C^α–regularity results for quasilinear parabolic systems*

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Abstract. C^α–regularity is proved for a class of quasilinear parabolic system with critical growth in the lower order term.

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0. INTRODUCTION

In this paper we consider diagonal parabolic systems $Lu = H(x, u, \nabla u)$ where the nonlinear coupling of the components of u is governed by the term H which may have quadratic growth in ∇u. By Struwe's counterexample [9] it is known that in this case *irregular* solutions may occur after some time t_0. On the other hand several regularity results for diagonal parabolic *systems* have been obtained under a *smallness assumption* for the quadratic growth of H in ∇u, cf. Ladyženskaya–Ural'seva–Solonnikov [8], Struwe [10], Giaquinta–Struwe [6] and others. This smallness assumption is *not* acceptable for the parabolic systems which arise as the *Bellman–equation* of *stochastic differential games*. These equations have a special structure which allows to prove C^α–regularity (and hence $H^{2,p}$–regularity) without a smallness condition. The situation is similar to the elliptic case treated by the authors in [2]. Since parabolic systems are even more applicable than elliptic systems, to the theory of stochastic control and stochastic differential games, it is natural to look for an extension of the results previously obtained by the authors.

This means that we can naturally consider the specific structure, introduced by the authors in the elliptic case, in the context of parabolic systems. Obtaining regularity results for such parabolic systems will imply solving a large class of stochastic differential games, with finite horizon (whereas elliptic systems cover only infinite horizon stationary problems).

Our specific structure is the following

$$
\begin{aligned}
\frac{\partial u^1}{\partial t} + A(t)u^1 &= H^1(x,t,u,Du) \\
\frac{\partial u^2}{\partial t} + A(t)u^2 &= H^2(x,t,u,Du)
\end{aligned}
\tag{1}
$$

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where $A(t)$ is a second order elliptic operator (possibly depending on time), with Dirichlet boundary conditions, and H^1, H^2 satisfy ($u = (u^1, u^2)$)

$$(2) \qquad \begin{aligned} \left|H^1(x,t,u,Du)\right| &\le K\,|Du|^2 + f \\ \left|H^2(x,t,u,Du)\right| &\le K\,|Du|\,\left|Du^2\right| + f \end{aligned}$$

where K is a constant and $f \in L^p$.

We can also add to H^2 an operator of the form $H_0^2(x,t,u,Du)$ satisfying

$$\left|H_0^2(x,t,u,Du)\right| \le \varepsilon\left|Du^1\right|^2 + K_{20}$$

where ε is small. To simplify the presentation we shall not consider such a term in the sequel.

Our techniques combine the use of special test functions introduced in the elliptic case, with techniques of Struwe [10] to deal with the parabolic Green function.

1. Setting of the problem

1.1. Notation.

Let Ω be a smooth bounded domain of R^n and $Q = \Omega \times [0,T]$. Let $z_0 = (x_0, t_0) \in Q$, we shall consider the ball

$$B_R(x_0) = \{x \mid |x - x_0| \le R\}$$

and

$$\tilde{B}_R(x_0) = B_R(x_0) \cap \Omega$$

and the cylinder

$$\begin{aligned} Q_R(z_0) &= B_R(x_0) \cap \left[\left(t_0 - R^2\right)^+, t_0\right] \\ \tilde{Q}_R(z_0) &= \tilde{B}_R(x_0) \cap \left[\left(t_0 - R^2\right)^+, t_0\right]. \end{aligned}$$

In general, there will be no ambiguity in the point z_0 and we shall omit it, writing $B_R, Q_R, \ldots$.

The gradient with respect to x is denoted D and the derivative in time by ∂_t.

1.2. Assumptions.

Let $a_{ij}(x,t)$, $i,j = 1,\ldots,n$ be given functions satisfying

$$(1.1) \qquad a_{ij} \in L^\infty(Q)$$

$$(1.2) \qquad a_{ij}\xi_i\xi_j \ge \alpha\,|\xi|^2, \quad \forall \xi \in R^n, \alpha > 0.$$

We define

$$A = A(t) = -\frac{\partial}{\partial x_i} a_{ij} \frac{\partial}{\partial x_j}.$$

Let next H^1, H^2 be non linear functions satisfying

$$\left|H^1(x,t,\lambda,p)\right| \le K\left|p\right|^2 + f \tag{1.3}$$

$$\left|H^2(x,t,\lambda,p)\right| \le K\left|p\right|\left|p_2\right| + f \text{ with } f \in L^q(Q), \tag{1.4}$$

$$q > \frac{n}{2}+1, K>0, \text{ and } \lambda \in R^2, p \in R^{2n}, p=(p_1,p_2).$$

We shall be interested in the system of partial differential equations

$$\begin{aligned} \frac{\partial u^1}{\partial t} + A(t)u^1 &= H^1(x,t,u,Du) \\ \frac{\partial u^2}{\partial t} + A(t)u^2 &= H^2(x,t,u,Du) \\ u^1 &= u^2|_\Sigma = 0 \quad , \quad \Sigma = \partial\Omega \times (0,T) \\ u^1(x,0) &= u^2(x,0) = 0. \end{aligned} \tag{1.5}$$

We have taken 0 as initial and boundary conditions, but of course the result can be extended to data which are sufficiently smooth and compatible.

