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Commentationes Mathematicae Universitatis Carolinae, Vol. 31 (1990), No. 4, 661--684

Persistent URL: http://dml.cz/dmlcz/106902

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Abstract. Support functionals in Musielak–Orlicz sequence spaces endowed with the Luxemburg norm are completely characterized. An explicit formula for regular support functionals is given. For obtaining a characterization of singular support functionals a generalized Banach limit is applied. Some necessary and sufficient conditions for smooothness of these spaces are given, too.

Keywords: Musielak–Orlicz spaces, support functionals, smoothness

Classification: 46B20, 46B25

0. Introduction.

This paper is divided into four parts. The first part is an introduction. The second part consists of some results concerning the existence and the general form of regular support functionals at points of the unit sphere $S(l^\Phi)$ of Musielak–Orlicz sequence spaces l^Φ. In the third part a general formula for singular support functionals at points of $S(l^\Phi)$ is given. In the last one a criterion for smoothness of Musielak–Orlicz sequence spaces is obtained.

Smoothness and uniform smoothness of Orlicz spaces equipped with the Luxemburg norm were first discussed by Rao in [**13**] and [**14**]. However, no completely full characterizations of these properties were obtained. Smoothness of Orlicz sequence spaces was considered by Ye in [**14**]. Next, Płuciennik and Ye considered in [**11**] smoothness of Musielak–Orlicz sequence spaces obtaining almost complete its characterization. Moreover, Chen obtained in [**2**] a characterization of smooth Orlicz function spaces endowed with the Orlicz norm in the case of a non-atomic finite measure. These problems were also considered in [**15**].

In the sequel $\mathbf{N}$ denotes the set of natural numbers, $\mathbf{R}$ denotes the reals, $\mathbf{R}^e$ denotes the interval $[-\infty, +\infty]$ and is called the set of extended reals, $\mathbf{R}_+$ denotes the set of nonnegative reals and Φ_i are Orlicz functions which means that Φ_i are vanishing and continuous at zero, left-continuous on whole $\mathbf{R}_+$, convex and even on $\mathbf{R}$, and not identically equal to zero. For any Musielak–Orlicz function $\Phi = (\Phi_i)_{i=1}^\infty$ we denote by Φ^* its complementary function in the sense of Young, i.e. $\Phi^* = (\Phi_i^*)_{i=1}^\infty$, where

$$\Phi_i^*(u) = \sup_{v>0}\{|u|v - \Phi_i(v)\} \quad (\forall\, u \in \mathbf{R}).$$

If Ψ is an Orlicz function and $u \in \mathbf{R}$, we denote by $\Psi^-(u)$ and $\Psi^+(u)$ the left and the right derivatives of Ψ at u, respectively. Given an Orlicz function Ψ, we define

$$b(\Psi) = \sup\{u \in \mathbf{R}_+ : \Psi(u) < +\infty\},$$

$$\begin{aligned}
\partial\Psi(u) =&[\Psi^-(u), \Psi^+(u)] && \text{if } u \geq 0 \text{ and } u < b(\Psi)\\
=&[\Psi^+(u), \Psi^-(u)] && \text{if } u < 0 \text{ and } u > -b(\Psi)\\
=&[\Psi^-(u), +\infty) && \text{if } u = b(\Psi) \text{ and } \Psi^-(b(\Psi)) < +\infty\\
=&(-\infty, \Psi^+(u)] && \text{if } u = -b(\Psi) \text{ and } \Psi^+(-b(\Psi)) > -\infty\\
=&\{+\infty\} && \text{if } u = b(\Psi) \text{ and } \Psi^-(b(\Psi)) = +\infty\\
=&\{-\infty\} && \text{if } u = -b(\Psi) \text{ and } \Psi^+(-b(\Psi)) = -\infty
\end{aligned}$$

It is not difficult to show that for any $u \in \mathbf{R}$ and $v \in \partial\Psi(u)$ we have $\Psi(u) + \Psi^*(v) = uv$.

Moreover, if Ψ is an Orlicz function with finite values then

$$\partial\Psi(u) = \{v \in \mathbf{R} : \Psi(u) + \Psi^*(v) = uv\} \quad (\forall\, u \in \mathbf{R}). \tag{0.1}$$

When Ψ is an Orlicz function which jumps to $+\infty$, equality (0.1) holds only for these $u \in \mathbf{R}$ which satisfy $\Psi(u) < +\infty$.

Let us denote by l^0 the space of all sequences of reals, and for any $x = (x_i)_{i=1}^{\infty} \in l^0$ and $A \subset \mathbf{N}$ define $x^A = \sum_{i\in A} x_i e_i$, where e_i are the i-th basic sequences, i.e. $e_i = (0, \dots, 0, 1, 0, \dots)$, where 1 stands on the i-th place. For any $x \in l^0$ we define also

$$x^{(n)} = (x_1, x_2, \dots, x_n, 0, \dots).$$

If $\Phi = (\Phi_i)_{i=1}^{\infty}$ is a Musielak–Orlicz function and $x = (x_i)_{i=1}^{\infty} \in l^0$, we define $\partial\Phi(x) = (\partial\Phi_i(x_i))_{i=1}^{\infty}$. Moreover, we define

$$\begin{aligned}
b_i =&\ b_i(\Phi_i),\\
a_i =&\ a_i(\Phi_i) = b_i && \text{when } \Phi_i(b_i) \leq 1,\\
&\qquad\quad\ = \Phi_i^{-1}(1) && \text{when } \Phi_i(b_i) > 1.
\end{aligned}$$

Given a Musielak–Orlicz function Φ, we define on l^0 a convex functional I_Φ by the formula

$$I_\Phi(x) = \sum_{i=1}^{\infty} \Phi_i(x_i) \quad (\forall\, x = (x_i) \in l^0).$$

The Musielak–Orlicz space l^Φ generated by a Musielak–Orlicz function Φ is defined in the following way

$$l^\Phi = \{x \in l^0 : I_\Phi(\lambda x) < +\infty \text{ for some } \lambda > 0\}.$$

This space endowed with the Luxemburg norm

$$\|x\|_\Phi = \inf\{\lambda > 0 : I_\Phi(x/\lambda) \leq 1\}$$

is a Banach space (cf. [6], [7] and [8]).

For any Musielak–Orlicz function Φ we define h^Φ to be a closure in l^Φ with respect to the norm topology defined above of the set h of all sequences in l^0 with finite number of coordinates different from 0. This space will be considered with the norm $\| \|_\Phi$ induced from l^Φ.

In the case when $b_i = b_i(\Phi_i) = +\infty$ for any $i \in \mathbb{N}$, we have

$$h^\Phi = \{x \in l^0 : I_\Phi(\lambda x) < +\infty \quad \text{for any } \lambda > 0\}.$$

Every functional $x^* \in (h^\Phi)^*$ (=the dual space of h^Φ) is of the form

$$x^*(y) = \sum_{i=1}^{\infty} z_i y_i \quad (\forall\, y = (y_i) \in h^\Phi), \tag{0.2}$$

where $z = (z_i) \in l^{\Phi^*}$ (cf. [6], [7], [8] and [9]). It is obvious by the Hölder inequality

$$|x^*(y)| \leq 2\|z\|_{\Phi^*}\|y\|_\Phi \tag{0.3}$$

that every linear functional defined by formula (0.2) is also continuous on l^Φ. Such functional are called regular.

A functional $x^* \in (l^\Phi)^*$ is said to be singular if $x^*(y) = 0$ for every $y \in h^\Phi$.

We denote by $B(l^\Phi)$ and $S(l^\Phi)$ the unit ball and the unit sphere of l^Φ, respectively.

We say that Φ satisfies the δ_2^0-condition if there exist constants $K, a > 0$, a number $m \in \mathbb{N}$ and a sequence (c_i) of nonnegative extended reals such that $\sum_{i=m}^{\infty} c_i < +\infty$ and for any $i \in \mathbb{N}$ and $u \in \mathbb{R}$ satisfying the inequality $\Phi_i(u) \leq a$, we have

$$\Phi_i(2u) \leq K\Phi_i(u) + c_i.$$

Let X be a Banach space and X^* be its dual space. Then $x^* \in X^*$ is said to be a support functional at $x \in X \setminus \{0\}$ if $\|x^*\| = 1$ and $x^*(x) = \|x\|$ (cf. [10]). We denote by $\mathrm{Grad}(x)$ the set of all support functionals at x. If X is a Musielak–Orlicz space and $x \in X \setminus \{0\}$, we denote by $\mathrm{RGrad}(x)$ and $\mathrm{SGrad}(x)$ the sets of all regular and all singular support functionals at x, respectively.

A Banach space X is said to be smooth if for any $x \in S(X)$ the set $\mathrm{Grad}(x)$ has only one element (cf. [2] and [10]).

1. Regular support functionals in l^Φ.

We start with the following

Lemma 1.1. (i) *For any $x^* \in (l^\Phi)^*$, we have $(x^*(e_i)) \in l^{\Phi^*}$.*

(ii) *The functional $\overline{x}^*$ defined by*

$$\overline{x}^*(x) = \sum_{i=1}^{\infty} x_i^*(e_i)x_i \quad (\forall\, x = (x_i) \in l^\Phi)$$

is continuous on l^Φ and the functional $\underline{x}^*$ defined by

$$\underline{x}^*(x) = x^*(x) - \overline{x}^*(x) \quad (\forall\, x \in l^\Phi) \tag{1.1}$$

is singular.

(iii) *For every $x^* \in (l^\Phi)^*$, we have*

$$x^* = \overline{x}^* + \underline{x}^*, \tag{1.2}$$

where $\overline{x}^*$ *and* $\underline{x}^*$ *are the regular and singular parts of* x^* *defined above, respectively. The representation* (1.2) *is unique for any* $x^* \in (l^\Phi)^*$.

