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POLARS ON PARTIALLY ORDERED GROUPS

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In the present study the relation of disjunctivity on partially ordered groups (denoted by the symbol ϱ) is defined which, in case of a lattice-order of a group, is equivalent to the usual disjunctivity on l-groups (see [3], [6], [7]). The system of all ϱ -polars on any partially ordered group is a complete Boolean lattice. At the end of the study, an example is given to prove that the ϱ -polars generally are lacking some of the other important properties of the polars on l-groups.

In § 1 definitions are given of some binary relations that are associated with the theory of disjunctivity on partially ordered groups and their mutual relation is examined. In § 2 the definition of an absolute of an element is generalized which has been introduced by L. Fuchs ([2]) for the case of a directed group, for any partially ordered group and some of the properties of these absolutes are proved here. In § 3 a new disjunctivity ϱ on partially ordered groups is defined on the basis of the results given in the preceding paragraphs and is compared with the disjunctivities contained in § 2. § 4 is devoted to the study of ϱ -polars on partially ordered groups. It is shown here that the system of all ϱ -polars forms a complete Boolean lattice on a partially ordered group and an example of a group is given whose ϱ -polars generally are neither its subgroups nor convex subsets.

By the symbol G we denote throughout the present study — unless stated otherwise — a partially ordered group, i.e., a group whose set of elements is partially ordered and from $g \leq h$ ($g, h \in G$) follows $g + x \leq h + x$ and $x + g \leq x + h$ for all $x \in G$. Additive notations will be used for group operations, and group elements will be denoted by small letters of the alphabet everywhere in the text.

If B is a non-empty subset of a group G , the following denotation will be introduced: $L(B) = \{g \in G: g \leq b \text{ for all } b \in B\}$, $U(B) = \{g \in G: g \geq b \text{ for all } b \in B\}$. If $B = \{b, c, \dots\}$, then we write $L(B) = L(b, c, \dots)$ and $U(B) = U(b, c, \dots)$. If $B, C, D \subseteq G$ and $B, C \neq \emptyset$, we denote $B + C = \{b + c: b \in B, c \in C\}$, $-C = \{-c: c \in C\}$, $B - C = B + (-C)$ and define $D + \emptyset = \emptyset + D = \emptyset$, $-\emptyset = \emptyset$.

The positive cone of a partially ordered group G , i.e., the set $\{g \in G: g \geq 0\}$ is denoted by the symbol $P(G)$ or P . Lattice-operations are denoted by the symbols \wedge , \bigwedge (the infimum) and \vee , \bigvee (the supremum).

Let G be a lattice-ordered group (an l-group). Then an absolute of an element $g \in G$ is defined as the supremum of elements g and $-g$, elements $g, h \in G$ being called disjunctive (denoted as $g \perp h$) when the infimum of their absolutes equals zero. The basic properties of absolutes of elements and the relation of disjunctivity on l-groups are given in [3], Chapter V.

The Definition of a Polar. Let a symmetric binary relation ω be defined on a set $Q \neq \emptyset$, and let $\emptyset \neq M \subseteq Q$. The set $\{q \in Q: m\omega q \text{ for all } m \in M\}$ is called a polar of the set M with respect to the relation ω or, briefly, an ω -polar of the set M , and is denoted by the symbol M^ω .

The properties of polars on l-groups with regard to the disjunctivity \perp are examined in the studies [1], [3], [6], and [7]. The system of all \perp -polars forms a complete Boolean lattice on an l-group G , and every \perp -polar is a convex l-subgroup of G .

If α, β are binary relations on a set M , then $\beta \subseteq \alpha$ means that from $x\beta y$ follows $x\alpha y$ ($x, y \in M$).

All other denotations and notions have been used in accordance with [3] and [4].

§1

In this paragraph, definitions of some disjunctivities on partially ordered groups are given and their comparison carried out.

Every symmetric binary relation on a non-empty subset of a partially ordered group is called *disjunctivity*.