We shall be considering functions u^1, u^2 satisfying (1.5), and

$$u^1, u^2 \in L^\infty(Q) \cap L^2\left(0,T,H_0^1(\Omega)\right). \tag{1.6}$$

Note that we can without loss of generality write

$$H^2(x,t,u,Du) = Q(x,t,u,Du).Du^2 + f_2 \tag{1.7}$$

where

$$\begin{aligned} &\left|Q(x,t,u,Du)\right| \le K\left|Du\right| \\ &\text{and} \\ &\left|f_2\right| \le f. \end{aligned} \tag{1.8}$$

Indeed, since from (1.4)

$$\left|H^2(x,t,u,Du)\right| \le K\left|Du\right|\left|Du^2\right| + f$$

there exists a measurable bounded function $\sigma(x,t)$ (depending on u) such that

$$H^2(x,t,u,Du) = \sigma K\left|Du\right|\left|Du^2\right| + f\sigma.$$

If we set

$$\begin{aligned} Q(x,t,u,Du) &= \sigma(x,t)K\left|Du\right|\frac{Du^2}{\left|Du^2\right|} \\ f_2 &= \sigma f \end{aligned}$$

then we recover the form (1.7).

1.3. Statement of the main result.

Our objective is to prove the following:

Theorem 1.1. *Let the Caratheodory conditions and the growth and ellipticity conditions* (1.1), (1.2), (1.3), (1.4) *for the data be satisfied. Then every solution u of the parabolic system* (1.5) *satisfying* (1.6) *belongs to $C^{\alpha,\alpha/2}(Q)$, for some $\alpha > 0$.*

We recall that

$$C^{\alpha,\frac{\alpha}{2}}(Q) = \left\{\varphi \,|\, |\varphi(x_1,t_1) - \varphi(x_2,t_2)| \le c\left(|x_1 - x_2|^\alpha + |t_1 - t_2|^{\alpha/2}\right)\right\}.$$

We shall use the characterization of $C^{\alpha,\frac{\alpha}{2}}(Q)$ as a Campanato space (see Campanato [3], Da Prato [4]), namely $\varphi \in L^2(Q)$ belongs to $C^{\alpha,\frac{\alpha}{2}}(Q)$ if and only if

$$\sup_{R,z} R^{-(n+2+2\alpha)} \int_{\tilde{Q}_R(z)} |u - u_{R,z}|^2 < \infty \tag{1.9}$$

where

$$u_{R,z} = \frac{\int_{\tilde{Q}_R(z)} u}{Meas\tilde{Q}_R(z)}. \tag{1.10}$$

Note that we assume implicitly that Ω satisfies the condition of type A (cf. Kufner–John–Fučík [7]),[1] namely

$$\begin{aligned} &Meas\tilde{B}_R(x_0) \ge AR^n \quad , \quad \forall x_0 \in \Omega, \forall R, \\ &\text{with } A > 0, \text{ independent of } x_0 \text{ and } R. \end{aligned} \tag{1.11}$$

Note that since $u \in L^2$, it is sufficient to take $R < R_1$ in the set (1.9).

2. Basic inequalities

2.1. Test functions.

There will be as in the elliptic case, a basic trick to handle the quadratic growth of H_1, H_2. We describe it first. The presentation is slightly simpler than our previous one.

Let

$$\begin{aligned} \theta(x) &= e^x - x - 1 \\ \theta'(x) &= e^x - 1. \end{aligned}$$

We shall consider the following functions

$$\begin{aligned} F_+^1 &= \exp\left[\theta\left(\lambda\left(u^1 - c^1\right)\right)\right] \\ F_-^1 &= \exp\left[\theta\left(-\lambda\left(u^1 - c^1\right)\right)\right] \\ F_+^2 &= \exp\left[\gamma\theta\left(\lambda\left(u^2 - c^2\right)\right)\right] \\ F_-^2 &= \exp\left[\gamma\theta\left(-\lambda\left(u^2 - c^2\right)\right)\right] \end{aligned}$$

[1] if Ω is smooth, it is satisfied.

where γ, λ are constants to be chosen later, *independently* of u^1, u^2, whereas c^1, c^2 are constants related to $u^1, , u^2$. It will be convenient to use the notation

$$F^1_{\nu_1} = \exp\left[\theta\left(\lambda\nu_1\left(u^1-c^1\right)\right)\right],$$
$$F^2_{\nu_2} = \exp\left[\gamma\theta\left(\lambda\nu_2\left(u^2-c^2\right)\right)\right], \quad \text{where } \nu_1, \nu_2 = \pm.$$

The asymmetry between F^1 and F^2, due to the presence of γ, stems from the fact that the two operators H^1, H^2 do not play a symmetric role.