PROOF : (i) Let us denote by $\tilde{x}^*$ the restriction of x^* to the subspace h^Φ and let $x \in h^\Phi$. Then $\|x - x^{(n)}\|_\Phi \to 0$ and

$$\tilde{x}^*(x^{(n)}) = \sum_{i=1}^{n} x^*(e_i)x_i.$$

Therefore,

$$\tilde{x}^*(x) = \lim x^*(x^{(n)}) = \lim \sum_{i=1}^{n} x^*(e_i)x_i = \sum_{i=1}^{\infty} x^*(e_i)x_i.$$

In view of the general representation of linear continuous functionals over h^Φ (cf. formula (0.2)), we conclude that $(x^*(e_i)) \in l^{\Phi^*}$.

(ii) This follows immediately from (i) and from the Hölder inequality (0.3).

(iii) Equality (1.2) is obvious. It is obvious also that x^* and $\overline{x}^*$ coincide on h. Therefore, they coincide also on h^Φ. Hence it follows that $\underline{x}^*$ is a singular functional. The uniqueness of the representation (1.2) is obvious. ∎

Lemma 1.2. *Assume that* $x \in S(l^\Phi)$ *and* $x^* \in \mathrm{RGrad}(x)$ *is represented by a sequence* $\lambda = (\lambda_i) \in l^{\Phi^*}$. *Then:*

(i) $\lambda_i x_i \geq 0$ *for any* $i \in \mathbb{N}$,

(ii) *if* $\lambda_{i_0} x_{i_0} > 0$ *and* $|x_{i_0}| < a_{i_0}$ *for some* $i_0 \in \operatorname{supp} x^*$, *then* $I_\Phi(x) = \alpha$, *where* $\alpha = \sup\{I_\Phi(y^{\operatorname{supp} x^*}) : \|y\|_\Phi \leq 1\}$.

PROOF : (i) Assume that $\lambda_{i_0} x_{i_0} < 0$ for a certain $i_0 \in \mathbb{N}$ and define

$$\overline{x} = \sum_{i \neq i_0} x_i e_i - x_{i_0} e_{i_0}.$$

Then we have $\|\overline{x}\|_\Phi = 1$ and $x^*(\overline{x}) > x^*(x)$, a contradiction.

(ii) Assume that $\lambda_{i_0} x_{i_0} > 0$ and $|x_{i_0}| < a_{i_0}$ for some $i_0 \in \mathbb{N}$ as well as $I_\Phi(x) < \alpha$. There is a number $c, c > |x_{i_0}|$, such that

$$\sum_{i \neq i_0} \Phi(x_i) + \Phi_{i_0}(c) \leq \alpha.$$

Defining

$$\overline{x} = \sum_{i \neq i_0} x_i e_i + c \operatorname{sgn}(\lambda_{i_0}) e_{i_0},$$

we get $\|\overline{x}\|_\Phi \leq 1$ and $x^*(\overline{x}) > x^*(x)$, a contradiction. This finishes the proof. ∎

Lemma 1.3. *Let* $\alpha = \sup\{I_\Phi(y) : \|y\|_\Phi \le 1\}, x \in S(l^\Phi)$ *and* $x^* \in \mathrm{RGrad}(x)$. *Then* $\mathrm{supp}\, x^* \subset A_x = \{i \in \mathbf{N} : |x_i| = a_i\}$ *whenever* $I_\Phi(x) < \alpha$.

PROOF : Denote by $\lambda = (\lambda_i)$ the sequence in l^{Φ^*} which generates the functional x^*. Assume now that $I_\Phi(x) < \alpha$ and there is $i_0 \in \mathrm{supp}\, x^*$ which satisfies $|x_{i_0}| < a_{i_0}$. Then there is a number $c, |x_{i_0}| < c < a_{i_0}$, such that

$$\sum_{i \ne i_0} \Phi_i(x_i) + \Phi_{i_0}(c) \le 1. \tag{1.3}$$

Defining

$$\overline{x} = \sum_{i \ne i_0} x_i e_i + c \,\mathrm{sgn}(\lambda_{i_0}) e_{i_0}, \tag{1.4}$$

we have $\|\overline{x}\|_\Phi = 1$ and $x^*(\overline{x}) = \sum_{i=1}^{\infty} \lambda_i \overline{x}_i > \sum_{i=1}^{\infty} \lambda_i x_i = 1$, a contradiction. Therefore, the Lemma is proved. ■

Lemma 1.4. *Let* $x \in S(l^\Phi)$ *and* $x^* \in \mathrm{RGrad}(x)$. *Then for every* $i, j \in \mathrm{supp}\, x^*$ *there exist* $d_i \in \partial\Phi_i(x_i), d_j \in \partial\Phi_j(x_j)$ *such that*

$$x^*(e_i) d_j = x^*(e_j) d_i. \tag{1.5}$$

PROOF : Denote shortly $x^*(e_i) = \lambda_i$. In virtue of Lemma 1.2(i) and the definition of $\partial\Phi_i(x_i)$ we know that $\lambda_i \alpha_j$ and $\lambda_j \alpha_i$ are of the same sign. Therefore, we may assume without loss of generality that $\lambda_i \ge 0, d_j \ge 0, \lambda_j \ge 0$, and $d_i \ge 0$. If the equality (1.5) does not hold, then

$$\lambda_i d_j \ne \lambda_j d_i \tag{1.6}$$

for every $d_i \in \partial\Phi(x_i), d_j \in \partial\Phi(x_j)$, where $i, j \in \mathrm{supp}\, x^*$. Hence it follows that

$$\lambda_i \Phi_j^-(x_j) > \lambda_j \Phi_i^+(x_i) \tag{1.7}$$

for some $i, j \in \mathrm{supp}\, x^*$. Indeed, if condition (1.6) is not satisfied, then

$$\lambda_i \Phi_j^-(x_j) > \lambda_j \Phi_i^+(x_i) \quad \text{or} \quad \lambda_i \Phi_j^-(x_j) < \lambda_j \Phi_i^+(x_i). \tag{1.8}$$

The first inequality of (1.8) is exactly inequality (1.7). If in the alternative (1.8) the second inequality holds, then it must be also

$$\lambda_i \Phi_j^+(x_j) < \lambda_j \Phi_i^-(x_i). \tag{1.9}$$

Indeed, in the opposite case it would be

$$\lambda_i \Phi_j^+(x_j) > \lambda_j \Phi_i^-(x_i), \tag{1.10}$$

because the equality is not possible by inequality (1.6). However, inequality (1.10) together with the second inequality in (1.8) and the fact that $\partial\Phi_k(u)$ are connected sets (intervals) for every $k \in \mathbb{N}$ yield that

$$\lambda_i d_j = \lambda_j d_i$$

for some $d_i \in \partial\Phi_i(x_i)$ and $d_j \in \partial\Phi_j(x_j)$, a contradiction. This proves that condition (1.6) implies the alternative of conditions (1.9) and (1.10). However, the meanings of these two conditions are the same. Only the roles of i and j are changed. Therefore, we may assume that condition (1.7) holds. This implies that $\Phi_i^+(x_i) < +\infty$ and $\lambda_i > 0$. Therefore $|x_i| < b_i$. First, we restrict ourselves only to the case when $\Phi_i^+(x_i) > 0$. Then, in virtue of inequality (1.7), we have

$$\lambda_i/\Phi_i^+(x_i) > \lambda_j/\Phi_j^-(x_j).$$

Thus, there exists a number $k > 0$ such that

$$\lambda_i/\Phi_i^+(x_i) > k > \lambda_j/\Phi_j^-(x_j).$$

Since Φ_i^+ is right-continuous and Φ_j^- is left-continuous, there exist $\overline{x}_i$ and $\overline{x}_j$ such that $x_i < \overline{x}_i < +\infty$, $0 < \overline{x}_j < x_j$, and

$$\lambda_i/\Phi_i^+(\overline{x}_i) > k > \lambda_j/\Phi_j^-(\overline{x}_j), \tag{1.11}$$

$$\int_{x_i}^{\overline{x}_i} \Phi_i^+(t)\,dt = \int_{\overline{x}_j}^{x_j} \Phi_j^-(t)\,dt. \tag{1.12}$$

Let $\overline{x} = (\overline{x}_n)_{n=1}^{\infty}$, where

$$\begin{aligned} \overline{x}_n &= x_n && \text{when } n \neq i \text{ and } n \neq j,\\ &= \overline{x}_i && \text{when } n = i,\\ &= \overline{x}_j && \text{when } n = j. \end{aligned}$$

In virtue of equality (1.12) we have

$$\begin{aligned} I_\Phi(\overline{x}) &= I_\Phi(x) + \Phi_i(\overline{x}_i) - \Phi_i(x_i) + \Phi_j(\overline{x}_j) - \Phi_j(x_j) = \\ &= I_\Phi(x) + \int_{x_i}^{\overline{x}_i} \Phi_i^+(t)\,dt - \int_{\overline{x}_j}^{x_j} \Phi_j^-(t)\,dt = I_\Phi(x) = 1. \end{aligned}$$

Therefore, $\|\overline{x}\|_\Phi = 1$. Moreover,

$$\begin{aligned} x^*(\overline{x}) - x^*(x) =& (\lambda_i\overline{x}_i - \lambda_i x_i) - (\lambda_j x_j - \lambda_j\overline{x}_j)\\ =& k(\overline{x}_i - x_i)\lambda_i/k - k(x_j - \overline{x}_j)\lambda_j/k\\ =& k\int_{x_i}^{\overline{x}_i} [\lambda_i/k - \Phi_i^+(t) + \Phi_i(t)]\,dt\\ & -k\int_{\overline{x}_j}^{x_j} [\Phi_j^-(t) + (\lambda_j/k - \Phi_j^-(t))]\,dt\\ =& k\{\int_{x_i}^{\overline{x}_i} (\lambda_i/k - \Phi_i^+(t))\,dt - \int_{\overline{x}_j}^{x_j} (\lambda_j/k - \Phi_j^-(t))\,dt\}\\ & +\{\int_{x_i}^{\overline{x}_i} \Phi_i^+(t)\,dt - \int_{\overline{x}_j}^{x_j} \Phi_j^-(t)\,dt\}. \end{aligned} \tag{1.13}$$

