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Boundary value problems with nonlinear boundary conditions in Banach spaces

GIUSEPPE MARINO, PAOLAMARIA PIETRAMALA

Abstract. Let X be a Banach space, $J = [a, b]$ a bounded real interval, $A(t, x)$ a bounded operator defined and continuous on the product $J \times X$, $f(t, x)$ a continuous function on $J \times X$, L a bounded linear operator with values in X and H a continuous operator, not necessarily continuous. In this paper, we study the existence of solutions of

$$x' = A(t, x)x + f(t, x)$$

which satisfy the condition

$$Lx = H(x).$$

Keywords: Evolution operator, boundary value problems, differential equations, nonlinear operator, fixed point theorems

Classification: 34K10

1. Introduction.

Consider a nonlinear differential problem with nonlinear boundary conditions of the type

$$\left\{ \begin{array}{l} x' = F(t, x) \\ Tx = y, \quad y \in X \ \text{ Banach space} \end{array} \right. \tag{1.1}$$

The most important works in this field, when $F(t, x)$ is of the form $A(t)x + f(t, x)$ (i.e. it is the perturbation of a linear bounded operator) and T is a bounded linear operator, are due to Scrucca [1], Conti [2], Opial [3], Bernfeld and Lakshmikantham [4], to which we refer for a nearly exhaustive reference.

The case of a nonlinear problem, that is, when $F(t, x)$ takes the form $A(t, x)x + f(t, x)$, has been studied by Conti [2], Kartsatos [5], Furi et al. [6] and Anichini [7]. In these papers T is a continuous but not necessarily linear operator. The methods used in these papers are based on fixed point arguments or topological degree theory.

Very recently, a further contribution to the subject has been given by Anichini–Conti [8]. By using a fixed point theorem for condensing maps due to Martelli [9], they prove the existence of solutions for (1.1), with $X = \mathbf{R}^n$ under the new assumption

$$|A(t, x)x| + |f(t, x)| \le g(t, |x|) \tag{1.2}$$

for a suitable function g.

In this paper, we give a substantial simplification of the arguments and estimates used in [8]; moreover, we improve their existence result, under assumptions of the kind in (1.2), but in a more general Banach space context. We rely on the classical fixed point theorem for compact maps due to Schaefer [10].

In the last section, we give some examples of how our main result (Theorem 3.1) can be successfully applied to some nonlinear boundary value problems.

2. Notations and preliminary results.

We use the following notations:

- $J = [a, b]$ is a compact interval on the real line $\mathbb{R}$.
- X is a Banach space with norm $|v|$, $v \in X$.
- $C(J, X)$ is the Banach space of continuous functions from J into X with the norm $\|x\|_\infty := \max\{|x(t)| : t \in J,\ x \in C(J, X)\}$.
- $B(X)$ is the Banach space of bounded linear operators from X into X with the norm $\|T\| := \sup\{|Tv| : |v| = 1\}$.

The following lemmas will be crucial in the proof of the main theorem:

Lemma 2.1 ([10]). *Let $S : X \to X$ be a continuous, compact map. If the set*

$$M := \{v \in X : cv = S(v) \text{ for some } c > 1\}$$

is bounded, then S has a fixed point.

Lemma 2.2 ([11, p. 32]). *Let $g(t, z)$ be a continuous function defined on $J \times \mathbb{R}$ such that the initial value problem for the equation*

$$z' = g(t, z)$$

has the unique solution $z(t)$ for $t \in J$. Then, if $|x'(t)| \leq g(t, |x(t)|)$ for every $t \in J$ and if $|x(a)| \leq z(a)$, we have $|x(t)| \leq z(t)$ for $t \in J$.

