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On weak solutions to a viscoelasticity model

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Abstract. The existence of global in time weak solutions of a viscoelasticity model is proved. There is no restriction on the dimension but it is supposed that the memory response is linear and a kernel has special properties.

Keywords: Weak solution, viscoelasticity, global existence, Galerkin approximations, a priori estimates, compact imbedding, monotone operators

Classification: 45K05, 73F99

1. Introduction.

The purpose of this paper is to prove the existence of global in time weak solutions of equations of motion for a model of viscoelastic body. We assume that the body occupies a reference configuration $\Omega \subset \mathbf{R}^N$ (Ω is a bounded domain with smooth boundary) and has unit reference density. We denote by $u(x,t)$ the displacement at the time t of the particle with the reference position x. The strain ϵ is given by

$$\epsilon(x,t) = \nabla_x u(x,t) \tag{1.1}$$

and the equation of balance of linear momentum has the form

$$u_{tt}(x,t) = \operatorname{div}_x \sigma(x,t) + f(x,t), \tag{1.2}$$

where σ is the stress and f is a body force. The body is characterized by constitutive assumptions which relate the stress to the motion. General constitutive theories are discussed for example in Coleman & Noll [**2**], Coleman & Mizel [**1**] and Saut & Joseph [**11**]. For the comprehensive account see the recent monograph Renardy & Hrusa & Nohel [**10**].

We shall limit our attention to constitutive relations of the type

$$\sigma(x,t) = \int_{-\infty}^{t} k(t-s)G(\epsilon(x,t),\epsilon(x,s))ds. \tag{1.3}$$

Here k is a given nonincreasing positive function which satisfies certain growth conditions at 0 and ∞. We suppose that the tensor function G has the special form

$$G(a,b) = g(a) + h(b). \tag{1.4}$$

Moreover, our crucial assumption is that h is linear. Assumptions on g are stated below.

Substitution of the constitutive relations into (1.2) yields

$$(1.5) \qquad u_{tt}(x,t) = \operatorname{div}_x g(\nabla u(x,t)) - \int_{-\infty}^{t} k(t-s)\Delta u(x,s)ds + f(x,t),$$

$x \in \Omega, t \geq 0$. We consider this equation together with the Dirichlet boundary condition

$$(1.6) \qquad u \mid \partial\Omega = 0\,.$$

We shall write (1.5) and (1.6) in the form

$$(1.7) \qquad u_{tt} + \phi(u) + k * \Delta u = f\,.$$

We seek a vector function u which satisfies (1.7) in weak sense together with the initial conditions

$$(1.8) \qquad u(0) = u_0, \qquad u_t(0) = u_1.$$

We remark that little is known about the existence of weak solutions for viscoelastic models. Recently, Nohel & Rogers & Tzavaras [9] established the global existence of weak solutions to the initial value problem above in the special case $\Omega = \mathbf{R}$ and $g = h$ in (1.4).

In Section 2 we introduce appropriate function spaces. Section 3 is devoted to the proof of the existence of weak solutions. The proof is standard: we use the Galerkin method to construct approximate solutions, establish a priori estimates and use compact imbeddings and the theory of monotone operators to prove convergence. The method of monotone operators was used in similar situation in [6].

The authors are indebted to John A. Nohel for the discussion of the preliminary version of this paper.

After this paper was finished we learned about the work of H.Engler [3] dealing with more general equation than (1.5) in one space dimension.

2. Appropriate spaces and operators.

Let $\Omega \subset \mathbf{R}^N$ be a bounded domain with smooth boundary. The spaces $V = H_0^1(\Omega)$, $H = L^2(\Omega)$ and $V' = H^{-1}(\Omega)$ of $\mathbf{R}^N$-valued functions are defined in the usual way. We denote by $(\cdot,\cdot)$ and $((\cdot,\cdot))$ respectively the scalar product in H and V. The corresponding norms are denoted by $|\cdot|$ and $\|\cdot\|$. The duality between V' and V is denoted by $\langle\cdot,\cdot\rangle$. The Laplace operator $\Delta : V \to V'$ is defined by

$$\langle -\Delta u, v\rangle = \int_\Omega \nabla u \cdot \nabla v = ((u,v)),$$

$u, v \in V$.