In virtue of equality (1.12) the value of the last bracket is equal to 0. Moreover,

$$\lambda_i/k > \Phi_i^+(t) \quad (\forall\, t \in [x_i, \overline{x}_i]),$$
$$\lambda_j/k < \Phi_j^-(t) \quad (\forall\, t \in [\overline{x}_j, x_j]).$$

Therefore,

$$\int_{x_i}^{\overline{x}_i} [\lambda_i/k - \Phi_i^+(t)]\, dt > 0, \tag{1.14}$$

$$\int_{\overline{x}_j}^{x_j} [\lambda_i/k - \Phi_j^-(t)]\, dt < 0. \tag{1.15}$$

Combining equality (1.13) and inequalities (1.14) and (1.15), we get

$$x^*(\overline{x}) > x^*(x) = 1,$$

which contradicts to the fact that $x^* \in \mathrm{RGrad}(x)$. This finishes the proof in the case of $\Phi_i^+(x_i) > 0$.

Assume now that $\Phi_i^+(x_i) = 0$. Then it must be $\Phi_i^+(\overline{x}_i) > 0$ for any $\overline{x}_i > x_i$. Indeed, in the opposite case we have $\Phi_i^+(\overline{x}_i) = 0$ for some $\overline{x}_i > x_i$. Defining $\overline{x} = \sum_{j \neq i} x_j e_j + \overline{x}_i e_i$, we have $\overline{x} \in S(l^\Phi)$ and $x^*(\overline{x}) > x^*(x)$, a contradiction. Therefore, we can find $\overline{x}_i > x_i$ and $\overline{x}_j < x_j$ in such a way that equalities (1.12) and (1.13) hold, $\lambda_i/k > \Phi_i^+(t)$ for any $t \in (x_i, \overline{x}_i]$ and $\lambda_j/k < \Phi_j^-(t)$ for any $t \in [\overline{x}_j, x_j]$. Now, we can repeat the proof from the case of $\Phi_i^+(x_i) > 0$. ■

Corollary 1.5. *Let Φ and $x \in S(l^\Phi)$ be such that $\partial\Phi_i(x_i) = +\infty$ or $\partial\Phi_i(x_i) = -\infty$ for a certain $i \in \mathbb{N}$. Then for every $x^* \in \mathrm{RGrad}(x)$ it must be* $\operatorname{supp} x^* \subset \{i \in \mathbb{N} : |x_i| = b_i\}$.

PROOF : This follows immediately from equality (1.5) of Lemma 1.4. ■

Lemma 1.6. (i) *Let Φ and $x \in S(l^\Phi)$ be such that $A_x = \{i \in \mathbb{N} : |x_i| = a_i\} \neq \emptyset$. Let $(\lambda_i)_{i \in A_x}$ be a family of nonnegative numbers such that $\sum_{i \in A_x} \lambda_i = 1$. Then the functional x^* defined by the formula*

$$x^*(y) = \sum_{i \in A_x} \lambda_i y_i / x_i \quad (\forall\, y = (y_i) \in l^\Phi) \tag{1.16}$$

is a support functional at x.

(ii) *If additionally $A_x^\infty = \{i \in A_x : |x_i| = b_i\} \neq \emptyset$, then every $x^* \in \mathrm{RGrad}(x)$ is of the form*

$$x^*(y) = \sum_{i \in A_x^\infty} \lambda_i y_i / x_i \quad (\forall\, y = (y_i) \in l^\Phi), \tag{1.17}$$

where $\lambda_i \geq 0$ for any $i \in A_x^\infty$ and $\sum_{i \in A_x^\infty} \lambda_i = 1$.

PROOF : (i) We have $x^*(x) = \sum_{i\in A_x} \lambda_i = 1$. Assume that $y \in B(l^\Phi)$. Then $|y_i| \le a_i$ for every $i \in \mathbb{N}$. Therefore,

$$|x^*(y)| \le \sum_{i\in A_x} \lambda_i |y_i|/a_i \le \sum_{i\in A_x} \lambda_i = 1,$$

which means that $\|x^*\| = 1$, i.e. $x^* \in \mathrm{RGrad}(x)$.

(ii) Assume that $A_x^\infty \neq \emptyset$ and $x^* \in \mathrm{RGrad}(x)$. Then, in virtue of Corollary 1.5, we have $\mathrm{supp}\, x^* \subset A_x^\infty$. Therefore,

$$x^*(y) = \sum_{i\in A_x^\infty} z_i y_i \quad (\forall\, y = (y_i) \in l^\Phi), \tag{1.18}$$

where z_i are some reals such that

$$x^*(x) = \sum_{i\in A_x^\infty} z_i x_i = 1, \tag{1.19}$$

$$\|x^*\| = 1. \tag{1.20}$$

In virtue of Lemma 1.2(i), we have $\lambda_i = z_i x_i \ge 0$. Moreover, in view of (1.19), we have $\sum_{i\in A_x^\infty} \lambda_i = 1$. Since $z_i = \lambda_i/x_i$ for every $i \in A_x^\infty$, we can write formula (1.18) in the form (1.16). ∎

Lemma 1.7. *Let $x \in S(l^\Phi)$. If $d(x, h^\Phi) = \inf\{\|x-y\|_\Phi : y \in h^\Phi\} < 1$, then* $\mathrm{Grad}(x) = \mathrm{RGrad}(x)$.

PROOF : We have $\|x - y\|_\Phi < 1$ for some $y \in h^\Phi$. Let $x^* \in \mathrm{Grad}(x)$. If $\underline{x}^*$ in (1.1) is nonzero, then

$$\begin{aligned} 1 = \|x^*\| = \|\overline{x}^*\| + \|\underline{x}^*\| = x^*(x) = \overline{x}^*(x) + \underline{x}^*(x-y) &\le \|\overline{x}^*\| + \|\underline{x}^*\|\ \|x-y\|_\Phi \\ &\le \|\overline{x}^*\| + \|\underline{x}^*\|\ \|x-y\|_\Phi, \end{aligned}$$

a contradiction. ∎

Remark 1.8. If Φ and $x \in S(l^\Phi)$ are such that $A_x = \{i \in \mathbb{N} : |x_i| = a_i\} \neq \emptyset$, then $\mathrm{RGrad}(x) \neq \emptyset$.

Indeed, in view of Lemma 1.6, the functional x^* is defined by the formula

$$x^*(y) = y_i/x_i \quad (\forall\, y = (y_i) \in l^\Phi),$$

where $i \in A_x$, belongs to $\mathrm{RGrad}(x)$.

The problem arises whether or not for every $x \in S(l^\Phi)$, the condition $I_\Phi(x) = \alpha$, where $\alpha = \sup\{I_\Phi(y) : \|y\|_\Phi \le 1\}$, implies that $\mathrm{RGrad}(x) \neq \emptyset$. The answer to this problem is negative. A counterexample will be given after the theorem written below.

Theorem 1.9. *Let Φ be a Musielak–Orlicz function, $x \in S(l^\Phi), x^* \in (l^\Phi)^*, A =$ supp x^*. Then $x^* \in$ RGrad(x) if and only if*

$$I_\Phi(x) = \alpha, \text{ where } \alpha = \sup\{I_\Phi(y) : \|y\|_\Phi \le 1, \operatorname{supp} y \subset A\}, \tag{1.21}$$

$$x^*(y) = (\sum_{i\in A} d_i y_i)/(\sum_{i\in A} d_i x_i) \quad (\forall\, y = (y_i) \in l^\Phi), \tag{1.22}$$

where

$$d_i \in \partial\Phi(x_i) \text{ for any } i \in A \text{ and } \sum_{i\in A} d_i x_i < +\infty. \tag{1.23}$$

PROOF : If $|x_i| = a_i$ for any $i \in A$, then condition (1.21) is satisfied. If $|x_{i_0}| < a_{i_0}$ for some $i_0 \in A$, then the necessity of condition (1.21) was proved in Lemma 1.2.

Let $x^* \in$ RGrad(x) and denote $x^*(e_i) = \lambda_i$. In virtue of Lemma 1.4 there exists a constant $k > 0$ such that $\lambda_i = kd_i$ for $i \in A$, where $d_i \in \partial\Phi_i(x_i)$. Therefore, x^* is of the form

$$x^*(y) = k\sum_{i\in A} d_i y_i \quad (\forall\, y = (y_i) \in l^\Phi). \tag{1.24}$$

By the assumption,

$$1 = x^*(x) = k\sum_{i\in A} d_i x_i\,,$$

whence it follows that $d = (d_i)_{i\in A}$ satisfies condition (1.23) and $k = 1/(\sum_{i\in A} d_i x_i)$. Combining this with formula (1.24), we obtain formula (1.22).