On consulting the literature quoted below in the list, I encountered the definitions of disjunctivity as follows. They are designated $\alpha, \varepsilon, \delta$, and defined thus:

- (1) $g, h \in G$; then $g\alpha h$ when $P + g + h = (P + g) \cap (P + h)$;
- (2) $g, h \in P(G)$; then $g\varepsilon h$ when $g \wedge h = 0$;
- (3) $g, h \in G$; then $g\delta h$ when there exist $r, s \in P(G)$ so that

$$r \geq g \geq -r, \quad s \geq h \geq -s, \quad \text{and} \quad r \wedge s = 0.$$

The first definition is given in [5], the remaining two in [8]. Let us remark that the expression as under definition (1) may be replaced by an equivalent expression $P = (P - g) \cap (P - h)$, since P is an invariant set in G .

1.1. Lemma. For elements $g, h \in G$ holds $P = (P - g) \cap (P - h)$ if and only if $g \wedge h = 0$.

Proof. Let $P = (P - g) \cap (P - h)$. Then $g, h \geq 0$, and if u is any lower bound of these elements, it follows from $u \leq g, h$ that $0 \leq -u + g, -u + h$, so that $-u = (-u + g) - g = (-u + h) - h \in (P - g) \cap (P - h) = P$, i.e., $u \leq 0$. Thus $g \wedge h = 0$. Let us suppose, vice versa, that $g \wedge h = 0$. If $u \in (P - g) \cap (P - h)$ is any element

whatsoever, it evidently holds that $u + g, u + h \in P$, so that $-u \leq g, h$, and, since $g \wedge h = 0$, $-u \leq 0$ and $u \in P$. Hence $(P - g) \cap (P - h) \subseteq P$. The inverse inclusion readily follows from the fact that from $p \in P$ follows evidently $p + g, p + h \in P$.

It is obvious from the lemma that the definition of the disjunctivity α is, in the case of an l-group G , equivalent on the set $P(G)$ to the definition of the disjunctivity \perp . In addition to this, even the relation between the disjunctivities α and ε is clear now, because only elements from $P(G)$ may be in the relation α .

It may be further easily established that the relation δ on an l-group is equal to the disjunctivity \perp , and it may be shown that, in the case of a partially ordered group G , for elements $g, h \in P(G)$, geh holds if and only if $g\delta h$ ([8], page 88).

If we denote by \perp^P, δ^P the restriction of the relations \perp and δ to the positive cone $P(G)$, then the following proposition holds:

1.2. Propozition. *If G is a partially ordered group, then $\alpha = \varepsilon = \delta^P \subseteq \delta$. If G is an l-group, then $\delta = \perp$ and $\alpha = \perp^P = \varepsilon$.*

§ 2

Absolutes of elements on directed groups are defined in double way in [2], their relation to absolutes defined on l-groups is shown and some of their properties shown. It may be shown that absolutes of the two types of [2] will retain their basic properties even when their definitions are extended to any partially ordered group. We shall further deal with an extension of one of these definitions and shall later on make use of the results obtained in defining and studying a new relation of disjunctivity of partially ordered groups.

2.1. Lemma. *Let X, Y be non-empty subsets of a partially ordered group G , and let $g, h \in G$ be any elements. The following relations hold:*

- (A) $-U(X) = L(-X)$;
 (B) $g + U(X) + h = U(g + X + h)$; (B') $g + L(X) + h = L(g + X + h)$;
 (C) $g - U(X) + h = L(g - X + h)$; (C') $g - L(X) + h = U(g - X + h)$;
 (D) $g - U(g, h) + h = L(g, h)$; (D') $g - L(g, h) + h = U(g, h)$;
 (E) $U(X) + U(Y) \subseteq U(X + Y)$; (E') $L(X) + L(Y) \subseteq L(X + Y)$.