Let us prove the following theorem:

Theorem 2.1. *Let $A : J \times X \to B(X)$ and $f : J \times X \to X$ be two continuous functions such that:*

(i) *$|v| \leq r$ implies that there exists $R = R(r) > 0$ such that*

$$\|A(t, v)\| + |f(t, v)| \leq R \quad \text{for } \ t \in J;$$

(ii) *$|A(t, v)v| + |f(t, v)| \leq g(t, |v|)$, for $t \in J$ and $v \in X$, where g is the function defined in Lemma 2.2;*

(iii) *If $u \in C(J, X)$ and x_u solves the Cauchy linear problem*

$$\begin{cases} x'(t) = A(t, u(t))x(t) + f(t, u(t)) \\ x(a) = x_0\,, \end{cases}$$

then the set $\{x_u(t) : u \text{ in a bounded set } B \text{ of } C(J, X)\}$ is relatively compact for any $t \in J$.

Then the initial value problem of nonlinear ordinary differential equation

$$\begin{cases} x' = A(t, x)x + f(t, x) \\ x(a) = x_0 \end{cases}$$

has at least one solution.

PROOF : Let $u \in C(J,X)$ be given. The maps $A_u : J \to B(X)$ and $f_u : J \to X$ defined respectively by $A_u(t) := A(t,u(t))$ and $f_u(t) := f(t,u(t))$ are continuous maps and so it is well known ([**12**, p. 196]) that the linear problem

$$\begin{cases} x'(t) = A_u(t)x(t) + f_u(t) \\ x(a) = x_0 \end{cases}$$

has a unique solution x_u that we can write as

$$x_u(t) = x_0 + \int_a^t A_u(s)x_u(s)\,ds + \int_a^t f_u(s)\,ds\,. \tag{2.1}$$

Hence we can define the map $S : C(J,X) \to C(J,X)$ by defining $S(u)$ to be the unique function x_u solution of (2.1). Our claim will be proved if we are able to show the existence of a fixed point for S.

First, we show that S is a continuous map. For this purpose, let $u_n \to u_0$ in $C(J,X)$ and $S(u_n) = x_{u_n}$. Then

$$\begin{aligned} |x_{u_n}(t) - x_{u_0}(t)| &\le \int_a^t |A_{u_n}(s)x_{u_n}(s) - A_{u_0}(s)x_{u_0}(s) \pm A_{u_n}(s)x_{u_0}(s)|\,ds+ \\ &+ \int_a^t |f_{u_n}(s) - f_{u_0}(s)|\,ds \le \int_a^t \|A_{u_n}(s)\|\;|x_{u_n}(s) - x_{u_0}(s)|\,ds+ \\ &+\|x_{u_0}\|_\infty \int_a^b \|A_{u_n}(s) - A_{u_0}(s)\|\,ds + \|f_{u_n} - f_{u_0}\|_\infty(b-a) \end{aligned}$$

for which, by the Gronwall inequality, we have

$$\begin{aligned} |x_{u_n}(t) - x_{u_0}(t)| \le (\|x_{u_0}\|_\infty &\int_a^b \|A_{u_n}(s) - A_{u_0}(s)\|\,ds+ \\ &+ \|f_{u_n} - f_{u_0}\|_\infty(b-a))\exp(\int_a^t \|A_{u_n}(s)\|\,ds). \end{aligned}$$

Now, $u_n \to u_0$ in $C(J,X)$ implies that there exists an $r > 0$ such that $\|u_n\|_\infty \le r$, and so, from hypothesis (i), it follows that there exists an $R > 0$ such that $\|A_{u_n}\|_\infty := \max\{\|A_{u_n}(s)\| : s \in J\} \le R$. Hence

$$\begin{aligned} \|x_{u_n} - x_{u_0}\|_\infty \le (\|x_{u_0}\|_\infty\,\|A_{u_n} - A_{u_0}\|_\infty+ \\ +\|f_{u_n} - f_{u_0}\|_\infty)(b-a)\exp(R(b-a)). \end{aligned}$$

On the other hand, under the assumptions of continuity of A and f, it follows that $\|A_{u_n} - A_{u_0}\|_\infty \to 0$ and $\|f_{u_n} - f_{u_0}\|_\infty \to 0$, so that $\|x_{u_n} - x_{u_0}\|_\infty \to 0$. Now we show that S is a compact map. From (2.1) it follows that

$$|(S(u))(t)| \le |x_0| + \int_a^t \|A_{u_n}(s)\|\;|(S(u))(s)|\,ds + \int_a^b |f_u(s)|\,ds,$$

so, again by Gronwall inequality,

$$|(S(u))(t)| \le (|x_0| + \int_a^b |f_u(s)|\,ds)\exp(\int_a^b \|A_u(s)\|\,ds).$$