Assume now that x^* is a functional from $(l^\Phi)^*$ satisfying conditions (1.21), (1.22) and (1.23). Then we have

$$\Phi(x_i) + \Phi^*(d_i) = d_i x_i \quad (\forall\, i \in A).$$

Therefore

$$I_\Phi(x^A) + I_{\Phi^*}(d) = \sum_{i\in A} d_i x_i\,, \tag{1.25}$$

whence it follows that $I_{\Phi^*}(d) < +\infty$, i.e. $d \in l^{\Phi^*}$, whence $d/k \in l^{\Phi^*}$, where $k = 1/\sum_{i\in A} d_i x_i$. Thus, the linear functional x^* defined by formula (1.22) is continuous over l^Φ. It is evident that $x^*(x) = 1$. Moreover, for every $y \in B(l^\Phi)$, we have $I_\Phi(y^A) \le \alpha$. Thus, in virtue of equality (1.25) and the Young inequality

$$|\sum_{i\in A} d_i y_i| \le I_\Phi(y^A) + I_{\Phi^*}(d) \le \alpha + I_{\Phi^*}(d),$$

we obtain $|x^*(y)| \le 1$. Therefore, $x^* \in$ RGrad(x), which finishes the proof. ∎

Now, we are ready to give a counterexample announced before Theorem 1.9.

Example 1.10. Let $\Phi = (\Phi_i)_{i=1}^{\infty}$, where

$$\begin{aligned}\Phi_i(u) &= |u| \quad \text{if } |u| \leq 2^{-i-1}\\ &= 2^i(|u| - 2^{-i-1}) + 2^{-i-1} \quad \text{if } |u| > 2^{-i-1}.\end{aligned}$$

Define $x = (x_i)_{i=1}^{\infty}$, where $x_i = 2^{-i-1} + 2^{-2i-1}$. We have $\Phi_i(x_i) = 2^{-i}$ for every $i \in \mathbb{N}$. Therefore $I_\Phi(x) = 1 = \alpha$, whence $\|x\|_\Phi = 1$. It is easily seen that Φ_i are smooth at x_i for any $i \in \mathbb{N}$ and $\partial\Phi_i(x_i) = \{2^i\}$. Moreover,

$$\sum_{i=1}^{\infty} 2^i x_i = \sum_{i=1}^{\infty} 2^i(2^{-i-1} + 2^{-2i-1}) \geq \sum_{i=1}^{\infty} 2^i 2^{-i-1} = \sum_{i=1}^{\infty} 1/2 = +\infty .$$

Therefore, in virtue of Theorem 1.9, we have $\mathrm{RGrad}(x) = \emptyset$.

2. Singular support functionals.

We start with the following

Lemma 2.1. *Let Φ and $x \in S(l^\Phi)$ be such that $I_\Phi(x) < 1$, $|x_i| < a_i$ for every $i \in \mathbb{N}$. Let $x^* \in \mathrm{Grad}(x)$ and A be a subset of $\mathbb{N}$. If $0 < \|x^A\|_\Phi < 1$, then $x^*(y^A) = 0$ for every $y \in B(l^\Phi)$.*

PROOF : It follows by Lemma 1.2(ii) that $x^* \in \mathrm{SGrad}(x)$. We divide the proof into two steps.

I. First, we shall prove that

$$x^*(x^A) = 0. \tag{2.1}$$

Assume for a contrary that $x^*(x^A) > 0$. Define

$$y = x^A / \|x^A\|_\Phi .$$

We have $\|y\|_\Phi = 1$. Choose $k \in \mathbb{N}$ in such a manner that $I_\Phi(y^{A_k}) < 1/2$, where $A_k = \{n \in \mathbb{N} : n \geq k\}$, and define

$$\begin{aligned}\overline{y}_i &= y_i \quad \text{if } i \in A \text{ and } i \geq k,\\ &= 0 \quad \text{otherwise.}\end{aligned}$$

Since $y - \overline{y}$ has only finite number of coordinates different from 0 and x^* is singular, we have

$$x^*(\overline{y}) = x^*(y) = x^*(x^A)/\|x^A\|_\Phi .$$

Let $B = \mathbb{N} \setminus A$ and $l \in \mathbb{N}$, $l > k$, be such that

$$\sum_{\substack{i \in B\\ i \geq l}} \Phi_i(x_i) < 1/2.$$

Define $w = (w_i)$, where

$$\begin{aligned} w_i &= y_i \quad \text{if } i \in A \text{ and } i \geq k, \\ &= x_i \quad \text{if } i \in B \text{ and } i \geq l, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Then $I_\Phi(w) < 1/2 + 1/2 = 1$, whence $\|w\|_\Phi \leq 1$. Moreover,

$$\begin{aligned} x^*(w) = x^*(y^A) + x^*(x^B) = x^*(x^A)/\|x^A\|_\Phi + x^*(x^B) > \\ > x^*(x^A) + x^*(x^B) = x^*(x) = 1\,, \end{aligned}$$

because the elements $w - (y^A + x^B)$ and $x - (x^A + x^B)$ have only finite number of coordinates different from 0 and x^* is singular. This contradicts to the fact that $x^* \in \mathrm{Grad}(x)$.

II. Assume now that there exists $y \in B(l^\Phi)$, $y \neq x$, $y \neq 0$, such that $x^*(y^A) \neq 0$. We may assume without loss of generality that $x^*(y^A) > 0$ and $\|y^A\|_\Phi = 1$ (considering $y^A/\|y^A\|_\Phi$ instead of y^A if it is necessary). Choose $m \in \mathbb{N}$ in such a way that

$$\sum_{\substack{i=m \\ i \in A}}^{\infty} \Phi_i(y_i) < 1/2.$$

Define $w = (w_i)_{i=1}^{\infty}$, where

$$\begin{aligned} w_i &= y_i \quad \text{when } i \in A \text{ and } i \geq m, \\ &= x_i \quad \text{when } i \in B \text{ and } i \geq l, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Then

$$I_\Phi(w) = \sum_{\substack{i=m \\ i \in A}}^{\infty} \Phi_i(y_i) + \sum_{\substack{i=l \\ i \in B}}^{\infty} \Phi_i(x_i) < 1/2 + 1/2 = 1,$$

whence $\|w\|_\Phi \leq 1$. Moreover, in view of singularity of x^*, we have

$$x^*(w) = x^*(y^A) + x^*(x^B). \tag{2.2}$$

In virtue of equality (2.1), we have

$$x^*(x^B) = x^*(x) - x^*(x^A) = x^*(x) = 1.$$

Combining this with equality (2.2), we get

$$x^*(w) = x^*(y^A) + x^*(x) = x^*(y^A) + 1 > 1,$$

what contradicts to the fact that $x^* \in \mathrm{Grad}(x)$. ■

Define for any Musielak-Orlicz function Φ:

$$\mathrm{Sg}(l^\Phi) = \{x \in S(l^\Phi) : d(x, h^\Phi) = 1\}, \text{ where } d(x, h^\Phi) = \inf\{\|x - y\|_\Phi : y \in h^\Phi\}.$$

Lemma 2.2. *Let $x \in \mathrm{Sg}(l^\Phi), N_x = \mathrm{supp}\ x$, and $x^* \in \mathrm{SGrad}(x)$. Let $y \in S(l^\Phi)$ and*

$$N^+ = \{i \in N_x : x_i y_i \geq 0\},$$
$$N^- = \{i \in N_x : x_i y_i \leq 0\},$$
$$N^0 = \{i \in N_x : x_i y_i = 0\}.$$

Then:

(i) $x^*(y^Q) \geq 0$ *whenever* $Q \subset N^+$,
(ii) $x^*(y^Q) \leq 0$ *whenever* $Q \subset N^-$,
(iii) $x^*(y^Q) = 0$ *whenever* $Q \subset N^0$.

Proof : (ii) Let $Q \subset N^-$ and $i \in Q$. Then $x_i y_i \leq 0$ and therefore

$$|x_i + y_i| \leq \max(|x_i|, |y_i|),$$

whence

$$\Phi_i(x_i + y_i) \leq \max(\Phi_i(x_i), \Phi_i(y_i)) \leq \Phi_i(x_i) + \Phi_i(y_i).$$

Thus,

$$I_\Phi(x + y^Q) \leq I_\Phi(x) + I_\Phi(y^Q) \leq 2.$$

There exists a natural number k such that

$$\sum_{i=k}^{\infty} \Phi_i(x_i + y_i^Q) \leq 1.$$

Define $w = (w_i)_{i=1}^\infty$, where

$$w_i = \begin{cases} x_i + y_i^Q & \text{if } i \geq k, \\ 0 & \text{if } i < k. \end{cases}$$

We have $I_\Phi(w) \leq 1$, whence $\|w\| \leq 1$ and $x^*(w) \leq 1$. Therefore,

$$1 \geq x^*(w) = x^*(x + y^Q) = x^*(x) + x^*(y^Q) = 1 + x^*(y^Q),$$

whence $x^*(y^Q) \leq 0$.

(i) Take $Q \subset N^+$. Replacing y by $-y$, we obtain the situation as in (ii), whence $x^*(-y^Q) \leq 0$, i.e. $x^*(y^Q) \geq 0$.

Statement (iii) follows by (i) and (ii). ■

We are now near the position to give a formula for singular support functionals. In virtue of Lemma 1.7 it follows that for $x \in S(l^\Phi)$ singular support functionals at the point x may only exist if $d(x, h^\Phi) = 1$. Moreover, in view of the Hahn–Banach theorem we have $\mathrm{SGrad}(x) \neq \emptyset$ whenever $x \in S(l^\Phi)$ and $d(x, h^\Phi) = 1$. We should note here that for $x \in S(l^\Phi)$, we have $d(x, h^\Phi) = 1$ if and only if

$$(*) \qquad \sum_{i=m}^{\infty} \Phi_i(\lambda x_i) = +\infty \quad \text{for any } m \in \mathbb{N} \quad \text{and any } \lambda > 1.$$

To obtain a formula for $x^* \in \mathrm{SGrad}(x)$ if $x \in \mathrm{Sg}(l^\Phi)$, we shall first define on l^Φ a Banach functional ϱ_x generated by $x \in \mathrm{Sg}(l^\Phi)$. By using the analytical version of the Hahn–Banach theorem, we shall deduce that there is a linear minorant for ϱ_x which coincide with ϱ_x on the set $\{\lambda x : \lambda \in \mathbb{R}\}$, and we shall prove that every such a minorant is a singular support functional at x.