If $\nu = (\nu_1, \nu_2)$ set $F_\nu = F^1_{\nu_1} F^2_{\nu_1}$.

Let also $\psi(x,t)$ a function satisfying

(2.1) $\psi \geq 0$; ψ is sufficiently smooth to perform some differentiation, as below. Moreover $\psi|_\Sigma = 0$, $D\psi|_\Sigma = 0$ if c^1, c^2 are not 0. If they are both 0, this condition is not requested.

We test the first equation (1.5) with $\nu_1\lambda\theta'\left(\lambda\nu_1\left(u^1-c^1\right)\right)F_\nu\psi$ and the second equation (1.5) with $\nu_2\gamma\lambda\theta'\left(\lambda\nu_2\left(u^2-c^2\right)\right)F_\nu\psi$. These functions are 0 on Σ, by our choice (2.1).

We perform an integration by parts in x, as usual, and add up. It is easy to check the following relation

$$\begin{aligned}
&\int \psi\frac{\partial F_\nu}{\partial t} + \int a_{ij}\frac{\partial\psi}{\partial x_i}\frac{\partial F_\nu}{\partial x_j} + \int \psi\frac{a_{ij}}{F_\nu}\frac{\partial}{\partial x_i}F_\nu\frac{\partial}{\partial x_j}F_\nu + \\
&+\lambda^2\int\psi F_\nu a_{ij}\left[\frac{\partial u^1}{\partial x_j}\frac{\partial u^1}{\partial x_i}\theta''\left(\nu_1\lambda\left(u^1-c^1\right)\right)+\gamma\frac{\partial u^2}{\partial x_j}\frac{\partial u^2}{\partial x_i}\theta''\left(\nu_2\lambda\left(u^2-c^2\right)\right)\right] \\
&= \int\lambda\psi F_\nu\left[H^1\nu_1\theta'\left(\nu_1\lambda\left(u^1-c^1\right)\right)+H^2\nu_2\gamma\theta'\left(\nu_2\lambda\left(u^2-c^2\right)\right)\right].
\end{aligned} \tag{2.2}$$

Using (1.7), and noticing that

$$DF_\nu = \lambda F_\nu\left[\nu_1\theta'\left(\lambda\nu_1\left(u^1-c^1\right)\right)Du^1+\gamma\nu_2\theta'\left(\lambda\nu_2\left(u^2-c^2\right)\right)Du^2\right]$$

the right hand side of (2.2) writes

$$= \int\lambda\psi F_\nu\tilde{H}^1\nu_1\theta'\left(\nu_1\lambda\left(u^1-c^1\right)\right)+\int\lambda\psi F_\nu\gamma f_2\nu_2\theta'\left(\nu_2\lambda\left(u^2-c^2\right)\right) + \int\psi Q.DF_\nu$$

where we have set

$$\tilde{H}^1 = H^1 - Q.Du^1. \tag{2.3}$$

Writing

$$\frac{a_{ij}}{F_\nu}\frac{\partial}{\partial x_i}F_\nu\frac{\partial}{\partial x_j}F_\nu - Q.DF_\nu = a\left(F_\nu^{-1/2}DF_\nu-\frac{F_\nu^{1/2}}{2}a^{-1}Q\right)\left(F_\nu^{-1/2}DF_\nu - \frac{F_\nu^{1/2}}{2}a^{-1}Q\right)-\frac{1}{4}a^{-1}Q.QF_\nu$$

where $a \equiv a_{ij}$
we deduce from (2.2) our first basic inequality

$$\begin{aligned}
&\int \psi \frac{\partial F_\nu}{\partial t} + \int a_{ij} \frac{\partial \psi}{\partial x_i} \frac{\partial F_\nu}{\partial x_j} + \lambda^2 \int \psi F_\nu a_{ij} \left[\frac{\partial u^1}{\partial x_j} \frac{\partial u^1}{\partial x_i} \theta'' \left(\nu_1 \lambda \left(u^1 - c^1\right)\right) \right.\\
&\left. + \gamma \frac{\partial u^2}{\partial x_j} \frac{\partial u^2}{\partial x_i} \theta'' \left(\nu_2 \lambda \left(u^2 - c^2\right)\right) \right] \leq \int \lambda \psi F_\nu \tilde{H}^1 \nu_1 \theta' \left(\nu_1 \lambda \left(u^1 - c^1\right)\right) \\
&+ \int \lambda \psi F_\nu \gamma f_2 \nu_2 \theta' \left(\nu_2 \lambda \left(u^2 - c^2\right)\right) + \int \frac{\psi F_\nu}{4} a^{-1} Q.Q.
\end{aligned} \tag{2.4}$$