Hence $\|u\|_\infty \le r$ yields

$$\|S(u)\|_\infty \le (|x_0| + R(b-a))\exp(R(b-a)) =: k, \tag{2.2}$$

so that S maps bounded sets into bounded sets. Moreover

$$(S(u))'(t) = A_u(t)(S(u))(t) + f_u(t)$$

and therefore

$$\|(S(u))'\|_\infty \le \|A_u\|_\infty \|S(u)\|_\infty + \|f_u\|_\infty .$$

It follows that $\|u\|_\infty \le r$ and (2.2) imply that $\|(S(u))'\|_\infty \le R(k+1)$ and this, together with (iii), is enough to conclude that S maps bounded sets into relatively compact sets, i.e. S is a compact map. Finally, let M be the set in Lemma 2.1. Let $x \in M$. Then, for some $c > 1$,

$$cx(t) = x_0 + \int_a^t A(s, x(s))cx(s)\,ds + \int_a^t f(s, x(s))\,ds,$$

so that

$$x'(t) = A(t, x(t))x(t) + c^{-1}f(t, x(t)) \quad \text{and} \quad x(a) = c^{-1}x_0 . \tag{2.3}$$

Let us consider the initial value problem

$$\begin{cases} z' = g(t,z) \\ z(a) = |x_0|. \end{cases}$$

Clearly, from (2.3) and hypothesis (ii) we have

$$|x'(t)| \le |A(t, x(t))x(t)| + c^{-1}|f(t, x(t))| \le g(t, |x(t)|).$$

Moreover, $|x(a)| = c^{-1}|x_0| < |x_0| = z(a)$, so that from Lemma 2.2 we get $|x(t)| \le z(t)$ for any x in M and this shows that M is bounded. From Lemma 2.1 the claim follows. ■

We note that the hypothesis (iii) of the previous theorem is certainly satisfied if X is finite-dimensional or if the set

$$\{E_u(t,s)B_1 : t - s > 0, u \text{ in a bounded set } B \text{ of } C(J,X)\}$$

is relatively compact in X, where $B_1 := \{u \in C(J,X) : \|u\|_\infty \le 1\}$ and $E_u(t,s)$ is the evolution operator of $A(t, u(t))$ (see Remark 2 below). Also, we emphasize that (iii) is equivalent to (iii)': The set

$$\{E_u(t,a)x_0 + \int_a^t E_u(t,s)f(s, u(s))\,ds : u \text{ in a bounded set } B \text{ of } C(J,X)\}$$

is relatively compact for any t in J fixed.

3. Main result.

Let us consider the following boundary value problem

$$(\mathrm{NLL})\begin{cases} x' = A(t,x)x + f(t,x) \\ Lx = H(x) \end{cases}$$

Assume that the following hypotheses hold:

(h$_1$) $A : J \times X \to B(X), (t,v) \mapsto A(t,v)$ is a continuous function for which $\forall r > 0\ \exists r_1 = r_1(r) > 0$ such that $|v| \le r$ implies that $\|A(t,v)\| \le r_1 \forall t \in J$.

(h$_2$) $f : J \times X \to X, (t,v) \mapsto f(t,v)$ is a continuous function.

(h$_3$) $|A(t,v)v| + |f(t,v)| \le g(t,|v|), \forall t \in J$ and $\forall v \in X$, where g is the function defined in Lemma 2.2.

(h$_4$) $L : C(J,X) \to X$ is a linear and continuous operator.