Consider the orthogonal basis of H consisting of the eigenfunctions $w_n \in V$ of $-\Delta$. We assume

$$-\Delta w_n = \lambda_n w_n, \quad |w_n| = 1, \qquad n = 1,2,\cdots,$$

where $0 < \lambda_1 < \lambda_2 \leq \lambda_3$ is the sequence of eigenvalues of $-\Delta$.

We denote by P_m the orthogonal projection (in H) of H onto the linear hull V_m of the first m eigenfunctions. The following statement is obvious:

Lemma 1. *The operators P_m can be extended to the orthogonal projections in V'.*

The extension of P_m to V' will be denoted also by P_m.

Let $u \in V$ and let $c_k = (w_k, u)$. We define

$$[u]_s^2 = \sum_{k=1}^{\infty} \lambda_k^s c_k^2.$$

We shall assume $-\frac{1}{2} < s < \frac{1}{2}$. We can consider $[\cdot]_s$ as a norm on V and the completion of V in this norm is denoted by $H^s(\Omega)$.

Let E be a Hilbert space and let $a < b \in \mathbf{R}$. The space $L^2(a,b;E)$ is defined in the usual way. For $0 < \nu < \frac{1}{2}$ and $u \in L^2(a,b;E)$ we define

$$\| u \|_\nu^2 = \int_0^1 dt \cdot t^{-(2\nu+1)} \int_{\mathbf{R}} | u(\tau - t) - u(\tau) |_E^2 \, d\tau.$$

(We extend u by zero outside (a,b).) The space of all $u \in L^2(a,b;E)$ for which $\| u \|_\nu$ is finite is denoted by $\mathcal{H}^\nu(a,b;E)$. For $u \in L^2(a,b;E)$ we denote by $\hat{u}$ the Fourier transform of u. It is well-known (see e.g. [5]) that

$$[\int_{\mathbf{R}} (1+\sigma^2)^\nu \, | \hat{u}(\sigma) |_E^2 \, d\sigma]^{\frac{1}{2}} \tag{2.2}$$

is an equivalent norm on $\mathcal{H}^\nu(a,b;E)$ .We put $\mathcal{H}^0(a,b;E) = L^2(a,b;E)$ and for $0 \le \alpha < \frac{1}{2}$ define $\mathcal{H}^{1+\alpha}(a,b;E)$ as the space of all $u \in L^2(a,b;E)$ with distributional derivatives u' belonging to $\mathcal{H}^\alpha(a,b;E)$. The norm on $\mathcal{H}^{1+\alpha}(a,b;E)$ is defined by

$$\| u \|_{1+\alpha} = \| u \|_{L^2} + \| u' \|_\alpha \, .$$

We also introduce the spaces

$$\mathcal{H}_-^{1+\alpha}(a,b;E) = \{u \in \mathcal{H}^{1+\alpha}(a,b;E), u(a) = 0\}$$
$$\mathcal{H}_0^{1+\alpha}(a,b;E) = \{u \in \mathcal{H}^{1+\alpha}(a,b;E), u(a) = u(b) = 0\}.$$

Throughout this article we assume $0 < \nu < \frac{1}{2}$.

Lemma 2. *The natural imbedding*

$$\mathcal{H}_-^{1+\nu}(a,b;V') \cap L^\infty(a,b;H) \hookrightarrow \mathcal{H}_-^1(a,b;V')$$

is compact.

PROOF : Let (u_m) be a bounded sequence in

$$\mathcal{H}_-^{1+\nu}(a,b;V') \cap L^\infty(a,b;H).$$

We fix a smooth function θ vanishing on $(b+1,\infty)$ and $=1$ in a neighbourhood of b and define

$$v_m(t) = \begin{cases} u_m(t), & t \in [a,b] \\ \theta(t)u_m(b), & t \in [b,b+1]. \end{cases}$$

The sequence v_m is bounded in $\mathcal{H}_0^{1+\nu}(a,b+1;V') \cap L^\infty(a,b+1;H)$. Let $\beta \in (0,1)$, $\nu' \in (0,\nu)$ satisfy $\beta(1+\nu) > 1+\nu'$.We notice that

$$\lambda_k^{-\beta}(1+\sigma^2)^{1+\nu'} \le \lambda_k^{-1}(1+\sigma^2)^{1+\nu} + 1$$

and using the definition of the norm $[\cdot]_\nu$ and the expression (2.2) we see that

$$\mathcal{H}_0^{1+\nu}(a,b+1;V') \cap L^\infty(a,b+1;H).$$

is continuously imbedded into $\mathcal{H}_0^{1+\nu'}(a,b+1;H^{-\beta}(\Omega))$. The last space is compactly imbedded into $\mathcal{H}_0^1(a,b+1;V')$.(See the proof of Theorem 1.5.2. in [4],for example.) ■

