5. However, if card $P \geqq 6$ the problem seems to be unsolved up to now. Note that it is not difficult to get a negative answer in the case when a homeomorphims of topological spaces and the equivalence of corresponding set-systems are given by the same permutation $f \in \Phi(P)$. Such a modification of the mentioned problem can be expressed in the language of category theory. Related problems are treated in [4]. Denote by $\mathfrak{A}(P)$ a category, objects of which are A-spaces (P, u), where P is a fixed set of the cardinality at least 4, $u \in \mathscr{A}(P)$ and morphisms are homeomorphisms. Let $\mathfrak{S}(P)$ denote a category with objects (P, S), $S \subset \exp P$, where P is a fixed set. Morphisms between (P, S) and (P, T) are permutations $f \in \Phi(P)$ such that $X \in S$ implies $f(X) \in T$. By $U_{\mathfrak{A}}$, $(U_{\mathfrak{S}})$ there will be denoted the forgetful functor from $\mathfrak{A}(P)$, $(\mathfrak{S}(P))$ into the category of sets.

Proposition 2. *Let P be a set,* card $P \geqq 3$, $F : \mathfrak{A}(P) \to \mathfrak{S}(P)$ *such a functor that* $U_{\mathfrak{S}} \circ F(f) = U_{\mathfrak{A}}(f)$ *for each* $f \in$ mor $\mathfrak{A}(P)$. *Then there exists a pair* $(P, u) \in$ ob $A(P)$, $(P, v) \in$ ob $A(P)$ *so that* $(P, u) \neq (P, v)$ *and* $F(P, u) = F(P, v)$.

Proof. Let P be a set of the cardinality at least 3. Let a_1, a_2, a_3 be different points. Put $Q = P - \{a_1, a_2, a_3\}$. Consider A-topologies u, v on the set P such that $u\{a_1\} = \{a_1, a_2\} = v\{a_2\}$, $u\{a_2\} = \{a_2, a_3\} = v\{a_3\}$, $u\{a_3\} = \{a_1, a_3\} = v\{a_1\}$ and $uX = vX = X$ for each $X \subset Q$. Denote by $S(u)$, $S(v)$ such set-systems that $(P, S(u)) = F(P, u)$, $(P, S(v)) = F(P, v)$. Consider an arbitrary set $X \in S(u)$. If $X \subset Q$ or card $(X - Q) = 3$ we consider a morphism $f \in [(P, u), (P, v)]_{\mathfrak{A}}$ which satisfies the conditions $U_{\mathfrak{A}}(f) \mid_Q = \mathrm{id}_Q$, $U_{\mathfrak{A}}(f)(a_1) = a_1$, $U_{\mathfrak{A}}(f)(a_2) = a_3$, $U_{\mathfrak{A}}(f)(a_3) = a_2$. Then $X = F(f)(X) \in S(v)$. If card $(X - Q) = 1$, e.g. $X - Q = \{a_2\}$ then using the homeomorphism $g : (P, u) \to (P, v)$ such that $U_{\mathfrak{A}}(g)(a_2) = a_2$, $U_{\mathfrak{A}}(g)(a_3) = a_1$, $U_{\mathfrak{A}}(g)(a_1) = a_3$ and $U_{\mathfrak{A}}(g) \mid_Q = \mathrm{id}_Q$, we get $X = g(X) = F(g)(X)$. Let card $(X - Q) = 2$. Let $a \in \{a_1, a_2, a_3\}$ be a point which does not belong to X. Considering a morphism $h \in [(P, u), (P, v)]_{\mathfrak{A}}$ such that $U_{\mathfrak{A}}(h) \mid_Q = \mathrm{id}_Q$ and $U_{\mathfrak{A}}(h)(a) = a$, $U_{\mathfrak{A}}(h)(b) = c$,

$U_{\mathfrak{A}}(h)(c) = b$, where $\{a, b, c\} = \{a_1, a_2, a_3\}$, we have $X = F(h)(X) \in S(v)$. Therefore $S(u) \subset S(v)$, hence $S(u) = S(v)$, whereas $u \neq v$, q.e.d.

Note that the above proposition and its proof can be modified for the case of connected compact A-topologies (for definitions see [2] 20 **B**.1. and 41 A.3.).

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