(h$_5$) $H : C(J,X) \to X$ is a continuous operator for which:
(i) $\forall r > 0\ \ \exists r_2 = r_2(r) > 0$ such that $\|u\|_\infty \le r$ implies that $|H(u)| \le r_2$;
(ii) $\exists d > 0 : |K_u(H(u) - L\int_a^{(\cdot)} E_u(\cdot,s)f(s,u(s))\,ds)(a)| \le d\ \ \forall u \in C(J,X)$, where K_u is the operator defined in (h$_6$) and $E_u(t,s)$ is the evolution operator of $A(t,u(t))$ (see Remark 2 below).

(h$_6$) For every given u in $C(J,X)$ there exists a linear and continuous operator $K_u : X \to \operatorname{Ker} D_u$, where $D_u := (d/dt) - A(t,u(t))$, such that
(i) $K : C(J,X) \to B(X, \operatorname{Ker} D_u), u \mapsto K_u$ is a continuous function;
(ii) $\forall r > 0\ \ \exists m = m(r) > 0$ such that $\|u\|_\infty \le r$ implies that $\|K_u\| \le m$;
(iii) $(I - LK_u)(H(u) - L\int_a^{(\cdot)} E_u(\cdot,s)f(s,u(s))\,ds) = 0\ \ \forall u \in C(J,X)$.

(h$_7$) If $u \in C(J,X)$, let $z_u(t) := \int_a^t E_u(t,s)f(s,u(s))\,ds$; *then the set*

$$\{(K_u(H(u) - Lz_u))(t) + z_u(t) : u \text{ in a bounded set } B \text{ of } C(J,X)\}$$

is relatively compact for any $t \in J$.

Remark 1. From (h$_1$), (h$_2$), (h$_3$), it follows that

$\forall r > 0\ \ \ \exists R = R(r) > 0$ such that $|v| \le r$ implies that

$$\|A(t,v)\| + |f(t,v)| \le R\ \ \ \forall t \in J \blacklozenge$$

Remark 2. From (h$_1$), we are able to claim the existence, for any fixed u, of a unique operator function $E_u : J \times J \to B(X), (t,s) \mapsto E_u(t,s)$, defined and continuous on $J \times J$ such that

$$E_u(t,s) = I + \int_s^t A_u(w)E_u(w,s)\,dw \tag{3.1}$$

(evolution operator of A_u), where $A_u(t) := A(t,u(t))$ ([1]). From (3.1), one has

$$E_u(t,t) = I,\ \ E_u(t,s)E_u(s,r) = E_u(t,r)\ \ \ \forall (t,s,r) \in J \times J \times J \tag{3.2}$$

and moreover

$$(\partial E_u(t,s)/\partial t) = A_u(t)E_u(t,s) \text{ almost everywhere for } t \in J, s \in J. \tag{3.3}$$

From this, it follows that the (Carathéodory) solutions of the linear homogeneous equation $D_u x = 0$ are defined in J and define a space isomorphic to X via the map $j_s : X \to \operatorname{Ker} D_u, j_s(x) := E_u(\cdot,s)x \blacklozenge$

Remark 3. From (h_1) and (h_2), it also follows that if $u \in C(J,X)$ then

(i) A_u belongs to $C(J,B(X))$;

(ii) f_u belongs to $C(J,X)$ (here $f_u(t) := f(t,u(t))$);

(iii) $\|u_n - u_0\|_\infty \to 0$ implies that

$$\|A_{u_n} - A_{u_0}\|_\infty \to 0 \text{ and } \|f_{u_n} - f_{u_0}\|_\infty \to 0 \blacklozenge \tag{3.4}$$

Remark 4. The hypothesis (h_7) is already used in several works (see [**1**], [**13**], in which the operator A depends only on $t \in J$). In any case, (h_7) is certainly satisfied if X is finite-dimensional or H is a compact operator and the set

$$\{E_u(t,s)B_1 : t-s > 0 \text{ and } u \text{ in a bounded set } B \text{ in } C(J,X)\}$$

is relatively compact ([**14**])$\blacklozenge$

Finally, we quote the following result which is useful in the proof of our main theorem.