2.2. Basic estimate.
We apply now (2.4) with the four possible values of ν and add up. Define

$$X_0 = \left(F_+^1 + F_-^1\right)\left(F_+^2 + F_-^2\right) \tag{2.5}$$

we obtain

$$\begin{aligned}
&\int \psi \frac{\partial X_0}{\partial t} + \int a_{ij} \frac{\partial \psi}{\partial x_i} \frac{\partial}{\partial x_j} X_0 + \lambda^2 \int \psi a_{ij} \left[\frac{\partial u^1}{\partial x_j} \frac{\partial u^1}{\partial x_i} \left(F_+^2 + F_-^2\right) \right.\\
&\left(F_+^1 \theta'' \left(\lambda \left(u^1 - c^1\right)\right) + F_-^1 \theta'' \left(-\lambda \left(u^1 - c^1\right)\right)\right) + \gamma \frac{\partial u^2}{\partial x_j} \frac{\partial u^2}{\partial x_i} \left(F_+^1 + F_-^1\right) \\
&\left.\left(F_+^2 \theta'' \left(\lambda \left(u^2 - c^2\right)\right) + F_-^2 \theta'' \left(-\lambda \left(u^2 - c^2\right)\right)\right)\right] \leq \\
&\int \lambda \psi \tilde{H}^1 \left(F_+^2 + F_-^2\right)\left(F_+^1 \theta' \left(\lambda \left(u^1 - c^1\right)\right) - F_-^1 \theta' \left(-\lambda \left(u^1 - c^1\right)\right)\right) + \\
&\int \lambda \psi \gamma f_2 \left(F_+^1 + F_-^1\right)\left(F_+^2 \theta' \left(\lambda \left(u^2 - c^2\right)\right) - F_-^2 \theta' \left(-\lambda \left(u^2 - c^2\right)\right)\right) + \\
&\frac{1}{4} \int \psi X_0 a^{-1} Q.Q.
\end{aligned} \tag{2.6}$$

We note that

$$\left|\tilde{H}^1\right| \leq 2K \left|Du\right|^2 + f \quad , \quad \left|f_2\right| \leq f.$$

Set also

$$X_1 = \left(e^{\lambda\left(u^1 - c^1\right)} F_+^1 + e^{-\lambda\left(u^1 - c^1\right)} F_-^1\right)\left(F_+^2 + F_-^2\right) \tag{2.7}$$

$$X_2 = \left(e^{\lambda\left(u^2 - c^2\right)} F_+^2 + e^{-\lambda\left(u^2 - c^2\right)} F_-^2\right)\left(F_+^1 + F_-^1\right). \tag{2.8}$$

We check easily (see our previous paper)

$$\begin{aligned}
&4 \leq X_0 \leq X_1 \quad , \quad X_0 \leq X_2 \\
&\left|F_+^1 \left(e^{\lambda\left(u^1 - c^1\right)} - 1\right) - F_-^1 \left(e^{\lambda\left(u^1 - c^1\right)} - 1\right)\right| \leq e^{\lambda\left(u^1 - c^1\right)} F_+^1 + e^{-\lambda\left(u^1 - c^1\right)} F_-^1
\end{aligned} \tag{2.9}$$

and the same inequality with the index 1 replaced by 2.

We obtain from (2.6)

$$\begin{aligned}&\int \psi \frac{\partial X_0}{\partial t} + \int a_{ij} \frac{\partial \psi}{\partial x_i} \frac{\partial X_0}{\partial x_j} + \lambda^2 \alpha \int \psi \left[X_1 \left| Du^1 \right|^2 + \gamma X_2 \left| Du^2 \right|^2 \right] \\ &\leq 2K\lambda \int \psi \left| Du \right|^2 X_1 + \int \lambda \psi f \left[X_1 + \gamma X_2 \right] \\ &+ \frac{1}{4\alpha} K^2 \int \psi \left(\left| Du^1 \right|^2 X_1 + \left| Du^2 \right|^2 X_2 \right).\end{aligned} \tag{2.10}$$