To construct the operator ϕ in (1.7.) we fix a convex function $F : \mathbf{R}^{N\times N} \to \mathbf{R}$ of the class C^2 satisfying

$$F(0) = 0, \quad \frac{\partial F}{\partial p_{ij}}(0) = 0, \qquad i,j = 1,\cdots,N,$$
$$|\frac{\partial^2 F}{\partial p_{ij}\partial p_{kl}}| \le M, \qquad i,j,k,l = 1,\cdots,N,$$
$$\sum_{i,j,k,l=1}^{N} \frac{\partial^2 F}{\partial p_{ij}\partial p_{kl}}\xi_{ij}\xi_{kl} \ge \mu|\xi|^2$$

for some positive μ and M. We define the operator $\varphi : V \to V'$ by the formula

$$\langle\varphi(u), v\rangle = \int_\Omega \frac{\partial F}{\partial p_{ij}}(\nabla u)\frac{\partial v_i}{\partial x_j},$$

$u, v \in V$ Clearly φ is Lipschitz continuous and satisfies

$$\mu \parallel u - v \parallel^2 \le \langle\varphi(u) - \varphi(v), u - v\rangle \le M \parallel u - v \parallel^2, \tag{2.5}$$

$u, v \in V$. For fixed $T \in (0,\infty)$ we introduce the operator
$\phi : L^2(0,T;V) \to L^2(0,T;V')$ by

$$\langle\phi(u), v\rangle_T = \int_0^T \langle\varphi(u(t)), v(t)\rangle dt, \tag{2.6}$$

$u, v \in L^2(0,T;V)$, where $\langle\cdot,\cdot\rangle_T$ denotes the duality between $L^2(0,T;V)$ and $L^2(0,T;V')$.

Lemma 3. *The operator ϕ maps $\mathcal{H}^\nu(0,T;V)$ into $\mathcal{H}^\nu(0,T;V')$ and*

$$\| \phi(u) \|_{\mathcal{H}^\nu(0,T;V')} \leq M \| u \|_{\mathcal{H}^\nu(0,T;V)}$$

for all $u \in \mathcal{H}^\nu(0,T;V)$.

PROOF : This follows easily from (2.5). ■

Let us fix $\alpha, \beta > 0$ and let

$$k(t) = \begin{cases} 0, & \text{for } t \leq 0 \\ \beta t^{-2\nu} e^{-\alpha t}, & \text{for } t > 0. \end{cases} \tag{2.7}$$

In what follows we could replace k by any function vanishing on $(-\infty, 0)$ and satisfying together with its derivative the same growth conditions at 0 and ∞ as the special k above.

We define the operator $\mathcal{K} : L^2(0,T;V) \to L^2(0,T;V')$ by

$$\mathcal{K}u(t) = \int_0^t -k(t-s)\Delta u(s) ds.$$

It is not difficult to see that

$$\langle \mathcal{K}u, u \rangle \leq \kappa \| u \|^2_{L^2(0,T;V)}, \tag{2.8}$$

where $\kappa = \int_0^\infty k$.

Lemma 4. *The operator $\mathcal{K}$ maps $\mathcal{H}^\nu(0,T;V)$ into $\mathcal{H}^\nu(0,T;V')$ and*

$$\| \mathcal{K}u \|_{\mathcal{H}^\nu(0,T;V')} \leq \kappa \| u \|_{\mathcal{H}^\nu(0,T;V)} .$$

PROOF : This is easy. ■

Lemma 5. *Let E be a Hilbert space and let $v : \mathbf{R} \to E$ satisfy*

$$\begin{gathered} v(t) = 0 \qquad \textit{for } t \leq 0, \\ \lim_{t \to +\infty} v(t) = v_\infty \qquad \textit{(strong limit)}, \\ v' \in L^1 \cap L^\infty(\mathbf{R}; E). \end{gathered} \tag{2.9}$$