Proof. (A) If $u \in -U(X)$, then $-u \geq x$, and so $u \leq -x$ for all $x \in X$, i.e., $u \in L(-X)$. Contrary to this, for $u \in L(-X)$ the relation $u \leq -x$, i.e., $-u \geq x$ holds for all $x \in X$, so that $-u \in U(X)$ and $u \in -U(X)$. (B) If there exists $u \in g + U(X) + h$, then $u = g + v + h$, whereby $v \geq x$ for all $x \in X$. Hence $u \geq g + x + h$ for all $x \in X$ and $u \in U(g + X + h)$. Conversely, $u \in U(g + X + h)$ implies $u \geq g + x + h$ and therefore $-g + u - h \geq x$ for all $x \in X$. Hence $-g + u - h \in U(X)$, so that $u \in g +$

$+ U(X) + h$. (B') follows from (B) and (A). (C) and (C') will be obtained immediately on using (A), (B), and (B'). (D) and (D') follow from (C) and (C') for the case $X = \{g, h\}$. (E) If there exists $u \in U(X) + U(Y)$, then $u = v + w$, $v \in U(X)$, $w \in U(Y)$. Hence it may be seen that $u \geq x + y$ for every pair of elements x, y ($x \in X, y \in Y$), so that $u \in U(X + Y)$. (E') will be proved similarly.

2.2. Definition. Let g be an element of a partially ordered group G . A positive part g^+ (a negative part g^-) of an element g are:

$$g^+ = U(g, 0), \quad (g^- = L(g, 0)).$$

Remark. A positive and a negative part of an element of a group G are certain subsets in G which may even be empty. If, however, one of the defined parts of an element is non-empty, the other part of this element, too, is non-empty which follows, e.g., from 2.1. (D).

We shall now make use of the properties of the sets $U(X)$ and $L(X)$ from 2.1. and show that positive and negative parts of elements as defined above possess, on partially ordered groups, similar properties as those of elements defined on l-groups (see [3], Chapter V).

2.3. Proposition. For any elements g, h of a partially ordered group G , the following relations hold:

- (A) $g^+ \subseteq P(G)$, $g^- \subseteq -P(G)$, and the equation in the first (the other) case holds if and only if $g \leq 0$ ($g \geq 0$);
- (B) $g^- = -(-g)^+$, $g^+ = -(-g)^-$;
- (C) $(g + h)^+ \supseteq g^+ + h^+$, $(g + h)^- \supseteq g^- + h^-$;
- (D) $(ng)^+ \supseteq n(g^+)$, $(ng)^- \supseteq n(g^-)$ for every positive integer n ;
- (E) $g - g^+ = -g^+ + g = g^-$, $g - g^- = -g^- + g = g^+$;
- (F) g^+ and $(-g^-)$ are permutable sets.

Proof. (A) The inclusions are obvious from the definition. It may be also seen immediately that from $g \leq 0$ or $g \geq 0$ follows $g^+ = P(G)$ or $g^- = -P(G)$. If $g^+ = P(G)$, then $0 \in g^+$ holds, that is, $g \leq 0$ and, similarly, from $g^- = -P(G)$ follows $g \geq 0$. (B) Regarding to 2.1. (A) we get $g^- = L(g, 0) = -U(-g, 0) = -(-g)^+$ and $g^+ = U(g, 0) = -L(-g, 0) = -(-g)^-$. (C) From 2.1. (E) follows $g^+ + h^+ = U(g, 0) + U(h, 0) \subseteq U(g + h, g, h, 0) \subseteq U(g + h, 0) = (g + h)^+$ and, similarly, regarding to 2.1. (E') in the other case. (D) By means of the relations which are corollaries of 2.1. (E) and 2.1. (E'), we get $n(g^+) = U(g, 0) + \dots + U(g, 0) \subseteq U(ng, (n-1)g, \dots, g, 0) \subseteq U(ng, 0) = (ng)^+$ and, analogically, $n(g^-) \subseteq (ng)^-$. (E) From 2.1. (D) follows $g - g^+ = g - U(g, 0) = L(g, 0) = g^-$, $-g^+ + g = -U(g, 0) + g = L(g, 0) = g^-$ and by means of 2.1. (D') the other part of (E) will be proved. (F) On using successively 2.3. (B), 2.1. (D'), and 2.1. (A) we get $g^+ + (-g^-) = g^+ + (-g)^+ = U(g, 0) + U(-g, 0) = (-L(0, g) + g) +$

$$+ (-g - L(-g, 0)) = -L(0, g) - L(-g, 0) = -L(0, g) + U(g, 0) = -g^- + g^+.$$

We shall further make use of the notion and properties of positive and negative parts of elements in definitions and proofs of the properties of absolutes of elements on partially ordered groups.