Now use the fact that

$$\begin{aligned}&F_+^2 + F_-^2 \leq e^{\lambda(u^2 - c^2)} F_+^2 + e^{-\lambda(u^2 - c^2)} F_-^2 \\ &e^{\lambda(u^1 - c^1)} F_+^1 + e^{-\lambda(u^1 - c^1)} F_-^1 \leq e^{\lambda \left\| u^1 - c^1 \right\|} \left(F_+^1 + F_-^1 \right)\end{aligned}$$

where $\left\| u^1 - c^1 \right\|$ is the L^∞ norm, to conclude that

$$X_1 \leq e^{\lambda \left\| u^1 - c^1 \right\|} X_2.$$

We can then carry all terms in $\left| Du^1 \right|^2$, $\left| Du^2 \right|^2$ to the left hand side of (2.10), to obtain

$$\begin{aligned}&\int \psi \frac{\partial X_0}{\partial t} + \int a_{ij} \frac{\partial \psi}{\partial x_i} \frac{\partial X_0}{\partial x_j} + \int \psi \left[X_1 \left| Du^1 \right|^2 \left(\lambda^2 \alpha - 2K\lambda - \frac{K^2}{4\alpha} \right) \right. \\ &\left. + X_2 \left| Du^2 \right|^2 \left(\gamma \lambda^2 \alpha - 2K\lambda e^{\lambda \left\| u^1 - c^1 \right\|} - \frac{K^2}{4\alpha} \right) \right] \\ &\leq \int \lambda \psi f \left[X_1 + \gamma X_2 \right].\end{aligned}$$

We then fix λ and γ as follows

$$\begin{aligned}&\lambda^2 \alpha - 2K\lambda - \frac{K^2}{4\alpha} \geq c_0 > 0 \\ &\gamma \lambda^2 \alpha - 2K\lambda e^{\lambda \left\| u^1 - c^1 \right\|} - \frac{K_2}{4\alpha} \geq c_0.\end{aligned}$$

In the sequel c^1, c^2 will always be average of u^1, u^2 if not 0, and thus majorized by the L^∞ norm of u^1, u^2, hence we can assert

Proposition 2.1. *For any ψ satisfying* (2.1) *the following basic inequality holds*

$$k_0 \int_Q \psi \left| Du \right|^2 \leq - \int_Q \left(\psi \frac{\partial X_0}{\partial t} + a_{ij} \frac{\partial \psi}{\partial x_i} \frac{\partial X_0}{\partial x_j} \right) + k_1 \int_Q \psi f \tag{2.11}$$

where $k_0 > 0$, $k_1 > 0$ depend only on the L^∞ norm of $\|u\|$ and the constants K and α.

3. Proof of Theorem 1.1

3.1. Green function.

Let $z_0 = (x_0, t_0) \in Q$ and $\theta > 0$, we shall consider the Green function $G_{x_0,t_0+\theta}(x,t)$, abbreviated as G_θ defined for $t < t_0 + \theta$, $x \in R^n$, and which satisfies formally the equation

$$\begin{aligned}(3.1)\qquad &\frac{\partial G_\theta}{\partial t} + \frac{\partial}{\partial x_i} a_{ij} \frac{\partial}{\partial x_j} G_\theta = 0\\ &G_\theta(x_0, t_0 + \theta) = \delta(x - x_0).\end{aligned}$$

The properties of G_θ have been given by Aronson [1], and namely one can describe the behaviour of G_θ near the singularity by the estimates

$$\begin{aligned}(3.2)\qquad &k_1(t_0 + \theta - t)^{-n/2} \exp\left(-\frac{\delta_1 |x - x_0|^2}{t_0 + \theta - t}\right) \leq G_\theta(x,t)\\ &\leq k_2 \exp\left(-\delta_2 \frac{|x - x_0|^2}{t_0 + \theta - t}\right)(t_0 + \theta - t)^{-n/2}\end{aligned}$$

where $k_1, k_2, \delta_1, \delta_2$ are positive and depend only on α and the L^∞ norm of the a_{ij}.

3.2. Basic inequality.

We can replace in (2.10) X_0 by $X_0 + constant$. Take in particular $X_0 - 4$. It is positive and 0 on the boundary if c^1, c^2 are 0.

Consider two cut off functions as follows

$$\tau(x) = \begin{cases} 1 & \text{if} \quad |x| \leq 1 \\ 0 & \text{if} \quad |x| \geq 2 \end{cases}$$

$\tau \in C_0^\infty(R^n), 0 \leq \tau \leq 1$

$$\beta(t) = \begin{cases} 1 & \text{if} \quad t \geq -1 \\ 0 & \text{if} \quad t \leq -4 \end{cases}$$

