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Abstract. In the present paper, a necessary and sufficient condition for an l^1-product of two Banach spaces to be 2-uniformly convex is given.

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Introduction.

The notion of the n-dimensional volume enclosed by $n+1$ vectors $x_1,\dots,x_{n+1}$ in a Banach space E was introduced by Silverman [**Si**] in the following way:

$$A(x_1,x_2,\dots,x_{n+1}) = \sup\left\{\det\left|\begin{pmatrix} 1 & \dots & 1 \\ f_1(x_1) & \dots & f_1(x_{n+1}) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ f_n(x_1) & \dots & f_n(x_{n+1})\end{pmatrix}\right|, f_i\in E^*, \|f_i\|\le 1\right\}.$$

Let, for $k=1,2,\dots,n-1, d_k = \operatorname{dist}(x_k,[x_{k+1},x_{k+2},\dots,x_{n+1}])$ where $[x_{k+1},\dots,x_{n+1}]$ is the affine span of $\{x_{k+1},\dots,x_{n+1}\}$, and $d_n=\|x_n-x_{n+1}\|$. Then we have the following inequalities (see [**G-S**] and [**B-S**]): $d_1d_2\dots d_n \le A(x_1,\dots,x_{n+1}) \le n^{n/2}d_1d_2\dots d_n$. In particular we shall use the above inequality for $n=2$:

Lemma 1 (see [**G-S**]). *We have:* $A(x,y,z) \le 2\|x-y\| \operatorname{dist}(z,[x,y])$ *for all* $x,y,z\in E$.

The following generalization of uniformly convex Banach spaces is due to Sullivan [**Su**]. Let first introduce the modulus of k-convexity of a Banach space E as follows:

$$\delta^k_E(\varepsilon) = \inf\{1-\frac{1}{k+1}\|\sum_{i=1}^{k+1}x_i\| \,:\, \|x_i\|\le 1,\, A(x_1,x_2,\dots x_{k+1})>\varepsilon\}.$$

We say that a Banach space E is $k-$UC if $\delta^k(\varepsilon)>0$ for all $\varepsilon>0$. Observe that $1-$UC spaces are exactly the uniformly convex spaces introduced by Clarkson [**C**], since $A(x_1,x_2)=\|x_1-x_2\|$.

Recall that for a sequence of Banach spaces (E_n) and $1\le p\le\infty$, the p-direct sum, $(\sum\oplus E_n)_p$, is the space of all sequences $\{x_n\}$, where for each $n, x_n\in E_n$ and

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$\sum \|x_n\|^p < \infty$ if $1 \leq p < \infty$ or $\sup \|x_n\| < \infty$ if $p = \infty$. The norm is respectively given by

$$\|(x_n)\| = (\sum \|x_n\|^p)^{1/p} \, (1 \leq p < \infty) \quad \text{or} \quad \|(x_n)\| = \sup \|x_n\| \; (p = \infty).$$

The following proposition was proved in [**G-S**]:

Proposition. *Let $1 < p < \infty$, and (E_n) a sequence of Banach spaces. Then $(\sum \oplus E_n)_p$ is 2– UC if and only if all but one of the E_n's are 1– UC with a common modulus of convexity (i.e. $\inf_n \delta^1_E(\varepsilon) > 0$ for all $\varepsilon > 0$) and the remaining space is 2– UC.*

Generalizations of these results have been announced in [**Y-W**]. Of course the above proposition is not true in the cases $p = 1, \infty$ (take, for example, $(\sum_n^\infty \oplus l_2)_1$ or $(\sum_n^\infty \oplus l_2)_\infty$). We want to show that there are, however, some positive results for the direct sums also in the case $p = 1$.

Main result.