2.4. Definition. A set $g^+ - g^-$ is called an absolute of an element g of a partially ordered group G and denoted $g^+ - g^- = |g|$.

Remark. In this way, with each element of a group G is arranged a certain subset of this group which may be, in general, also empty. It is shown in [2] that, in the case of an l-group, an absolute $|g|$ as just defined is equal to the set $U(g, -g)$ and hence its relation to the absolute defined on l-groups is also obvious (on an l-group, $|g|$ is the set of all upper bounds of an element $g \vee -g$).

Like positive and negative parts, absolutes of elements on partially ordered groups have similar properties as absolutes on l-groups.

2.5. Lemma. Let g be an element of a partially ordered group G . The following relations hold:

- (A) $|g| = -g^- + g^+$; (B) $|g| = g^+ + (-g)^+ = (-g)^+ + g^+$;
 (C) if $g \in P(G)$, then $|g| = U(g)$.

Proof. (A) and (B) follow at once from 2.3. (F) and 2.3. (B). (C) If $g \geq 0$, then the following relation holds: $|g| = U(g, 0) + U(-g, 0) = U(g) + P$, so that $|g| = U(g)$.

2.6. Proposition. For any elements g, h of a partially ordered group G the following relations hold:

- (A) $|g| \subseteq P(G)$, and the equation holds if and only if $g = 0$;
 (B) $|-g| = |g|$; (C) $|ng| \supseteq n|g|$ for every positive integer n ;
 (D) $|g + h| \supseteq |g| + |h| + |g|$; (E) $|g - h| = U(g, h) - L(g, h)$;
 (F) if G is commutative, then $|g + h| \supseteq |g| + |h|$ holds.

Proof. (A) The inclusions $|g| \subseteq P(G)$ and $|0| = P(G)$ are obvious (see 2.5. (B)). Further, if $P = |g| = g^+ + (-g)^+$, then necessarily $g^+ = P = (-g)^+$, because if 0 is an element of the sum of two subsets of the positive cone P , then 0 is an element in each of them, so that from 2.3. (A) follows $g \leq 0$, $-g \leq 0$, i.e., $g = 0$. (B) follows from 2.5. (B). (C) According to 2.5. (B) and 2.3. (D) we get $n|g| = n(g^+ + (-g)^+) = ng^+ + n(-g)^+ \subseteq (ng)^+ + (-ng)^+ = |ng|$. (D) To prove this we apply 2.1. (E). The following relation holds: $|g| + |h| + |g| = (U(g, 0) + U(-g, 0) + U(h, 0)) + (U(-h, 0) + U(g, 0) + U(-g, 0)) \subseteq U(g + h, 0, \dots) + U(-h - g, 0, \dots) \subseteq U(g + h, 0) + U(-h - g, 0) = (g + h)^+ + (-(g + h))^+ = |g + h|$. (E) With respect to 2.1. (B) and 2.1. (B'), the following relation holds: $|g - h| = U(g - h, 0) -$

– $L(g - h, 0) = (U(g, h) - h) - (L(g, h) - h) = U(g, h) - L(g, h)$. (F) If G is commutative, the following relation follows from 2.3. (C): $|g| + |h| = g^+ + (-g)^+ + h^+ + (-h)^+ = g^+ + h^+ + (-g)^+ + (-h)^+ \subseteq (g + h)^+ + (-(g + h))^+ = |g + h|$.

§ 3

We shall now make use of the results from the preceding paragraph in defining a new disjunctivity on partially ordered groups, observe its relation to the disjunctivity \perp and compare it with the disjunctivities given in § 1.

3.1. Definition. For any element $g \in G$ we define a set $C(g) = L(|g|) \cap P(G)$, when $|g| \neq \emptyset$; $C(g) = P(G)$, when $|g| = \emptyset$.