Lemma 3.1 ([**11**, p. 36]). *Suppose that* $g_1, g_2 \in C(J,\mathbf{R})$, $g_3 \in L^1(J,\mathbf{R})$, $g_3 \geq 0$ *almost everywhere*, $g_1(t) \leq g_2(t) + \int_a^t g_3(s)g_1(s)\,ds$, $t \in J$.
Then $g_1(t) \leq g_2(t) + \int_a^t g_3(s)g_2(s)\exp(\int_s^t g_3(v)\,dv)\,ds$.

Our main result is:

Theorem 3.1. *Suppose that* (h_1)–(h_7) *hold. Then the problem* (NLL) *admits at least one solution.*

PROOF :

Step 1. $\|u\|_\infty \leq r \Rightarrow \exists\, r' = r'(r) > 0 : \|E_u\|_\infty := \max\{\|E_u(t,s)\| : (t,s) \in J \times J\} \leq r'$.
Indeed, from (3.1) we obtain, if $s \leq t$ (analogously if $t > s$):

$$\|E_u(t,s)\| \leq 1 + \int_s^t \|A_u(w)\|\,\|E_u(w,s)\|\,dw$$

which, by Gronwall's inequality, yields

$$\|E_u(t,s)\| \leq \exp\left(\int_s^t \|A_u(w)\|\,dw\right) \leq \exp(R(b-a)) =: r'$$

(the last inequality follows from Remark 1).

Step 2. $E_u(t,s)$ is continuous with respect to u, i.e. $\|u_n - u\|_\infty \to 0$ implies $\|E_{u_n} - E_u\|_\infty \to 0$.
Indeed, let $\|u_n - u\|_\infty \to 0$. Then there exists an $r > 0$ such that $\|u_n\|_\infty, \|u\|_\infty \leq r$. Moreover, if $s \leq t$ (analogously if $t > s$), we have from (3.1),

$$\|E_{u_n}(t,s) - E_u(t,s)\| \leq \int_a^t \|E_{u_n}(w,s)\|\,\|A_{u_n}(w) - A_u(w)\|\,dw +$$
$$+ \int_s^t \|A_u(w)\|\,\|E_{u_n}(w,s) - E_u(w,s)\|\,dw.$$

This implies, by Lemma 3.1,

$$\|E_{u_n}(t,s) - E_u(t,s)\| \le \int_s^t \|E_{u_n}(w,s)\| \, \|A_{u_n}(w) - A_u(w)\| \, dw +$$
$$+ \int_s^t \|A_u(w)\| \Big(\int_s^w \|E_{u_n}(y,s)\| \, \|A_{u_n}(y) - A_u(y)\| \, dy \Big) \exp\Big(\int_w^z \|A_u(z)\| \, dz \Big) \, dw \le$$
$$\le \|A_u\|_\infty \|E_{u_n}\|_\infty \|A_{u_n} - A_u\|_\infty (b-a)^2 \exp(\|A_u\|_\infty (b-a)) +$$
$$+ \|E_{u_n}\|_\infty \|A_{u_n} - A_u\|_\infty (b-a) \le$$
$$\le \|A_{u_n} - A_u\|_\infty r'(b-a)(1 + R(b-a)\exp(R(b-a))),$$

from which we obtain

$$\|E_{u_n} - E_u\|_\infty \le \|A_{u_n} - A_u\|_\infty r'(b-a)(1 + R(b-a)\exp(R(b-a))),$$

so that the claim follows from (3.4).

To prove that (NLL) has solutions , we consider, for any $u \in C(J,X)$, the map $S : C(J,X) \to C(J,X)$ defined by $S(u) := K_u H(u) - K_u L z_u + z_u$. We now prove that S has fixed points and that these are solutions of (NLL).