Definition 2.3. If $x \in \mathrm{Sg}(l^\Phi)$ then the collection $E(x) = (N_1, N_2, \ldots, N_k)$ of pairwise disjoint subsets of $\mathbb{N}$ such that $\operatorname{supp} x = \bigcup_{i=1}^{k} N_i$ is said to be a finite decomposition of $\operatorname{supp} x$.

The set of all finite decompositions of $\operatorname{supp} x$ for $x \in \mathrm{Sg}(l^\Phi)$ denote by $\mathcal{E}(x)$.

Lemma 2.4. *Let $x \in \mathrm{Sg}(l^\Phi)$ and $E(x) = (N_1, \ldots, N_k) \in \mathcal{E}(x)$. Then $\|x^{N_i}\|_\Phi = 1$ for some $i \in \{1, \ldots, k\}$.*

PROOF : Since $I_\Phi(\lambda x) = +\infty$ for every $\lambda > 1$, it follows that there exists a number $i \in \{1, \ldots, k\}$ such that $I_\Phi(\lambda x^{N_i}) = +\infty$ for every $\lambda > 1$, whence the equality $\|x^{N_i}\|_\Phi = 1$ follows. ∎

Denote by $\operatorname{ext} E(x)$ the set of all $N_i \in E(x)$ such that $\mathrm{Card}(N_i) = +\infty$ and $x^{N_i} \in S(l^\Phi)$. It follows from Lemma 2.4 that $\operatorname{ext} E(x) \neq \emptyset$ for any $x \in \mathrm{Sg}(l^\Phi)$.

Take an arbitrary element $x \in \mathrm{Sg}(l^\Phi)$ and define on l^Φ the following functionals:

$$(2.3) \qquad \begin{aligned} \sigma_x(y, E(x)) &= \sup_{N_l \in \operatorname{ext} E(x)} \ \overline{\lim_{i \in N_l, i \to \infty}} (y_i / x_i) \quad (\forall\, y = (y_i) \in l^\Phi), \\ \varrho_x(y) &= \inf_{E(x) \in \mathcal{E}(x)} \sigma_x(y, E(x)) \quad (\forall\, y = (y_i) \in l^\Phi). \end{aligned}$$

Lemma 2.5. *Let $x \in \mathrm{Sg}(l^\Phi)$. Then the functional ϱ_x defined by formula* (2.3) *is a Banach functional on l^Φ, i.e. ϱ_x is subadditive and positively homogeneous.*

PROOF : The equality $\varrho_x(\lambda y) = \lambda \varrho_x(y)$ for any $\lambda > 0$ and $y \in l^\Phi$ is obvious. Now, we shall prove the subadditivity of ϱ_x. Take arbitrary $y, z \in l^\Phi$ and $\varepsilon > 0$. We can find $E_1 = E_1(x) = (N_1, \ldots, N_k) \in \mathcal{E}(x)$ and $E_2 = E_2(x) = (N_1', \ldots, N_h') \in \mathcal{E}(x)$ such that

$$\begin{aligned} \varrho_x(y) + \varepsilon &> \sigma_x(y, E_1), \\ \varrho_x(z) + \varepsilon &> \sigma_x(z, E_2). \end{aligned}$$

Define a new decomposition of supp $x, E_0 = \{N_l \cap N_m'\}_{l=1}^{k}{}_{m=1}^{h}$. Denote $N_l \cap N_m' = N_{lm}$. If $\|x^{N_{lm}}\|_\Phi = 1$, then the inclusions $N_{lm} \subset N_l$ and $N_{lm} \subset N_m'$ yield $\|x^{N_l}\|_\Phi = \|x^{N_m'}\|_\Phi = 1$. Therefore,

$$
\begin{aligned}
\sigma_x(y, E_0) &= \sup_{N_{lm} \in \text{ext } E_0} \overline{\lim_{i \in N_{lm}, i \to \infty}} (y_i/x_i) \leq \\
&\leq \sup_{N_l \in \text{ext } E} \overline{\lim_{i \in N_l, i \to \infty}} (y_i/x_i) = \sigma_x(y, E_1).
\end{aligned} \tag{2.4}
$$

We can prove in the same way that $\sigma_x(t, E_0) \leq \sigma_x(t, E_2)$. Therefore,

$$
\begin{aligned}
\sigma_x(y+z) = \inf_{E \in \mathcal{E}(x)} \sigma_x(y+z, E) &\leq \sigma_x(y+z, E_0) \leq \sigma_x(y, E_0) + \sigma_x(z, E_0) \leq \\
&\leq \sigma_x(y, E_1) + \sigma_x(y, E_2) \leq \sigma_x(y) + \sigma_x(z) + 2\varepsilon
\end{aligned}
$$

The arbitrariness of $\varepsilon > 0$ yields $\sigma_x(y+z) \leq \sigma_x(y) + \sigma_x(z)$. ∎

Note. The functional σ_x defined by formula (2.3) and generated by an element $x \in \text{Sg}(l^\Phi)$ is linear over the subspace $l_x = \{\lambda x : \lambda \in \mathbf{R}\}$.

Definition 2.6. Let $x \in \text{Sg}(l^\Phi)$ and ϱ_x be the Banach functional defined by formula (2.3). Denote by B-lim(x) the set of all linear minorants for ϱ_x, i.e. the set of all linear continuous functionals x^* over l^Φ such that $x^*(y) = \varrho_x(y)$ for any $y \in l_x = \{\lambda x : \lambda \in \mathbf{R}\}$ and $x^*(y) \leq \varrho_x(y)$ for every $y \in l^\Phi$.

In virtue of the analytical version of the Hahn–Banach theorem such a linear continuous functional always exists. Therefore B-lim$(x) \neq \emptyset$ for every $x \in \text{Sg}(l^\Phi)$.

For every $x^* \in B$-lim(x) and every $y \in l^\Phi$, we write

$$x^*(y) = B\text{-}\lim(y_i/x_i).$$

Lemma 2.7. *Let x, y, N^+, N^- and N^0 be as in Lemma 2.2 and let $\varrho_x \in B$-lim(x). Then*

(i) $\varrho_x(y^{N^+}) = \varrho_x(y)$ *whenever* $\varrho_x(y) \geq 0$,
(ii) $\|x^{N^+}\|_\Phi < 1$ *and* $\varrho_x(y^{N^-}) = \varrho_x(y)$ *whenever* $\varrho_x(y) < 0$.

PROOF : (i) For any finite decomposition $E = E(x)$ of supp $x, E = (N_1, \ldots, N_k)$, define a new decomposition $E_0 = (N_i^+, N_i^-, N_i^0)_{i=1}^k$, where $N_i^+ = N_i \cap N^+, N_i^- = N_i \cap N^-, N_i^0 = N_i \cap N^0$. Note that we defined in such a way a mapping γ from $\mathcal{E}(x)$ into itself. Define

$$\mathcal{E}_0(x) = \gamma E = \{E_0 : E_0 = \gamma E \quad \text{for some } E \in \mathcal{E}(x)\}.$$

In the same way as in (2.4), we have

$$\sigma_x(y, E_0) \leq \sigma_x(y, E) \quad (\forall\, y \in l^\Phi).$$

Hence it follows that

$$\inf_{E_0 \in \mathcal{E}_0(x)} \sigma_x(y, E_0) \leq \inf_{E \in \mathcal{E}_0(x)} \sigma_x(y, E) = \sigma_x(y).$$

Since $\mathcal{E}_0(x) \subset \mathcal{E}(x)$, i.e. for any $A \in \mathcal{E}_0(x)$ there is $B \in \mathcal{E}(x)$ such that $A \subset B$, we have

$$\sigma_x(y) = \inf_{E_0 \in \mathcal{E}_0(x)} \sigma_x(y, E_0).$$

In the same way we can obtain

$$\varrho_x(y^{N^+}) = \inf_{E_0 \in \mathcal{E}_0(x)} \sigma_x(y^{N^+}, E_0).$$

Since $\varrho_x(y) > 0$ by the assumption, in virtue of the definitions of N_j^- and N_j^0, we have (defining the limits over finite sets to be equal to 0)

$$\overline{\lim_{i \in N_j^-, i \to \infty}} (y_i/x_i) \leq 0 \quad \text{and} \quad \lim_{i \in N_j^0, i \to \infty} (y_i/x_i) = 0.$$

Hence it follows that

$$\begin{aligned}\varrho_x(y) = \inf_{E_0 \in \mathcal{E}_0(x)} \varrho_x(y, E_0) &= \inf_{E_0 \in \mathcal{E}_0(x)} \overline{\lim_{i \in N^+ \cap \operatorname{ext} E_0, i \to +\infty}} (y_i/x_i) = \\ &= \inf_{E_0 \in \mathcal{E}_0(x)} \sigma_x(y^{N^+}, E_0) = \varrho_x(y^{N^+}).\end{aligned}$$

(ii) Assume that $\varrho_x(y) < 0$. First, we shall prove that $N^+ \notin \operatorname{ext} E(x)$. Assume for a contrary that $N^+ \in \operatorname{ext} E(x)$. In virtue of Lemma 2.4, we have $N_j^+ \in \operatorname{ext} E(x)$ for some $j \in \{1, \ldots k\}$. Therefore

$$\sigma_x(y, E_0) \geq \overline{\lim_{i \in N_j^0, i \to +\infty}} (y_i/x_i) \geq 0 \quad (\forall\, E_0 \in \mathcal{E}_0(x)).$$