Remark. Obviously $C(g) \neq \emptyset$ for every $g \in G$, because $|g| \subseteq P(G)$ (see 2.6. (A)) and thus $0 \in C(g)$.

3.2. Definition. The relation ϱ on a partially ordered group G , is defined in the following way:

let $g, h \in G$; then $g\varrho h$, when $C(g) \cap C(h) = 0$.

3.3. Lemma. The relation ϱ is symmetric and anti-reflexive.

Proof. The relation ϱ is obviously symmetric. If $g \in G$ and $g\varrho g$, then $C(g) = C(g) \cap C(g) = 0$, so that, for all $x \in G$, the following holds: $C(g) \cap C(x) = 0$, i.e., $g\varrho x$, and ϱ is anti-reflexive.

3.4. Proposition. If G is an l -group, then the relation ϱ is equal to the disjunctivity \perp .

Proof. For each element $g \in G$, the following relation holds: $L(|g|) = L(U(g, -g)) = L(g \vee -g)$ (see § 2). Thus $C(g)$ is the set of all positive lower bounds of the element $g \vee -g$ and hence it is evident that elements from G are in the relation ϱ if and only if they are in the relation \perp .

3.5. Lemma. If x and y are those elements of a partially ordered group for which there exists $x \wedge y$ and $x \wedge y = 0$, then $x \wedge (y + y) = 0$ holds.

Proof. Let $x \wedge y = 0$. Obviously, $x, y + y \geq 0$ and if u is any lower bound of elements x and $y + y$, the following relations hold: $u - y \leq y$ and $u - y \leq u \leq x$. Hence $u - y \leq 0$, so that $u \in L(x, y)$ and $u \leq 0$. Then $x \wedge (y + y) = 0$.

3.6. Proposition. If G is a partially ordered group, then $\alpha \subseteq \varrho$, $\varepsilon \subseteq \varrho$, $\delta \subseteq \varrho$.

Proof. Since $\alpha = \varepsilon \subseteq \delta$ (see 1.2.), it suffices to show $\delta \subseteq \varrho$. Let $g, h \in G$ and $g\delta h$, i.e., there exist $r, s \in G$, $r \geq g$, $-g, 0, s \geq h$, $-h, 0$ and $r \wedge s = 0$. Obviously $r + r \in |g| = U(g, 0) + U(-g, 0)$ and $s + s \in |h| = U(h, 0) + U(-h, 0)$. If

$u \in C(g) \cap C(h)$, then $u \leq r + r, s + s$ holds, so that $u \leq 0$, for, according to 3.5., $(r + r) \wedge (s + s) = 0$. Hence $u = 0$ and $g \varrho h$.

We shall now give an example of a partially ordered group on which the disjunctivities δ and ϱ are different.

3.7. Example. Let us denote, by the symbol H , an additive group of all complex numbers having integer real and imaginary parts on which partial ordering is defined by the following rule:

$x + yi \geq 0$, when $x > 0$ and $y > 0$ or $x + yi = 0$ ([3], Chapter II, § 3).

We shall show that the elements $1 + 2i$ and $2 + 2i$ are in the relation ϱ , but are not in the relation δ . With regard to 2.5.(C), the following relations hold: $C(1 + 2i) = L(1 + 2i) \cap P = \{1 + 2i, 0\}$, $C(2 + 2i) = L(2 + 2i) \cap P = \{2 + 2i, 1 + i, 0\}$, so that $C(1 + 2i) \cap C(2 + 2i) = 0$ and the elements are in the relation ϱ . However, they are obviously not in the relation δ , because, e.g., the number i is a lower bound of the sets $U(1 + 2i)$, $U(2 + 2i)$ and is not less than 0, so that there do not exist numbers $r \in U(1 + 2i)$, $s \in U(2 + 2i)$ with the property $r \wedge s = 0$.

§ 4

This paragraph is devoted to the study of some of the properties of ϱ -polars on partially ordered groups. It will be the aim of our further considerations to show especially that the system of all ϱ -polars forms a complete Boolean lattice on every partially ordered group. In proving this we shall make use of the following theorem.