Step 3. For any $u \in C(J,X)$, $S(u)$ is a solution of the linearized problem

$$(\mathrm{NL})_u \begin{cases} x' = A_u(t)x + f_u(t) \\ Lx = H(u). \end{cases}$$

Indeed, since the range of K_u is contained in $\operatorname{Ker} D_u$ (see (h_6)), we have $D_u K_u y = 0 \ \ \forall\, y \in X$, in such a way that $D_u S(u) = D_u z_u$. Hence, from (3.2) and (3.3), it follows that

$$(S(u))'(t) = A_u(t)((S(u))(t)) + f_u(t) \tag{3.5}$$

Moreover, from (iii) of (h_6) we have

$$LS(u) = H(u) - Lz_u + Lz_u = H(u).$$

An obvious consequence of Step 3 is that the fixed points of S are solutions of (NLL). The existence of fixed points of S will follow from Lemma 2.1.

Step 4. S is a continuous map.
Indeed, let $u_n \to u_0$. There exists an $r > 0$ such that $\|u_n\|_\infty, \|u_0\|_\infty \le r$. Now, $\|E_{u_n} - E_{u_0}\|_\infty \to 0$ (Step 2) and $\|f_{u_n} - f_{u_0}\|_\infty \to 0$ (Remark 3), so that $E_{u_n}(t,s) f_{u_n}(s) \to E_{u_0}(t,s) f_{u_0}(s)$ uniformly in (t,s) and therefore $\|z_{u_n} - z_{u_0}\|_\infty \to 0$. Moreover,

$$\|K_{u_n} L z_{u_n} - K_{u_0} L z_{u_0}\|_\infty \le \|K_{u_n}\| \, \|L\| \, \|z_{u_n} - z_{u_0}\|_\infty + \|K_{u_n} - K_{u_0}\| \, \|L\| \, \|z_{u_0}\|_\infty,$$

so that (i) and (ii) of (h_6) yield $\|K_{u_n} L z_{u_n} - K_{u_0} L z_{u_0}\|_\infty \to 0$. Analogously, one can see that $\|K_{u_n} H(u_n) - K_{u_0} H(u_0)\|_\infty \to 0$.

Step 5. S maps bounded sets into relatively compact sets.
Indeed, if $\|u\|_\infty \leq r$, from (h_5) and (h_6) we have

$$\|S(u)\|_\infty \leq m(r_2 + \|L\| \ \|z_u\|_\infty) + \|z_u\|_\infty$$

and so Remark 1 and Step 1 yield

$$\|S(u)\|_\infty \leq m(r_2 + \|L\| r' R(b-a)) + r' R(b-a).$$

Moreover, from (3.5) we have

$$\|(S(u))'\|_\infty \leq \|A_u\|_\infty \|S(u)\|_\infty + \|f_u\|_\infty \leq R(m(r_2 + \|L\| r' R(b-a)) + r' R(b-a) + 1)$$

and this, together with (h_7), is enough to conclude that S is a compact map.

At this point, we consider the set M in Lemma 2.1.

Step 6. $u \in M$ implies that $|u(a)| < d$.
Indeed, $u \in M$ implies that $cu = K_u H(u) - K_u L z_u + z_u$, so that $u(a) = c^{-1}(K_u(H(u) - Lz_u))(a)$. Thus the claim follows from (h_5).

Step 7. S has fixed points.
Indeed, in order to apply Lemma 2.1, we consider the initial value problem

$$\begin{cases} z' = g(t,z) \\ z(a) = d. \end{cases} \tag{3.6}$$

Now, $u \in M$ implies $cu'(t) = A_u(t)cu(t) + f_u(t)$, so that from ($h_3$) we have $|u'(t)| \leq g(t, |u(t)|)$. By Lemma 2.2, $|u(t)| \leq z(t)$, where $z(t)$ is the unique solution of (3.6) in J. This is sufficient to conclude that the set M is bounded, so that, from Lemma 2.1, we can affirm the existence of fixed points for S. ∎

4. Applications.

Example 1. Let $J = [0,1], X = \mathbf{R}^2$ normed by $\|(x_1, x_2)\| := |x_1| + |x_2|$ and $B(X) = M_{2\times 2}$ be the Banach algebra of the real 2×2 matrices $B = (b_{ij})$ normed by $\|B\| := \max |b_{ij}|$.Moreover, let g_1 and g_2 be two functions from $J \times \mathbf{R}$ into $\mathbf{R}$. We assume

(a) g_1 is a bounded continuous function on $J \times \mathbf{R}, \|g_1\|_\infty := \sup\{|g_1(t,x)| : (t,x) \in J \times \mathbf{R}\}$.
(b) g_2 is a continuous function for which there exist a function $h \in C(J, \mathbf{R})$ and a constant $\beta > 0$ such that $|g_2(t,x)| \leq h(t) + \beta|x|$.