Hence

$$\varrho_x(y) = \inf_{E_0 \in \mathcal{E}_0(x)} \sigma_x(y, E_0) \geq 0,$$

a contradiction. Therefore $N^+ \notin \operatorname{ext} E(x)$. In the same way we obtain $N^0 \notin \operatorname{ext} E(x)$, whence $N_j^+ \notin \operatorname{ext} E(x)$ and $N_j^0 \notin \operatorname{ext} E(x)$ for any $j \in \{1, \ldots k\}$. Therefore,

$$\sigma_x(y, E_0) = \sup_{N_j^- \in \operatorname{ext} E_0} \overline{\lim_{i \in N_j^-, i \to +\infty}} (y_i/x_i) = \sigma_x(y^{N^-}, E_0),$$

and $\varrho_x(y) = \varrho_x(y^{N^-})$, which finishes the proof. ∎

Theorem 2.8. *Let $x \in \mathrm{Sg}(l^{\Phi})$. Then $x^* \in \mathrm{SGrad}(x)$ if and only if $x^* \in B\text{-}\lim(x)$.*

PROOF : Sufficiency. Take an arbitrary $x^* \in B\text{-}\lim(x)$. It is obvious that x^* is singular. Since $x^*(x) = 1$, we have $\|x^*\| \geq 1$. Take an arbitrary $y \in S(l^{\Phi}), \varepsilon > 0$, and define

$$N_1 = \{i \in \mathrm{supp}\, x : |y_i| \geq (1+\varepsilon)|x_i|\},\ N_2 = \mathrm{supp}\, x \setminus N_1.$$

The pair $E_\varepsilon = (N_1, N_2)$ is a finite decomposition of supp x. Let $\overline{y} = (\overline{y}_i)_{i=1}^{\infty}$, where $\overline{y}_i = y_i \,\mathrm{sgn}(x_i)$ ($\mathrm{sgn}(0) = 0$ by the definition). Obviously $\overline{y} \in S(l^{\Phi})$. If $\|x^{N_1}\| = 1$, then $\|\overline{y}^{N_1}\|_{\Phi} \geq (1+\varepsilon)\|x^{N_1}\|_{\Phi} = 1+\varepsilon$, a contradiction. Therefore, $\|x^{N_1}\|_{\Phi} < 1$, and

$$\sigma_x(\overline{y}, E_\varepsilon) = \overline{\lim_{i \in N_2, i \to +\infty}} (y_i/x_i) \leq 1+\varepsilon.$$

This yields

$$\varrho_x(y) \leq \inf_{\varepsilon > 0} \varrho_x(\overline{y}, E_\varepsilon) = 1,$$

whence the inequality $\|x^*\| \leq 1$ follows. Thus, in virtue of $\|x^*\| \geq 1$, we have $\|x^*\| = 1$, i.e. $x^* \in \mathrm{Grad}(x)$.

Necessity. Take an arbitrary $x^* \in \mathrm{SGrad}(x)$. We shall prove that $x^* \in B\text{-}\lim(x)$, i.e. $x^*(y) \leq \varrho_x(y)$ for any $y \in l^{\Phi}$. If this inequality does not hold, there exists $y \in S(l^{\Phi})$ such that

$$x^*(y) > \varrho_x(y).$$

We shall show that this yields a contradiction. Let us consider for this purpose two cases separately.

1) $\varrho_x(y) \geq 0$. Define $N^+ = \{i \in N_x : x_i y_i > 0\}, N^- = \{i \in N_x : x_i y_i < 0\}, N^0 = \{i \in N_x : x_i y_i = 0\}$, where $N_x = \mathrm{supp}\, x$.? In view of Lemma 2.7, we have $\varrho_x(y) = \varrho_x(y^{N^+})$. Moreover, by Lemma 2.2, $x^*(y^{N^-}) \leq 0$ and $x^*(y^{N^0}) = 0$. Thus

$$x^*(y^{N^+}) = x^*(y) - x^*(y^{N^-}) - x^*(y^{N^0}) \geq x^*(y) > \varrho_x(y) = \varrho_x(y^{N^+}).$$

Therefore, we may assume without loss of generality in this case that $x_i y_i \geq 0$ for any $i \in N_x$. By the definition of $\varrho_x(y)$, there exists a finite decomposition $E = \{N_i\}_{i=1}^{k}$ of N_x such that

$$x^*(y) > \varrho_x(y, E) \geq \varrho_x(y) \geq 0. \tag{2.5}$$

Take a positive number ε such that $\lambda + \varepsilon < x^*(y)$, where $\lambda = \sigma_x(y, E)$. Define a new finite decomposition of $N_x, E_\varepsilon = \{N_i^\varepsilon, \bar{N}_i^\varepsilon$, where $N_i^\varepsilon = N_i \cap N^\varepsilon, \bar{N}_i^\varepsilon = N_i \cap N'_\varepsilon$, and

$$N^\varepsilon = \{i \in N_x : 0 \leq y_i/x_i < \lambda + \varepsilon\}, \quad N'_\varepsilon = N_x \setminus N^\varepsilon.$$

Since $x_i y_i \geq 0$, we have $N_x = N^\varepsilon \cup N'_\varepsilon$. If $N'_\varepsilon \in \mathrm{ext}\, E_\varepsilon$, by Lemma 2.4, there exists i_0 such that $\bar{N}_{i_0}^\varepsilon \in \mathrm{ext}\, E_\varepsilon$. We have

$$\overline{\lim_{i \in \bar{N}_{i_0}^\varepsilon, i \to +\infty}} (y_i/x_i) \geq \lambda + \varepsilon.$$

Since $\bar{N}^{\varepsilon}_{i_0} \subset N_{i_0}$, we have

$$1 \geq \|x^{N_{i_0}}\|_{\Phi} \geq \|x^{\bar{N}^{\varepsilon}_{i_0}}\|_{\Phi} = 1,$$

whence

$$\lambda = \sigma_x(y, E) = \sup_{N_i \in \text{ext}\, E} \overline{\lim_{j \in N_i, j \to +\infty}} (y_j/x_j) \geq \overline{\lim_{i \in \bar{N}^{\varepsilon}_{i_0}, i \to +\infty}} (y_i/x_i) \geq \lambda + \varepsilon,$$

a contradiction. Therefore, $N'_{\varepsilon} \notin \text{ext}\, E_{\varepsilon}$. By Lemma 2.4, we conclude that $N^{\varepsilon} \in \text{ext}\, E_{\varepsilon}$. In virtue of Lemma 2.1, we have $x^*(x^{N'_{\varepsilon}}) = 0$. Therefore,

$$x^*(y) = x^*(y^{N'_{\varepsilon}}) + x^*(y^{N^{\varepsilon}}) = x^*(y^{N^{\varepsilon}}).$$

Moreover,

$$\|y^{N^{\varepsilon}}\|_{\Phi} \leq \|(\lambda + \varepsilon)x^{N^{\varepsilon}}\|_{\Phi} = (\lambda + \varepsilon)\|x^{N^{\varepsilon}}\|_{\Phi} = \lambda + \varepsilon,$$

whence,

$$x^*(y^{N^{\varepsilon}}) \leq \|y^{N^{\varepsilon}}\|_{\Phi}\, \|x^*\| \leq \lambda + \varepsilon,$$

what contradicts to the inequality $\lambda + \varepsilon < x^*(y) = x^*(y^{N^{\varepsilon}})$ written just after inequality (2.5).

2) $\varrho_x(y) < 0$. It follows from Lemma 2.7(ii) that $\|x^{N^+}\|_{\Phi} < 1$ and $\varrho_x(y^{N^-}) = \varrho_x(y)$. In view of Lemma 2.1, we have $x^*(y^{N^+}) = 0$. Therefore, $x^*(y) = x^*(y^{N^-})$ and

$$x^*(-y^{N^-}) = -x^*(y^{N^-}) = -x^*(y) < -\varrho_x(y) = -\varrho_x(y^{N^-}). \tag{2.6}$$

Putting $z = -y^{N^-}$, we have $z_i x_i \geq 0$ for any $i \in \mathbb{N}$. By (2.6),

$$\begin{aligned} x^*(z) < -\varrho_x(-z) &= \inf_{E \in \mathcal{E}(x)} \sup_{N_i \in \text{ext}\, E} \overline{\lim_{j \in N_i, j \to +\infty}} (-z_j/x_j) = \\ &= \sup_{E \in \mathcal{E}(x)} \inf_{N_i \in \text{ext}\, E} \underline{\lim}_{j \in N_i, j \to +\infty} (z_j/x_j). \end{aligned}$$

So, there exists a finite decomposition $E = \{N_i\}_{i=1}^k$ of N_x such that

$$x^*(z) < \inf_{N_i \in \text{ext}\, E} \underline{\lim}_{j \in N_i, j \to +\infty} (z_j/x_j).$$

Take a positive number ε such that $x^*(z) < \lambda - \varepsilon$ and $\lambda - \varepsilon > 0$. In virtue of $x_i z_i \geq 0$, applying Lemma 2.2, we get $x^*(z) \geq 0$. Define a finite decomposition E_{ε} of N_x by $E_{\varepsilon} = \{N^{\varepsilon}_i, N'_i\}_{i=1}^k$, where $N^{\varepsilon}_i = N_i \cap N^{\varepsilon}, N'_i = N_i \cap N'$, and

$$N^{\varepsilon} = \{i \in N_x : 0 \leq z_i/x_i < \lambda - \varepsilon\}, \quad N' = \{i \in N_x : z_i/x_i \geq \lambda - \varepsilon\}.$$