Then

$$\int_0^\infty (k * v(s), v'(s))_E ds$$
$$= \frac{\kappa}{2} |v_\infty|^2_E + \frac{1}{2} \int_0^\infty ds\, k'(s) \int_0^\infty dt\, |v(t) - v(t-s)|^2_E. \tag{2.10}$$

PROOF : It is not difficult to see that the following computation is legal

$$\int_0^\infty (k*v(s),v'(s))_E\,ds - \frac{\kappa}{2}|v_\infty|_E^2$$
$$= \int_0^\infty ds \int_0^\infty dt\, k(t)(v(s-t)-v(s),v'(s))_E$$
$$= \int_0^\infty ds \int_0^\infty dt\, k(s-t)(v(t)-v(s),\frac{d}{ds}(v(s)-v(t)))_E$$
$$= -\frac{1}{2}\int_0^\infty ds \int_0^\infty dt\, k(s-t)\frac{d}{ds}|v(s)-v(t)|_E^2$$
$$= \frac{1}{2}\int_0^\infty dt \int_0^\infty ds\, k'(s-t)|v(s)-v(t)|_E^2$$
$$= \frac{1}{2}\int_0^\infty ds \int_0^\infty dt\, k'(t)|v(s)-v(s-t)|_E^2.$$

The proof is finished. ∎

3. Construction of solutions.

Our next step consist in the construction of Galerkin approximations for the problem (1.7),(1.8). We assume that the forcing term f satisfies

$$f \in L^2(0,T;H)\cap \mathcal{H}^\nu(0,T;V') \tag{3.1}$$

for some fixed $T \in (0,+\infty)$.

Lemma 6. *Suppose $\mu > \kappa$. For each $m \in \mathbf{N}$ there is a unique function $u_m \in H_2^2(0,T;V_m)$ satisfying*

$$\langle (u_m''(t)+\varphi(u_m(t))+k*\Delta u_m(t)),w_j\rangle = (f(t),w_j)\,,\qquad j=1,\dots,m\,, \tag{3.2}$$

for a.e. $t\in(0,T)$ and

$$u_m(0)=u_m'(0)=0.$$

The functions u_m satisfy

$$\| u_m' \|_{L^\infty(0,T;H)}^2 + \| u_m \|_{L^\infty(0,T;V)}^2 \le c\,, \tag{3.3}$$

$$\| u_m \|_{\mathcal{H}^\nu(0,T;V)}^2 \le c\,, \tag{3.4}$$

where c does not depend on m.

PROOF : It is standard that (3.2) has a solution on a small interval $(0,\delta)$. (See e.g. [7]). Let us derive a priori estimates. Let $t\in(0,T)$ and suppose u_m is defined on $(0,t)$. Replacing w_j by u_m' in (3.2) and integrating over $(0,t)$ we obtain

$$\left(\frac{1}{2}|u_m'|^2 + \int_\Omega F(\nabla u_m)\right)\Big|_{t=0}^{t=t} = \int_0^t ((k*u_m,u_m')) + \int_0^t (f,u_m').$$

From Lemma 5 we see that

$$(3.5)\quad \int_0^t ((k * u_m, u_m')) = \frac{\kappa}{2} \| u_m(t) \|^2 + \frac{1}{2} \int_0^\infty d\tau\, k'(\tau) \int_0^\infty ds\, \| u_m^{(t)}(s-\tau) - u_m^{(t)}(s) \|^2 ,$$

where

$$u_m^{(t)}(s) = \begin{cases} u_m(s), & \text{if } s < t \\ u_m(t), & \text{if } s \geq t. \end{cases}$$

Since the second term on the right-hand side is negative (or nonpositive) and $u_m(0) = u_m'(0) = 0$, we see that

$$(3.6)\qquad \frac{1}{2}|u_m'(t)|^2 + \int_\Omega F(\nabla u_m(t)) - \frac{\kappa}{2} \| u_m(t) \|^2 \leq \int_0^t (f, u_m').$$

Now if $\mu > \kappa$ then

$$\int_\Omega F(\nabla u_m(t)) - \frac{\kappa}{2} \| u_m(t) \|^2 \geq \frac{\mu - \kappa}{2} \| u_m(t) \|^2 ,$$

and by the standard use of the Gronwall lemma we infer

$$(3.7)\quad |u_m'(t)|^2 + \| u_m(t) \|^2 - \frac{1}{2} \int_0^\infty d\tau\, k'(\tau) \int_0^\infty ds\, \| u_m^{(t)}(s-\tau) - u_m^{(t)}(s) \|^2 \leq c ,$$

where c is independent of m and $t \leq T$.