$\beta \in C^\infty(R), 0 \leq \beta \leq 1.$

We set

$$\begin{aligned}\tau_{R,x_0} &= \tau_R = \tau\left(\frac{x - x_0}{R}\right)\\ \beta_{R,x_0} &= \beta_R = \beta\left(\frac{t - t_0}{R^2}\right)\end{aligned}$$

and

$$\eta_{R,z_0}(x,t) = \eta_R = \tau_{R,x_0}\beta_{R,t_0}$$

hence

$$0 \leq \eta_R \leq s \quad , \quad \eta_R \in C^\infty(R^{n+1}),$$

$$\eta_R = \begin{cases} 1 & \text{if} \quad (x,t) \in Q_R \\ 0 & \text{if} \quad x \notin B_{2R} \text{ or } t \leq t_0 - 4R^2. \end{cases}$$

We are going to use (2.11) with $\psi = \eta_R^2 G_\theta$. We have

$$
\begin{aligned}
&-\int_Q \left[\eta_R^2 G_\theta \frac{\partial}{\partial t}(X_0-4) + a_{ij}\frac{\partial}{\partial x_i}\left(\eta_R^2 G_\theta\right)\frac{\partial}{\partial x_j}(X_0-4)\right] \\
&= -\int_Q \left[G_\theta \frac{\partial}{\partial t}(\eta_R^2(X_0-4))) + a_{ij}\frac{\partial}{\partial x_i}G_\theta\frac{\partial}{\partial x_j}((X_0-4)\eta_R^2)\right] \\
&\quad + \int_Q G_\theta\left[(X_0-4)\frac{\partial}{\partial t}\eta_R^2 - a_{ij}\frac{\partial X_0}{\partial x_j}\frac{\partial}{\partial x_i}\eta_R^2\right] \\
&\quad + \int_Q (X_0-4)a_{ij}\frac{\partial G_\theta}{\partial x_i}\frac{\partial \eta_R^2}{\partial x_j}.
\end{aligned}
\tag{3.3}
$$

The constants c^1, c^2 entering into the definition of X_0, (cf. (2.5)), will be chosen so that (hence depending on R)

$$
c^1, c^2 = 0 \quad \text{if} \quad B_{2R}(x_0)\cap(R^n - \Omega) \neq \emptyset. \tag{3.4}
$$

The choice implies that $(X_0-4)\eta_R^2 = 0$ on Σ.

We can then perform an integration by parts in the first term at the right hand side of (3.3). Using (3.1), we deduce that this term is equal to

$$
-\int_\Omega G_\theta\eta_R^2(X_0-4)(x,T)\,dx + \int_\Omega G_\theta\eta_R^2(X_0-4)(x,0)\,dx.
$$

Therefore we have proven the following basic inequality

Lemma 3.1. *One has the inequality*

$$
\begin{aligned}
k_0\int_Q |Du|^2 G_\theta\eta_R^2 &\le \int_\Omega G_\theta\eta_R^2(X_0-4)(x,0)\,dx \\
&+ \int_Q G_\theta\left[(X_0-4)\frac{\partial}{\partial t}\eta_R^2 - a_{ij}\frac{\partial X_0}{\partial x_j}\frac{\partial}{\partial x_i}\eta_R^2\right] \\
&+ \int_Q (X_0-4)a_{ij}\frac{\partial G_\theta}{\partial x_i}\frac{\partial}{\partial x_j}\eta_R^2 + k_1\int_Q fG_\theta\eta_R^2.
\end{aligned}
\tag{3.5}
$$

We emphasize that (3.5) holds for any pair c^1, c^2 subject to (3.3) and majorized by the L^∞ norm of u^1, u^2.

To proceed we have to estimate further $X_0 - 4$, and DX_0. Note that

$$
\begin{aligned}
X_0 - 4 &= \left(F_+^1 + F_-^1 - 2\right)\left(F_+^2 + F_-^2 - 2\right) \\
&\quad + 2\left(F_+^2 + F_-^2 - 2\right) + 2\left(F_+^1 + F_-^1 - 2\right).
\end{aligned}
$$

Using the properties

$$\begin{aligned}|\theta(x)| &\le e^{|x|}x^2\\ |e^x-1| &\le |x|\, e^{|x|}\end{aligned}$$

we easily see that

$$\left|F_+^1-1\right| \le c\left|u^1-c^1\right|^2$$

where c depends only on the L^∞ norm of u^1. Therefore we can assert that

$$|X_0-4| \le c\left[\left|u^1-c_R^1\right|^2+\left|u^2-c_R^2\right|^2\right] \tag{3.6}$$

where we have written c_R^1, c_R^2 instead of c^1, c^2 to emphasize that these constants are to be chosen, possibly dependent of R (in fact they will depend of R, see (3.3), although again majorized by the L^∞ norm of u^1, u^2). Similarly

$$\begin{aligned}DX_0 =&\lambda\left(F_+^1\theta'(\lambda(u^1-c^1))-F_-^1\theta'(-\lambda(u^1-c^1))\right)\left(F_+^2+F_-^2\right)Du^1\\ &+\gamma\left(F_+^2\theta'(\lambda(u^2-c^2))-F_-^2\theta'(-\lambda(u^2-c^2))\right)\left(F_+^1+F_-^1\right)Du^2\end{aligned}$$

and thus as easily checked

$$|DX_0| \le c\left[\left|Du^1\right|\left|u^1-c_R^1\right|+\left|Du^2\right|\left|u^2-c_R^2\right|\right]. \tag{3.7}$$