Since $x_i z_i \geq 0$ for any $i \in \mathbf{N}$, we have $N^\varepsilon \cup N' = N_x$. First, we shall prove that $N^\varepsilon \notin \operatorname{ext} E_\varepsilon$. If not, $N^\varepsilon \in \operatorname{ext} E_\varepsilon$ and by Lemma 2.4 there exists $i_0 \in \{1, \ldots, k\}$ such that $N^\varepsilon_{i_0} \in \operatorname{ext} E_\varepsilon$. Thus,

$$\lambda \leq \varliminf_{i \in N_{i_0}, i \to +\infty} (z_i/x_i) \leq \varliminf_{i \in N^\varepsilon_{i_0}, i \to +\infty} (z_i/x_i) < \lambda - \varepsilon,$$

which contradicts to $\varepsilon > 0$. Thus, $N^\varepsilon \notin \operatorname{ext} E_\varepsilon$. Since $x \in \operatorname{Sg}(l^\Phi)$, applying again Lemma 2.4, we get $N' \in \operatorname{ext} E_\varepsilon$. By $N^\varepsilon \notin \operatorname{ext} E_\varepsilon$ and Lemma 2.1, we have $x^*(x^{N^\varepsilon}) = 0$. So, $x^*(x^{N'}) = x^*(x) = 1$. We have for any $i \in N'$,

$$[z_i - (\lambda - \varepsilon)x_i]/x_i \geq 0 \quad \text{and} \quad [z_i - (\lambda - \varepsilon)x_i]x_i \geq 0.$$

Thus, in view of Lemma 2.2, we get $x^*(z^{N'} - (\lambda - \varepsilon)x^{N'}) \geq 0$, i.e.

$$x^*(z^{N'}) \geq (\lambda - \varepsilon)x^*(x^{N'}) = \lambda - \varepsilon.$$

But, on the other hand, by $N^\varepsilon \notin \operatorname{ext} E_\varepsilon$ and Lemma 2.1, we have $x^*(z^{N^\varepsilon}) = 0$. Therefore,

$$x^*(z^{N'}) = x^*(z) - x^*(z^{N^\varepsilon}) = x^*(z).$$

Combining this with the previous inequality, we get $x^*(z) \geq \lambda - \varepsilon$, which contradicts to the inequality $x^*(z) < \lambda - \varepsilon$. The theorem is proved. ■

Theorem 2.9. *Let $x^* = \overline{x}^* + \underline{x}^*$ be a linear continuous functional over l^Φ, where $\overline{x}^*$ and $\underline{x}^*$ are its regular and singular parts, respectively. Then*

$$\|x^*\| = \|\overline{x}^*\| + \|\underline{x}^*\|. \tag{2.7}$$

PROOF : As it was already proved, we have

$$\overline{x}^*(x) = \sum_{i=1}^{\infty} x^*(e_i)\, x_i,$$

and $\underline{x}^*(x) = 0$ for every $x \in h^\Phi$. We need to prove the inequality $\|x^*\| \geq \|\overline{x}^*\| + \|\underline{x}^*\|$. Take an arbitrary $\varepsilon > 0$. There exist $x^{(1)}, x^{(2)} \in S(l^\Phi)$ such that

$$\underline{x}^*(x^{(1)}) \geq \|\underline{x}^*\| - \frac{\varepsilon}{4}, \tag{2.8}$$

$$\overline{x}^*(x^{(2)}) \geq \|\overline{x}^*\| - \frac{\varepsilon}{4}. \tag{2.9}$$

We shall consider two cases.

I. $x^{(2)}$ has infinite number of coordinates different from 0. Since the series $\overline{x}^*(x^{(2)}) = \sum_{i=1}^{\infty} x_i^{(2)} \overline{x}^*(e_i)$ is convergent, there exists a number $k \in \mathbf{N}$ such that

$$\Big|\sum_{i=k}^{\infty} x_i^{(2)} \overline{x}^*(e_i)\Big| < \varepsilon/4. \tag{2.10}$$

We have $I_\Phi(x^{(1)}) = \sum_{i=1}^{\infty} \Phi_i(x_i^{(1)}) \le 1$, because $\|x^{(1)}\| = 1$. Thus, there exists a number $l \in \mathbb{N}, l > k$, such that

$$\sum_{i=l}^{\infty} \Phi_i(x_i^{(1)}) < \sum_{i=k}^{\infty} \Phi_i(x_i^{(2)}), \tag{2.11}$$

$$|\sum_{i=l}^{\infty} x_i^{(1)} x^*(e_i)| < \frac{\varepsilon}{4}. \tag{2.12}$$

Define

$$\begin{aligned} x^{(3)} &= (x_i^{(3)})_{i=1}^{\infty} \\ \text{where} \quad x_i^{(3)} &= x_i^{(1)} && \text{when } i \ge l, \\ &= x_i^{(2)} && \text{when } i < k, \\ &= 0 && \text{otherwise.} \end{aligned}$$

Applying (2.11) and (2.12), we get

$$I_\Phi(x^{(3)}) = \sum_{i=1}^{k-1} \Phi_i(x_i^{(2)}) + \sum_{i=l}^{\infty} \Phi_i(x_i^{(1)}) \le \sum_{i=1}^{\infty} \Phi_i(x_i^{(2)}) \le 1,$$

whence $\|x^{(3)}\|_\Phi \le 1$. Therefore, in virtue of (2.8), (2.9), (2.11) and (2.12), we have

$$\begin{aligned} \|x^*\| &\ge x^*(x^{(3)}) = \underline{x}^*(x^{(3)}) + \overline{x}^*(x^{(3)}) = \\ &= \underline{x}^*(x^{(1)}) + \sum_{i=1}^{k-1} x_i^{(2)} x^*(e_i) + \sum_{i=l}^{\infty} x_i^{(1)} x^*(e_i) = \\ &= \underline{x}^*(x^{(1)}) + \overline{x}^*(x^{(2)}) - \sum_{i=k}^{\infty} x_i^{(2)} x^*(e_i) - \sum_{i=l}^{\infty} x_i^{(1)} x^*(e_i) \ge \\ &\ge \underline{x}^*(x^{(1)}) + \overline{x}^*(x^{(2)}) - |\sum_{i=k}^{\infty} x_i^{(2)} x^*(e_i)| - |\sum_{i=l}^{\infty} x_i^{(1)} x^*(e_i)| \ge \\ &\ge (\|\underline{x}^*\| - \frac{\varepsilon}{4}) + (\|\overline{x}^*\| - \frac{\varepsilon}{4}) - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} = \|\underline{x}^*\| + \|\overline{x}^*\| - \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this means that $\|x^*\| \ge \|\overline{x}^*\| + \|\underline{x}^*\|$.

II. $x^{(2)}$ has only finite number of coordinates different from 0, i.e. there exists a number $k \in \mathbb{N}$ such that $x_i^{(2)} = 0$ for any $i \ge k$. Let $\varepsilon \in (0,2)$ and $\lambda = 1 - \varepsilon/4$. We have $I_\Phi(\lambda x^{(2)}) \le \|\lambda x^{(2)}\|_\Phi < 1$. Choose a natural number l, $l \ge k$, in such a way that

$$\sum_{i=l}^{\infty} \Phi_i(x_i^{(1)}) < 1 - \lambda \sum_{i=1}^{k-1} \Phi_i(x_i^{(2)}), \tag{2.13}$$

$$|\sum_{i=l}^{\infty} x_i^{(1)} x^*(e_i)| < \varepsilon/4. \tag{2.14}$$

Define

$$\begin{aligned} x^{(3)} &= (x_i^{(3)})_{i=1}^{\infty}, \\ \text{where} \quad x_i^{(3)} &= x_i^{(1)} && \text{when } i \geq l, \\ &= \lambda x_i^{(2)} && \text{when } i < k, \\ &= 0 && \text{otherwise.} \end{aligned}$$

Then, in view of (2.13) and (2.14), we have

$$I_\Phi(x^{(3)}) = \sum_{i=1}^{k-1} \Phi_i(\lambda x_i^{(2)}) + \sum_{i=l}^{\infty} \Phi_i(x_i^{(1)}) \leq 1,$$

whence $\|x^{(3)}\|_\Phi \leq 1$. We have

$$\begin{aligned} \|x^*\| &\geq x^*(x^{(3)}) = \underline{x}^*(x^{(3)}) + \overline{x}^*(x^{(3)}) = \\ &= \underline{x}^*(x^{(1)}) + \lambda \sum_{i=1}^{k-1} x_i^{(2)} x^*(e_i) + \sum_{i=l}^{\infty} x_i^{(1)} x^*(e_i) = \\ &= x^*(x^{(1)}) + \lambda x^*(x^{(2)}) + \sum_{i=l}^{\infty} x_i^{(1)} x^*(e_i) \geq \\ &\geq (\|\underline{x}^*\| - \frac{\varepsilon}{4}) + (1 - \frac{\varepsilon}{4})(\|\overline{x}^*\| - \frac{\varepsilon}{4}) - \frac{\varepsilon}{4} \geq \\ &\geq \|\underline{x}^*\| + \|\overline{x}^*\| - \varepsilon\,. \end{aligned}$$

The arbitrariness of ε in (0,2) yields $\|x^*\| \geq \|\underline{x}^*\| + \|\overline{x}^*\|$. ■

Note. For Orlicz spaces over a non-atomic measure space Theorem 2.9 was proved by T. Ando in [**1**]. For Orlicz sequence spaces Theorem 2.9 was proved by M. Nowak in [**9**].

Theorem 2.10. *Let $x \in S(l^\Phi)$ and $x^* \in$ Grad(x). Then $x^* \in$ SGrad(x) if and only if*

(2.15) $$x^* = \alpha x_1^* + \beta x_2^*,$$

where $\alpha, \beta \geq 0; \alpha + \beta = 1, x_1^* \in$ RGrad(x) and $x_2^* \in$ SGrad(x).

Proof : It is obvious that $x^* \in$ Grad(x) whenever $x^* = \alpha x_1^* + \beta x_2^*$, where $x_1^*, x_2^* \in$ Grad(x) and $\alpha, \beta \geq 0, \alpha + \beta = 1$.