We consider the matrix function $A : J \times \mathbf{R}^2 \to M_{2\times 2}$ defined by $A\left(t, \binom{x_1}{x_2}\right) = \begin{pmatrix} 0 & 1 \\ 0 & g(t,x_1) \end{pmatrix}$ and let $E_u(t,s) = \begin{pmatrix} E_u^{11}(t,s) & E_u^{12}(t,s) \\ E_u^{21}(t,s) & E_u^{22}(t,s) \end{pmatrix}$ be the evolution operator of A which depends on $(u_1, u_2) = u \in C(J, \mathbf{R}^2)$.

We want to look for the solutions of the second order nonlinear ordinary differential equation

$$x'' - g_1(t,x)x' = g_2(t,x) \tag{4.1}$$

with boundary conditions of the type

$$\left\{\begin{array}{l}\int_0^1(\int_s^1 \exp(\int_s^w g_1(v,x(v))\,dv)\,dw)g_2(s,x(s))\,ds = x(1)\\ \int_0^1(\int_0^t \exp(\int_s^t g_1(v,x(v))\,dv)g_2(s,x(s))\,ds)\,dt = \int_0^1 x'(t)\,dt.\end{array}\right. \tag{4.2}$$

The equation (4.1) can be written as $y' = A(t,y)y + f(t,y)$, where $y = \binom{y_1}{y_2}$ belongs to $\mathbf{R}^2$ and $f\left(t,\binom{y_1}{y_2}\right) = \binom{0}{g_2(t,y_1)}$.

Finally, we introduce the operators L and H from $C(J,\mathbf{R}^2)$ in $\mathbf{R}^2$ by

$$L\begin{pmatrix}u_1\\u_2\end{pmatrix} = \begin{pmatrix}u_1(1)\\ \int_0^1 u_2(t)\,dt\end{pmatrix},\ H\begin{pmatrix}u_1\\u_2\end{pmatrix} =$$

$$= \begin{pmatrix}\int_0^1(\int_s^1 \exp(\int_s^w g_1(v,u_1(v))\,dv)\,dw)g_2(s,u_1(s))\,ds\\ \int_0^1(\int_0^t \exp(\int_s^t g_1(v,u_1(v))\,dv)g_2(s,u_1(s))\,ds)\,dt\end{pmatrix}.$$

Now, the ordinary differential problem (4.1)–(4.2) can be equivalently formulated as (NLL). To prove the existence of solutions it is sufficient to see that (h_1)–(h_7) are satisfied.

Step 1. (h_1), (h_2), (h_4), (h_7) are satisfied.
Obvious.

Step 2. (h_3) is satisfied.
Let $x = (x_1, x_2)$. Then $\|A(t,x)x\| + \|f(t,x)\| = |x_2|(1 + |g_1(t,x_1)|) + |g_2(t,x_1)| \leq \|x\|(1 + \|g_1\|_\infty) + \|h\|_\infty + \beta\|x\|$. Put $\hat{a} := 1 + \|g_1\|_\infty + \beta, \hat{b} := \|h\|_\infty, g(t,z) := \hat{a}z + \hat{b}$. We obtain $\|A(t,x)x\| + \|f(t,x)\| \leq g(t,\|x\|)$, where of course g satisfies the hypotheses of Lemma 2.2 and the unique solution of the initial value problem $z' = g(t,z), z(0) = z_0$ is given by

$$z(t) = (z_0 + (\hat{b}/\hat{a}))\exp(\hat{a}t) - (\hat{b}/\hat{a}).$$

Step 3. (h_5) is satisfied.
It is enough to note that

$$E_u(t,s) = \begin{pmatrix}1 & \int_s^t \exp(\int_s^w g_1(v,u_1(v))\,dv)\,dw\\ 0 & \exp(\int_s^t g_1(v,u_1(v))\,dv)\end{pmatrix}.$$