To be more precise we have the following:

Theorem 1. *Let E and F Banach spaces. Then the space $(E \oplus F)_1$ is 2– UC if and only if E and F are 1– UC.*

Proof : Suppose $(E \oplus F)_1$ is 2– UC and, by absurdity that E is not 1– UC. Then there exists $\varepsilon > 0, \{x'_n\}, \{x''_n\} \subset E$ such that

$$\|x'_n\| \leq 1, \quad \|x''_n\| \leq 1, \quad \|x'_n - x''_n\| > \varepsilon, \quad \|x'_n + x''_n\| \to 2.$$

Take $a_n = (\frac{1}{2}x'_n, y)$, $b_n = (\frac{1}{2}x''_n, y)$ and $c_n = (0, 2y)$ where $y \in F$ and $\|y\| = \frac{1}{2}$. Then

$$\begin{aligned} &A(a_n, b_n, c_n) \geq \|a_n - b_n\| \operatorname{dist}(c_n, [a_n, b_n]) \geq \\ &\geq \frac{1}{2}\|x'_n - x''_n\|\{\inf_\lambda(\|0 - \frac{1}{2}(\lambda x'_n + (1-\lambda)x''_n)\| + \|2y - (\lambda y + (1-\lambda)y)\|)\} \geq \\ &\geq \frac{\varepsilon}{2}\{\inf_\lambda(\frac{1}{2}\|\lambda x'_n + (1-\lambda)x''_n\| + \|y\|)\} \geq \frac{\varepsilon}{4}. \end{aligned}$$

Now $\|a_n + b_n + c_n\| = \|\frac{1}{2}x'_n + \frac{1}{2}x''_n\| + \|4y\| \to 3$, so $(E \oplus F)_1$ is not 2– UC.

To prove the converse implication we need the following:

Lemma 2. *Let x and y two distinct vectors in a uniformly convex Banach space. Suppose that $x, y \neq 0$ and $\|y\| \leq \|x\| \leq 1$. Define y^x the "radial projection" of y on the sphere of radius $\|x\|$, i.e. $y^x = \frac{\|x\|}{\|y\|}\, y$, then we have:*

$$\|\frac{1}{2}(x+y)\| \leq \frac{1}{2}(\|x\| + \|y\|) - \|y\|\delta^1(\|x - y^x\|).$$

PROOF (See [**B**, p. 191]): Take now $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $z = (z_1, z_2)$ on the unit sphere of $(E \oplus F)_1$ such that $A(x, y, z) > \varepsilon$. Suppose $\|x_1\| < \|y_1\| < \|z_1\|$ (so $\|z_2\| < \|y_2\| < \|x_2\|$).

We divide the proof in three cases:

I. First of all, we show that it is impossible that:

$$\|x_1^{y_1} - y_1\|, \|x_1^{z_1} - z_1\|, \|y_1^{z_1} - z_1\|, \|z_2^{y_2} - y_2\|, \|z_2^{x_2} - x_2\|, \|y_2^{x_2} - x_2\|,$$

are all less than $\varepsilon/16$. In fact we have, by using Lemma 1:

$$\begin{aligned}
&A(x,y,z) \leq 2\|x-z\| \operatorname{dist}(y[x,z]) \leq \\
&\leq 2(\|x\| + \|z\|) \inf_{\lambda} \|y - (\lambda x + (1-\lambda)z)\| \leq \\
&\leq 4 \inf_{\lambda} \{\|y_1 - (\lambda x_1 + (1-\lambda)z_1)\| + \|y_2 - (\lambda x_2 + (1-\lambda)z_2)\|\} \leq \\
&\leq 4\{\|y_1 - kx_1 - (1-k)z_1\| + \|y_2 - kx_2 - (1-k)z_2\|\} = \\
&= 4\{\|k(y_1^{x_1} - x_1) + (1-k)(y_1^{z_1} - z_1)\| + \|k(y_2^{x_2} - x_2) + \\
&+ (1-k)(y_2^{z_2} - z_2)\|\} \leq \\
(*) \qquad &\leq 4\{k\|y_1^{x_1} - x_1\| + (1-k)\|y_1^{z_1} - z_1\| + k\|y_2^{x_2} - x_2\| + \\
&+ (1-k)\|y_2^{z_2} - z_2\|\} \leq \\
&\leq 4\{k\frac{\|x_1\|}{\|y_1\|}\|y_1 - x_1^{y_1}\| + (1-k)\|y_1^{z_1} - z_1\| + k\|y_2^{x_2} - x_2\| + \\
&+ (1-k)\frac{\|z_2\|}{\|y_2\|}\|y_2 - z_2^{y_2}\|\} \leq \\
&\leq 4\{k\frac{\varepsilon}{16} + (1-k)\frac{\varepsilon}{16} + k\frac{\varepsilon}{16} + (1-k)\frac{\varepsilon}{16}\} = \varepsilon/2,
\end{aligned}$$

where $k = \frac{\|z_1\| - \|y_1\|}{\|z_1\| - \|x_1\|} = \frac{\|z_2\| - \|y_2\|}{\|z_2\| - \|x_2\|}$, $k \in (0,1)$. This is a contradiction.