Theorem (A). *Let Q be a non-empty set on which are defined two relations: a symmetric, anti-reflexive binary relation ω and a reflexive, transitive binary relation $<$ with the minimal element 0 (i.e., $0 < x$, for all $x \in Q$) which satisfy the following conditions ($g, h, z \in Q$):*

(a) $h < g, g\omega h \Rightarrow h < 0$;

(b) $0\omega 0$;

(c) $g\omega h, z < g \Rightarrow z\omega h$;

(d) $g \text{ non } \omega h \Rightarrow$ there exists $z \in Q$, $z < g$, $z < h$ and $z \text{ non } < 0$. Then the system of all ω -polars on Q forms a complete Boolean lattice with respect to the set-theoretical inclusion. For any system of ω -polars $\{A_\alpha\}$ the following relations hold: $\bigwedge_\alpha A_\alpha = \bigcap_\alpha A_\alpha$, $\bigvee_\alpha A_\alpha = (\bigcap_\alpha A_\alpha^\omega)^\omega$ and the complement (in terms of Boolean lattice) of an ω -polar A is A^ω ([6], page 64).

4.1. Definition. *On a partially ordered group G , we define a binary relation \sim in the following way:*

for $g, h \in G$, $g \sim h$, when $C(g) = C(h)$.

Remark. The relation \sim is obviously reflexive, symmetric, and transitive. We shall denote, by the symbol \bar{G} , the quotient set of a group G with respect to the equivalence relation \sim , and, by the symbol K_g , the class of an element $g \in G$ (i.e., the set $\{x \in G: x \sim g\}$). The class of 0 is denoted by the symbol K_0 .

Remark. From the definition of the relation \sim may be seen that $C(g) \subseteq C(h)$ ($g, h \in G$) implies $C(x) \subseteq C(y)$ for every $x \in K_g$ and every $y \in K_h$. It will be easily established from gqh follows xqy for all $x \in K_g, y \in K_h$.

4.2. Definition. Let $K_g, K_h \in \bar{G}$. On the set \bar{G} , binary relations \bar{q} and $<$ are defined in the following way:

$$K_g \bar{q} K_h, \text{ when } gqh; \quad K_g < K_h, \text{ when } C(g) \subseteq C(h).$$

4.3. Lemma. The set \bar{G} with the relations \bar{q} and $<$ satisfies the assumptions of the Theorem (A).

Proof. The relation \bar{q} is symmetric and anti-reflexive, because the relation q has such properties according to 3.3. The relation $<$ is obviously reflexive and transitive, and, since $|0| = P$ (see 2.6. (A)), $C(0) = 0$ holds, so that $C(0) \subseteq C(g)$ for all $g \in G$ and thus K_0 is the minimal element of the set \bar{G} (with respect to the quasi-order $<$). The properties (a), (b), (c), (d) are still to be proved. (a) Let $K_h < K_g$ and $K_g \bar{q} K_h$. Then $C(h) \subseteq C(g)$ and $C(g) \cap C(h) = 0$, so that $C(h) = 0$. Hence $C(h) \subseteq C(0)$, so that $K_h < K_0$. (b) $K_0 \bar{q} K_0$ follows from $C(0) = 0$. (c) If $K_g \bar{q} K_h$ and $K_z < K_g$, the following relations hold: $C(g) \cap C(h) = 0$ and $C(z) \subseteq C(g)$, so that $C(z) \cap C(h) \subseteq C(g) \cap C(h) = 0$ and $K_z \bar{q} K_h$. (d) Let $K_g \text{ non } \bar{q} K_h$. Then there exists $z \in C(g) \cap C(h)$, $z > 0$. According to 2.5.(C), $|z| = U(z)$, so that $C(z) = L(z) \cap P$ and hence $z \in C(z)$. Thus $C(z)$ is not a subset of $C(0) = 0$ and $K_z \text{ non } < K_0$. It remains to show $K_z < K_g$ and $K_z < K_h$. If $|g| = \emptyset$, then $C(z) \subseteq C(g) = P$, i.e., $K_z < K_g$. Thus let $|g| \neq \emptyset$. If $u \in C(z)$, then $0 \leq u \leq z$, so that $u \in C(g)$, because $z \in C(g)$. Hence $C(z) \subseteq C(g)$ and $K_z < K_g$. The relation $K_z < K_h$ is proved in a similar way.