Step 4. (h_6) is satisfied.
Let $(u_1,u_2) = u$ be an element of $C(J,\mathbf{R}^2)$. We define the operator $K_u : \mathbf{R}^2 \to C(J,\mathbf{R}^2)$ by

$$K_u\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}(b/\int_0^1 p_u(s)\,ds)\int_0^t p_u(s)\,ds + a - b\\ (b/\int_0^1 p_u(s)\,ds)p_u(t)\end{pmatrix},$$

where $p_u(s) := \exp(\int_0^s g_1(v,u_1(v))\,dv)$.

It is a routine calculation to verify that the range of K_u is $\operatorname{Ker} D_u$ and that $LK_u = I$ on $\mathbf{R}^2$, so that (iii) of (h_6) is obviously satisfied. Moreover, $u \mapsto K_u$ is a continuous function, as it is easy to verify by the definition of K_u. Finally, for each $u \in C(J, \mathbf{R}^2)$, we have $\|K_u\| \leq 2 + \|g_1\|_\infty (1 - \exp(-\|g_1\|_\infty)) \exp(\|g_1\|_\infty)$, so that (ii) is satisfied, too$\blacklozenge$

Example 2. Let J, g_1, g_2, A, f be as in the previous example. Fix $t_0 \in J$. Then the problem

$$(4.3) \qquad \begin{cases} x'' - g_1(t,x)x' = g_2(t,x) \\ x(0) = \sin\left(|1 - x(t_0) + x'(t_0)|\right)^{1/2} \\ x'(0) = \cos\left(\int_0^1 x(t)\,dt - 2 + \int_0^1 x'(t)\,dt\right) \end{cases}$$

can be written as (NLL) with H and L defined by

$$H\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \sin\left(|1 - x(t_0) + x'(t_0)|\right)^{1/2} \\ \cos\left(\int_0^1 x(t)\,dt - 2 + \int_0^1 x'(t)\,dt\right) \end{pmatrix}, \; L\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix}.$$

Then (h_1)–(h_7) are satisfied by taking

$$K_u\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b\int_0^t p_u(s)\,ds + a \\ bp_u(t) \end{pmatrix}$$

and so problem (4.3) has a solution.

In general, if f_1, f_2 are two bounded continuous functions from $C(J, \mathbf{R}) \times C(J, \mathbf{R})$ into $\mathbf{R}$, then the ordinary differential problem

$$(4.4) \qquad \begin{cases} x'' - g_1(t,x)x' = g_2(t,x) \\ x(0) = f_1(x, x') \\ x'(0) = f_2(x, x') \end{cases}$$

can be written as (NLL). As above, one can verify that (h_1)–(h_7) are satisfied, so that the problem (4.4) admits solutions$\blacklozenge$

Remark 4. The previous examples also work with the weaker assumptions: $|g_1(t,x)| \leq a_1(t) + b_1|x|$ for some $a_1 \in C(J, \mathbf{R})$, $b_1 \in \mathbf{R}^+$ and $(1 + \|a_1\|_\infty + \beta)^2 - 4b_1\|h\|_\infty \geq 0$. In this case g is defined by $g(t,z) := b_1 z^2 + (1 + \|a_1\|_\infty + \beta)z + \|h\|_\infty$. If, moreover, the following inequality

$$(4.5) \qquad g_2(t, u(t)) \leq \mathrm{G}\exp(\|g_1(\cdot, u(\cdot))\|_\infty t)$$

holds for some $G > 0$ and for every $u \in C(J, \mathbf{R})$, then the Nicoletti problem

$$\begin{cases} x'' - g_1(t,x)x' = g_2(t,x) \\ x(t_1) = r_1, \; x(t_2) = r_2 \end{cases}$$

has solutions. (Here (4.5) assures that (h_5) (ii) holds)$\blacklozenge$

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