Using (3.6), (3.7) in (3.5) yields, taking account of the properties of the cut off functions

Lemma 3.2. *One has the inequality*

$$\begin{aligned}k_0 = \int_{\tilde Q_R(z_0)} |Du|^2 G_\theta \le c \int_{\tilde B_{2R}(x_0)} G_\theta(x,0)\,dx\,|c_R|^2\\ + c\int_{\tilde Q_{2R}(z_0)-\tilde Q_R(z_0)}\left(\frac{|u-c_R|^2}{R^2}+|Du|^2\right)G_\theta\\ + c\int_{(t_0-4R^2)^+}^{t_0}\int_{\tilde B_{2R}-\tilde B_R}\eta_R^2\,|u-c_R|^2\,|DG_\theta|^2\,G_\theta^{-1}+cR^\beta,\end{aligned} \tag{3.8}$$

if $t_0 < 4R^2$.

Proof : One just notice that in the one before the last integrals at the right hand side of (3.5) the integrand can be majorized by

$$c\left(\frac{|u-c_R|^2}{R^2}G_\theta+\eta_R^2\,|u-c_R|^2\,|DG_\theta|^2\,G_\theta^{-1}\right).$$

Moreover from (3.2)

$$\int fG_\theta\eta_R^2 \le \int_{\tilde{Q}_{2R}} fG_\theta \le c\left(\int_{Q_{2R}} G_\theta^{p'}\right)^{1/p'}$$

$$\le \left(\int_{(t_0-4R^2)^+}^{t_0} \frac{dt}{(t_0+\theta-t)^{n/2(p'-1)}}\right)^{1/p'} \le cR^{2-(n+2)/p}$$

and thus the value of β in (3.7) is

$$\beta = 2 - \frac{n+2}{p} > 0. \tag{3.9}$$

■

3.3. Auxiliary result.

Following Struwe [**10**], we need to estimate the quantity

$$Z = \int_{(t_0-4R^2)^+}^{t_0} \int_{\tilde{B}_{2R}-\tilde{B}_R} \eta_R^2 |u-c_R|^2 |DG_\theta|^2 G_\theta^{-3/2} \tag{3.10}$$

(notice $G_\theta^{-3/2}$ and not G_θ^{-1}; we recover this later).

Let ρ be a new cut off function such that

$$\begin{aligned} &\rho \in C_0^\infty(R^n) \quad , \quad \rho = 0 \text{ for } |x| \le 1/2 \\ &0 \le \rho \le \tau \quad , \quad \rho = \tau \text{ for } |x| \ge 1. \end{aligned} \tag{3.11}$$

Set $\rho_R(x) = \rho\left(\frac{x-x_0}{R}\right)$ and

$$\varphi_R(x,t) = \rho_R(x)\beta_R(t). \tag{3.12}$$

Note in particular that $\varphi_R = \eta_R$ on $(\tilde{B}_{2R} - \tilde{B}_R)) \times \left[(t_0-4R^2)^+, t_0\right]$.

We test (3.1) with $G_\theta^{-1/2}|u-c_R|^2\varphi_R^2$, and obtain

$$\begin{aligned} &\int_{(t_0-4R^2)^+}^{t_0} \int_{\tilde{B}_{2R}-\tilde{B}_{R/2}} a_{ij}\frac{\partial G_\theta}{\partial x_j}\frac{\partial G_\theta}{\partial x_i} G_\theta^{-3/2}|u-c_R|^2\varphi_R^2 \\ &=4 \int_{(t_0-4R^2)^+}^{t_0} \int_{\tilde{B}_{2R}-\tilde{B}_{R/2}} \left[a_{ij}\frac{\partial}{\partial x_j}\left(|u-c_R|^2\varphi_R^2\right)\frac{\partial}{\partial x_i}G_\theta^{1/2} \right. \\ &\left. -\left(\frac{\partial}{\partial t}G_\theta^{1/2}\right)|u-c_R|^2\varphi_R^2\right]. \end{aligned} \tag{3.13}$$