Remark. A set of all q -polars on G , or \bar{q} -polars on \bar{G} with the set-theoretical inclusion as the relation of partial ordering will be denoted by the symbols $R(G)$, or $R(\bar{G})$.

4.4. Lemma. The sets $R(G)$ and $R(\bar{G})$ are order-isomorphic.

Proof. Let us remark first that if A is any q -polar on a group G and $A \cap K_g \neq \emptyset$, $K_g \in \bar{G}$, then obviously $K_g \subseteq A$.

Let f be a mapping of the set $R(G)$ into the system of all subsets of the set \bar{G} such that, for every $A \in R(G)$, $f(A) = \{K_g \in \bar{G} : K_g \subseteq A\}$. From the introductory remark of this Proof may be seen that f is injective, and for $A, B \in R(G)$ the relation $A \subseteq B$ holds if and only if $f(A) \subseteq f(B)$. There is still to be proved that f is a mapping from $R(G)$ onto $R(\bar{G})$.

We shall first prove that $f(A) \in R(\bar{G})$ for every $A \in R(G)$. Let us denote by M a subset of G for which $M^e = A$ and let $\bar{M} = \{K_g \in \bar{G} : M \cap K_g \neq \emptyset\}$. If $K_g \in f(A)$, then $K_g \subseteq A$, hence $M \bar{q} K_g$ (that is, $m \bar{q} x$ for every $m \in M$, $x \in K_g$) and from the definition of the relation \bar{q} follows $\bar{M} \bar{q} K_g$, so that $K_g \in \bar{M}^e$. Further, if $K_h \in \bar{M}^e$, i.e., $\bar{M} \bar{q} K_h$, then $M \bar{q} K_h$ holds obviously, so that $K_h \subseteq A$ and $K_h \in f(A)$. Hence $\bar{M}^e = f(A)$, that is, $f(A) \in R(\bar{G})$.

Further, let \bar{A} be any polar from $R(\bar{G})$ and let $\bar{A} = \bar{N}^e$, where $\bar{N} \subseteq \bar{G}$. Let us denote by $N \subseteq G$ the join of all sets from \bar{N} . Then $f(N^e) = \bar{A}$. Because, if $K_g \in \bar{A}$, then $\bar{N} \bar{q} K_g$ holds, which implies $K_g \subseteq N^e$, so that $K_g \in f(N^e)$. Conversely, $K_g \in f(N^e)$ if and only if $K_g \subseteq N^e$, but then $N \bar{q} K_g$ and $K_g \in \bar{A}$.

Thus f is an order-isomorphism of $R(G)$ on $R(\bar{G})$ and the lemma is proved.

4.5. Theorem. $R(G)$ is a complete Boolean lattice. For any system of q -polars $\{A_\alpha\}$, the following relations hold: $\bigwedge_\alpha A_\alpha = \bigcap_\alpha A_\alpha$, $\bigvee_\alpha A_\alpha = (\bigcap_\alpha A_\alpha^e)^e$ and the complement (in terms of Boolean lattice) of a polar $A = A^e$.

Proof. With regard to Lemma 4.3., the proposition of the Theorem (A) holds for \bar{q} -polars on \bar{G} . Thus, according to Lemma 4.4., $R(G)$ is a complete lattice, whereby, if f is the isomorphism defined in 4.4. and $\{A_\alpha\}$ is any system of polars from $R(G)$, the following relation holds: $f(\bigvee_\alpha A_\alpha) = \bigvee_\alpha f(A_\alpha)$ and the like for the infimum. Hence $R(G)$ will easily be proved to be a complete Boolean lattice.