To estimate the $\mathcal{H}^\nu$ norm let us define

$$\tilde{u}_m(s) = \begin{cases} u_m(s), & \text{if } s \leq t \\ 0, & \text{if } s > t. \end{cases}$$

By Lemma 5 the $\mathcal{H}^\nu(0,T;V)$ norm of u_m is estimated by $\int_{\mathbf{R}}((k * \tilde{u}_m, \tilde{u}_m'))$. But this integral equals to

$$-((k * u_m(t), u_m(t))) + \int_0^t ((k * u_m, u_m'))$$

since, roughly speaking, $\tilde{u}_m'$ gives the Dirac measure at t. By (3.5) this amounts to

$$\frac{\kappa}{2} \| u_m(t) \|^2 + \frac{1}{2} \int_0^\infty d\tau\, k'(\tau) \int_0^\infty ds\, \| u_m^{(t)}(s-\tau) - u_m^{(t)}(s) \|^2 - ((k * u_m(t), u_m(t)))$$

and this expression is bounded by (3.7). Hence

$$\| u_m \|^2_{\mathcal{H}^\nu(0,T;V)} \leq c ,$$

where c does not depend on m and $t \leq T$. The proof is finished. ∎

Lemma 7. *The sequence u_m'' is compact in $L^2(0,T;V')$.*

PROOF : We notice that (3.2) can be rewritten as

$$u_m'' + \tilde{P}_m \phi(u) + \mathcal{K} u_m = \tilde{P}_m f\,,$$

where $\tilde{P}_m : L^2(0,T;V') \to L^2(0,T;V_m)$ is defined by $(\tilde{P}_m u)(t) = P_m(u(t))$ (see Lemma 1). We can now use Lemmas 1-4 together with the estimates (3.3) and (3.4) to infer that the sequence u_m' is bounded in $\mathcal{H}_-^{1+\nu}(a,b;V') \cap L^\infty(a,b;H)$ and hence compact in $\mathcal{H}_-^1(a,b;V')$. This implies that u_m'' is compact in $L^2(0,T;V')$. The proof is finished. ∎

Passing to a subsequence, if necessary, we can assume that

$$\begin{aligned} u_m &\rightharpoonup u \quad &\text{in} \quad L^2(0,T;V),\\ u_m'' &\to u'' \quad &\text{in} \quad L^2(0,T;V'). \end{aligned}$$

Theorem. *Let $\kappa < \mu$. Then u is a weak solution of (1.7).*

PROOF : The only problem is to show that $\phi(u_m) \rightharpoonup \phi(u)$ in $L^2(0,T;V')$. Since clearly $\mathcal{K}u_m \rightharpoonup \mathcal{K}u$ it is enough to show $\mathcal{B}u_m \rightharpoonup \mathcal{B}u$, where $\mathcal{B} = \phi - \mathcal{K}$. By our assumptions, $\mathcal{B}$ is strongly monotone and Lipschitz continuous. We can suppose $\mathcal{B}u_m \rightharpoonup \chi$ in $L^2(0,T;V')$. Clearly $u'' + \chi = f$. For any $v \in L^2(0,T;V)$ we have

$$\begin{aligned} \langle \chi - \mathcal{B}v, u - v\rangle_T &= \langle -u'' + f - \mathcal{B}v, u - v\rangle_T = \text{(by Lemma 7)}\\ &= \lim_{m\to\infty} \langle u_m'' + f - \mathcal{B}v, u_m - v\rangle_T = \langle \mathcal{B}u_m - \mathcal{B}v, u_m - v\rangle_T \geq 0. \end{aligned}$$

From this we can infer $\chi = \mathcal{B}u$ by "Minty's trick". (See, for example, [8]). The proof is finished. ∎

Remark. If $T = \infty$ and $f \in L^1(0,\infty;H)$ the procedure above yields a weak solution u on the interval $(0,\infty)$ which belongs to the space $\mathcal{H}^\nu(0,\infty;V)$. This follows easily from the estimate

$$\| u_m \|_{\mathcal{H}^\nu(0,\infty;V)} \leq \| f \|_{L^1(0,\infty;H)}\,,$$

which can be obtained in a similar way as (3.7).

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