II. Suppose now that for one pair, for example x_1, y_1, we have: $\|x_1^{y_1} - y_1\| > \varepsilon/16$ but $\|x_1\| < \varepsilon/16$. (The other case will follow easily from this case.) Then from $(*)$ we obtain:

$$\begin{aligned}
&A(x,y,z) \leq 4\{k\|y_1^{x_1} - x_1\| + (1-k)\|y_1^{z_1} - z_1\| + \\
&+ k\|y_2^{x_2} - x_2\| + (1-k)\|y_2^{z_2} - z_2\|\} \leq \\
&\leq 4\{k(\|y_1^{x_1}\| + \|x_1\|) + (1-k)\|y_1^{z_1} - z_1\| + \\
&+ k\|y_2^{x_2} - x_2\| + (1-k)\|y_2^{z_2} \cdot z_2\|\} \leq \\
&\leq 4\{k\frac{\varepsilon}{8} + (1-k)\frac{\varepsilon}{16} + k\frac{\varepsilon}{16} + (1-k)\frac{\varepsilon}{16}\} < \varepsilon
\end{aligned}$$

which is again an absurdity.

III. By exclusion, a pair exists (for example x_1, z_1) such that: $\|x_1^{y_1} - y_1\| > \varepsilon/16$ and $\|x_1\| > \varepsilon/16$. Then we have, by Lemma 2:

$$\begin{aligned}
&\|x+y+z\| = \|x_1+y_1+z_1\| + \|x_2+y_2+z_2\| \leq \\
&\leq 2\|\frac{x_1+y_1}{2}\| + \|z_1\| + \|x_2\| + \|y_2\| + \|z_2\| \leq \\
&\leq 2(\frac{\|x_1\|+\|y_1\|}{2} - \|x_1\|\delta_E^1(\|x_1^{y_1} - y_1\|)) + \|z_1\| + \|x_2\| + \|y_2\| + \|z_2\| \leq \\
&\leq 3 - \frac{\varepsilon}{8}\delta_E^1(\varepsilon/16).
\end{aligned}$$

■

Remark 1. The excluded cases, that is, the case in which one or more vectors are null or the case in which we have $\|x_1\| = \|y_1\| = \|z_1\|$ are more easily to settle down and are left to the reader.

Remark 2. Theorem 1 is not true in the case $p = \infty$. Take $(l_1 \oplus l_2)_\infty$ and the three vectors: $x = (e_1, e_1), y\, y = (e_2, e_1)$, $z = (e_3, e_1)$. Then $\|x\| = \|y\| = \|z\| = 1$ and:

$$\|\sum_{i=1}^{3}(e_i, e_1)\| = \text{Max}(\|e_1+e_2+e_3\|, \|3e_1\|) = 3,$$

$$\begin{aligned}
&A(x,y,z) \geq \|x-y\| \operatorname{dist}(z,[x,y]) = \\
&= \text{Max}(\|e_1-e_2\|, \|e_1-e_1\|)\inf_\lambda\{\text{Max}(\|e_3-\lambda e_1-(1-\lambda)e_2\|, \|e_1-\lambda e_1-(1-\lambda)e_1\|)\} = \\
&= \sqrt{2}\inf_\lambda(\|e_3-\lambda e_1-(1-\lambda)e_2\|) = \sqrt{3}.
\end{aligned}$$

Remark 3. We recall that E has uniformly normal structure (UNS) if the "self-Jung constant":

$$J_S(E) = \sup\{2r_A(A); A \subseteq E, A \text{ convex }, \operatorname{diam} A = 1\}$$

(where $r_A(A) = \inf_{x\in A}\{\sup_{y\in A}\|x-y\|\}$) is strictly less then 2. In [A] Amir proved that $k-$UC spaces have UNS, so a simple consequence of Theorem 1 is that if E and F are $1-$UC then $(E\oplus F)_1$ has UNS. The following more general question seems to be unsolved: If E and F have UNS, does the space $(E\oplus F)_1$ has UNS?

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