Like in the concluding part of the proof of Lemma 4.4., it may be further shown that for every $A \in R(G)$ the following relation holds: $f(A^e) = (f(A))^e$. From the properties of the isomorphism f follows that the intersection of any system of q -polars $\{A_\alpha\}$ is again a q -polar and obviously $f(\bigcap_\alpha A_\alpha) = \bigcap_\alpha f(A_\alpha)$. Hence $f(\bigwedge_\alpha A_\alpha) = \bigwedge_\alpha f(A_\alpha) = \bigcap_\alpha f(A_\alpha) = f(\bigcap_\alpha A_\alpha)$, so that $\bigwedge_\alpha A_\alpha = \bigcap_\alpha A_\alpha$ and, further, $f(\bigvee_\alpha A_\alpha) = \bigvee_\alpha f(A_\alpha) = \bigcap_\alpha (f(A_\alpha))^e = (\bigcap_\alpha f(A_\alpha^e))^e = (f(\bigcap_\alpha A_\alpha^e))^e = f((\bigcap_\alpha A_\alpha^e)^e)$, so that $\bigvee_\alpha A_\alpha = (\bigcap_\alpha A_\alpha^e)^e$. Finally, if we denote by B the complement to a polar $A \in R(G)$, then $f(B) = (f(A))^e$ holds. Thus $f(B) = f(A^e)$, so that $B = A^e$ and the theorem is proved.

Remark. Since $C(0) = 0$, it may be seen from the definition of the equivalence relation \sim that the set K_0 will be the minimal polar of $R(G)$ and the group G will be obviously the greatest element of $R(G)$.

4.6. Proposition. Any polar $A \in R(G)$ has the following property:

$$g \in A, x \in G, |x| \supseteq |g| \Rightarrow x \in A.$$

Proof. Let A be a q -polar of a set $M \subseteq G$ and let $g \in A$. If $x \in G$ and $|x| \supseteq |g|$, then $C(x) \subseteq C(g)$, so that, for all $m \in M$, we have $C(m) \cap C(x) \subseteq C(m) \cap C(g) = 0$, i.e., $m \bar{q} x$ and $x \in A$.

Remark. Since $|-g| = |g|$ for every $g \in G$ (see 2.6.(B)), then $g \in A$, $A \in R(G)$ implies $-g \in A$ (see 4.6.).

4.7. Example. We shall show that ϱ -polars on the partially ordered group H from 3.7. are generally neither subgroups nor convex subsets of this group.

Let us denote by the letter A a ϱ -polar of the one-element set $\{1 + 2i\}$. In 3.7. we have established that $g_1 = (2 + 2i)\varrho(1 + 2i)$, so that $g_1 \in A$ and we shall now show that $g_1 + g_1 \notin A$. The following relations hold: $C(1 + 2i) = \{1 + 2i, 0\}$, $C(g_1 + g_1) = L(4 + 4i) \cap P = \{0, 1 + 2i, \dots\}$, hence $C(1 + 2i) \cap C(g_1 + g_1) = \{1 + 2i, 0\}$, so that the elements are not in the relation ϱ , i.e., $g_1 + g_1 \notin A$ and A is not a subgroup of the group H .

Further, we shall show that A is not a convex subset of the group H , that means that there exist $g, h \in A$ and $x \in H$ such that $g \leq x \leq h$ and $x \notin A$. With regard to the above-mentioned remark the following holds: $-g_1 \in A$. Let us denote $g_2 = i$. Then $g_1 > g_2 > -g_1$, and we shall show that $g_2 \notin A$. Now the following relation holds: $|g_2| = U(i, 0) + U(-i, 0) = \{x + yi \in H: x \geq 1, y \geq 2\} + \{x + yi \in H: x \geq 1, y \geq 1\} = \{x + yi \in H: x \geq 2, y \geq 3\}$. Hence $C(g_2) = L(|g_2|) \cap P = \{1 + 2i, 1 + i, 0\}$. Thus we have $C(1 + 2i) \cap C(g_2) = \{1 + 2i, 0\}$, that is, $(1 + 2i) \text{ non } \varrho g_2$, so that $g_2 \notin A$ and A is not convex.

Remark. It might be shown on a further example that not even a ϱ -polar of a directed and convex normal subgroup of a group having normal and distributive partial ordering (see [3], Chapter V) is generally a subgroup of this group.

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