Take an arbitrary $x^* \in$ Grad(x). In virtue of Lemma 1.1, x^* can be uniquely represented in the form

$$x^* = \overline{x}^* + \underline{x}^*,$$

where $\overline{x}^*$ and $\underline{x}^*$ are the regular and singular parts of x^*, respectively. In view of Theorem 2.9, we get

$$\|x^*\| = \|\overline{x}^*\| + \|\underline{x}^*\|.$$

Hence it follows that

$$\overline{x}^*(x) = \|\overline{x}^*\| \quad \text{and} \quad \underline{x}^*(x) = \|\underline{x}^*\|.$$

Indeed, in the opposite case it would be

$$x^*(x) = \overline{x}^*(x) + \underline{x}^*(x) < \|\overline{x}^*\| + \|\underline{x}^*\| = 1$$

whence $x^*(x) < 1$, a contradiction. Denote $\alpha = \|\overline{x}^*\|$ and $\beta = \|\underline{x}^*\|$. Then $x_1^* = \frac{1}{\alpha}\overline{x}^* \in \mathrm{RGrad}(x)$ and $x_2^* = \frac{1}{\beta}\underline{x}^* \in \mathrm{SGrad}(x)$. It is obvious that $x^* = \alpha x_1^* + \beta x_2^*$. ∎

3. Smoothness of l^Φ.

The following theorem characterizes smooth Musielak–Orlicz sequence spaces.

Theorem 3.1. *A Musielak–Orlicz sequence space l^Φ is smooth if and only if*

(i) *Φ satisfies the δ_2^0-condition,*
(ii) *for every $i, j \in \mathbb{N}, i \neq j$, we have $\Phi_i(b_i) + \Phi_j(b_j) > 1$,*
(iii) *for every $i \in \mathbb{N}$ such that $\Phi_i^-(b_i) < +\infty$ and $\Phi_i(b_i) \leq 1$ does not exist $j \in \mathbb{N}$, $j \neq i$, and $c \geq 0$ such that $\partial\Phi_j(c) \neq \{0\}$ and $\Phi_i(b_i) + \Phi_j(c) \leq 1$,*
(iv) *Φ_i are smooth on the intervals $[0, a_i)$.*

Proof : Sufficiency. First, we shall prove that $l^\Phi = h^\Phi$ whenever Φ satisfies the δ_2^0-condition. We want to prove that for every $x \in l^\Phi$ there exists a sequence $(x^{(n)})_{i=1}^\infty$ with $x^{(n)} \in h$ such that $\|x - x^{(n)}\|_\Phi \to 0$ as $n \to +\infty$. Let $x \in l^\Phi$ and

$$x^{(n)} = \sum_{i=1}^{n} x_i e_i \qquad (\forall\, n \in \mathbb{N}).$$

There is a number $\lambda > 0$ such that

$$I_\Phi(\lambda x) = \sum_{i=1}^{\infty} \Phi_i(\lambda x_i) < +\infty,$$

whence it follows that

$$I_\Phi(\lambda(x - x^{(n)})) = \sum_{i=n+1}^{\infty} \Phi_i(\lambda x_i) \to 0 \text{ as } n \to +\infty. \tag{3.1}$$

We need to prove that condition (3.1) implies that

$$I_\Phi(2\lambda(x - x^{(n)})) \to 0 \text{ as } n \to +\infty. \tag{3.2}$$

Let $\varepsilon > 0$ be given. Condition (3.1) implies that there exists a number $n_0 \in \mathbb{N}$, $n_0 \geq m$, such that

$$\sum_{i=n_0+1}^{\infty} \Phi_i(\lambda x_i) < \min(a, \varepsilon/2k),$$

and

$$\sum_{i=n_0+1}^{\infty} c_i < \frac{\varepsilon}{2},$$

where a, k, m and $(c_i)_{i=1}^{\infty}$ are the numbers and the sequence from the δ_2^0-condition. Therefore,

$$\Phi_i(\lambda x_i) < \min(a, \varepsilon/2k) \quad (\forall\, i \geq n_0 + 1),$$

and, in consequence of the δ_2^0-condition, we get

$$\Phi_i(2\lambda x_i) \leq k\Phi_i(\lambda x_i) + c_i \quad (\forall\, i \geq n_0 + 1),$$

whence

$$\sum_{i=n_0+1}^{\infty} \Phi_i(2\lambda x_i) \leq k \sum_{i=n_0+1}^{\infty} \Phi_i(\lambda x_i) + \sum_{i=n_0+1}^{\infty} c_i < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.$$

Therefore

$$I_\Phi(2\lambda(x - x^{(n)})) < \varepsilon \quad (\forall\, n \geq n_0),$$

which means that condition (3.1) implies (3.2). This implication yields

$$I_\Phi(\alpha(x - x^{(n)})) \to 0 \ \text{ as } \ n \to \infty \quad (\forall\, \alpha > 0),$$

what is equivalent to

$$\|(x - x^{(n)}))\|_\Phi \to 0 \ \text{ as } \ n \to +\infty\,,$$

and the equality $l^\Phi = h^\Phi$ is proved.
Thus, condition (i) implies that $\mathrm{Grad}(x) = \mathrm{RGrad}(x)$ for any $x \in S(l^\Phi)$.

Now, conditions (ii), (iii) and (iv), and the representation formulae for $x^* \in \mathrm{RGrad}(x)$ when $x \in S(l^\Phi)$ given in Chapter 1 imply that $\mathrm{Card}(\mathrm{Grad}(x)) = 1$ for any $x \in S(l^\Phi)$, i.e. l^Φ is smooth.

Necessity. (i) If condition (i) is not satisfied then there are two elements $x, y \in l^\Phi$ such that $\|x\|_\Phi = \|y\|_\Phi = \|x + y\|_\Phi = 1$ and the supports of x and y are disjoint (cf. [5]). Therefore, the element $x+y$ is not smooth (cf. [4], the proof of Theorem 8).

Therefore, in the remaining part of the proof of necessity, we may assume (and we do it) that Φ satisfies the δ_2^0-condition. Assume now that condition (ii) is not satisfied, i.e. there exist $i, j \in \mathbb{N}, i \neq j$, such that $\Phi_i(b_i) + \Phi_j(b_j) \leq 1$. Then the element $x = b_i e_i + b_j e_j$ belongs to $S(l^\Phi)$ and the functionals

$$x_1^*(y) = y_i/b_i \quad (\forall\, y \in l^\Phi),$$
$$x_2^*(y) = y_j/b_j \quad (\forall\, y \in l^\Phi)$$

are two different elements of $\mathrm{Grad}(x)$. Therefore, l^Φ is not smooth.

(iii) Assume without loss of generality that condition (ii) holds. If condition (iii) does not hold, there exist $j \neq i$ and $c \geq 0$ such that $\partial\Phi_j(c) \neq \{0\}$ and $\Phi_i(b_i) + \Phi_j(c) \leq 1$. In view of (ii) there is $k \in \mathbb{N}, k \neq i, k \neq j$, and $d \geq 0$ such that

$$\Phi_i(b_i) + \Phi_j(c) + \Phi_k(d) = 1.$$

Define $x = b_i e_i + c e_j + d e_k$. Then $x \in S(l^\Phi)$, because $I_\Phi(x) = 1$. Take $\eta_i \in \partial\Phi_i(b_i), \eta_k \in \partial\Phi_k(d)$ and $a, b \in \partial\Phi_j(c), a \neq b$, and define the functionals

$$x_1^*(y) = \frac{\eta_i y_i + a y_j + \eta_k y_k}{\eta_i x_i + a x_j + \eta_k x_k} \quad (\forall\, y \in l^\Phi),$$
$$x_2^*(y) = \frac{\eta_i y_i + b y_j + \eta_k y_k}{\eta_i x_i + b x_j + \eta_k x_k} \quad (\forall\, y \in l^\Phi).$$

Obviously, $x_1^* \neq x_2^*$ and, in virtue of Theorem 1.9, $x_1^*, x_2^* \in \mathrm{Grad}(x)$, i.e. x is not smooth, and so l^Φ is not smooth, too.

(iv) Assume without loss of generality that condition (ii) holds. If condition (iv) does not hold, then there exist numbers $i \in \mathbb{N}$ and $u \in [0, a_i]$ such that Φ_i is not smooth at u, i.e. $\partial\Phi_i(u)$ is a nontrivial interval. In view of condition (ii) there exist two natural numbers $j, k; j \neq i, j \neq k, k \neq i$, and two positive numbers v, w such that

$$\Phi_i(u) + \Phi_j(v) + \Phi_k(w) = 1.$$

Define $x = u e_i + v e_j + w e_k$. Then $I_\Phi(x) = 1$, whence $\|x\|_\Phi = 1$. Take $c, d \in \partial\Phi(u), c \neq d, e \in \partial\Phi_j(v), f \in \partial\Phi_k(w)$, and define

$$x_1^*(y) = \frac{c y_i + e y_j + f y_k}{cu + ev + fw} \quad (\forall\, y \in l^\Phi),$$
$$x_2^*(y) = \frac{d y_i + e y_j + f y_k}{du + ev + fw} \quad (\forall\, y \in l^\Phi).$$

Then $x_1^* \neq x_2^*$ and, in virtue of Theorem 1.9, $x_1^*, x_2^* \in \mathrm{Grad}(x)$. Therefore, x is not smooth, and so l^Φ is not smooth, too. ∎

Remark 3.2. If Φ does not satisfy the δ_2^0-condition, then $h^\Phi \neq l^\Phi$.

PROOF : The assumption implies that l^Φ contains an isomorphically isometric copy of l^∞ (cf. [5]). Since h is not dense in l^∞, it follows that $h^\Phi \neq l^\Phi$. ∎

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(*Received* April 18, 1990)