On the other hand, testing the equations (1.5) by $(u^\ell - c_R^\ell)\varphi_R^2 G_\theta^{1/2}$, $\ell = 1,2$ and adding up, we have, noting again that from the choice (3.3) this function vanishes on Σ,

$$
\begin{aligned}
\cdot \int_{(t_0-4R^2)^+}^{t_0} \int_{\tilde B_{2R}-\tilde B_{R/2}} \Bigg[& \frac{1}{2}\frac{\partial}{\partial t}\left|u-c_R\right|^2 \varphi_R^2 G_\theta^{1/2} \\
& + \frac{1}{2} a_{ij}\frac{\partial}{\partial x_j}\left|u-c_R\right|^2 \frac{\partial}{\partial x_i}(\varphi_R^2 G_\theta^{1/2}) \\
& + a_{ij}\left(\frac{\partial u^1}{\partial x_j}\frac{\partial u^1}{\partial x_i} + \frac{\partial u^2}{\partial x_j}\frac{\partial u^2}{\partial x_i}\right)\varphi_R^2 G_\theta^{1/2} \\
& -(H^1(u^1-c^1)+H^2(u^2-c^2))\varphi_R^2 G_\theta^{1/2}\Bigg] = 0.
\end{aligned} \tag{3.14}
$$

Combining (3.13), (3.14) yields

$$
\begin{aligned}
& \int_{(t_0-4R^2)^+}^{t_0} \int_{\tilde B_{2R}-\tilde B_{R/2}} a_{ij}\frac{\partial G_\theta}{\partial x_j}\frac{\partial G_\theta}{\partial x_i} G_\theta^{-3/2}\left|u-c_R\right|^2 \varphi_R^2 = \\
& 4\int\int \left|u-c_R\right|^2 a_{ij}\frac{\partial}{\partial x_j}\varphi_R^2 \frac{\partial}{\partial x_i} G_\theta^{1/2} + 4\int\int G_\theta^{1/2}\Bigg[\frac{\partial \varphi_R^2}{\partial t}\left|u-c_R\right|^2 \\
& -a_{ij}\frac{\partial}{\partial x_j}\left|u-c_R\right|^2 \frac{\partial}{\partial x_i}\varphi_R^2\Bigg] \\
& -8\int\int \varphi_R^2 G_\theta^{1/2}\left(a_{ij}\frac{\partial u^\ell}{\partial x_j}\frac{\partial u^\ell}{\partial x_i} - H^\ell(u^\ell-c^\ell)\right) \\
& -4\int_\Omega \left|u-c_R\right|^2 \varphi_R^2 G_\theta^{1/2}(x,t_0) + 4\left|c_R\right|^2 \int_\Omega \rho_R^2 G_\theta^{1/2}(x,0),
\end{aligned} \tag{3.15}
$$

if $t_0 < 4R^2$.

Using Young's inequality to majorize

$$
\left|DG_\theta^{1/2} D\varphi_R^2\right|^2 \le \delta\left|DG_\theta\right|^2 G_\theta^{-3/2}\varphi_R^2 + \frac{1}{\delta}\left|D\varphi_R\right|^2 G_\theta^{1/2}
$$

and eating the first term by the left hand side of (3.15), we obtain, after noticing that Z is smaller than the left hand size of (3.15)

$$
\begin{aligned}
Z \le c & \int_{(t_0-4R^2)^+}^{t_0} \int_{\tilde B_{2R}-\tilde B_{R/2}} \left(\frac{\left|u-c_R\right|^2}{R^2} + \left|Du\right|^2\right) G_\theta^{1/2} \\
& + c\int_{\tilde B_{2R}-\tilde B_{R/2}} G_\theta^{1/2}(x,0)\,dx\,\left|c_R\right|^2 + cR^{\frac{n}{2}+\beta},
\end{aligned} \tag{3.16}
$$

if $t_0 < 4R^2$.

3.4. Use of the estimate (3.15).

One of the difficulties in applying (3.7) stems from the fact for θ small and t close to t_0 the behaviour of the Green function is not comparable to a negative power of R, R^{-n} like in the elliptic case.

To handle this difficulty, one introduces as in Struwe [10] a splitting of the time integral in the last integral at the right hand side of (3.7).

We write

$$\int_{(t_0-4R^2)^+}^{t_0} = \int_{(t_0-4R^2)^+}^{t_0\wedge(t_0+\theta-\varepsilon^2R^2)^+} + \int_{t_0\wedge(t_0+\theta-\varepsilon^2R^2)^+}^{t_0}$$

where ε is small (and to begin $\varepsilon < 2$).

We consider the two terms

$$I_\varepsilon = \int_{\tilde{B}_{2R}-\tilde{B}_R} \int_{(t_0-4R^2)^+}^{t_0\wedge(t_0+\theta-\varepsilon^2R^2)^+} \eta_R^2 \left|u-c_R\right|^2 \left|DG_\theta\right|^2 G_\theta^{-1}$$

$$II_\varepsilon = \int_{\tilde{B}_{2R}-\tilde{B}_R} \int_{t_0\wedge(t_0+\theta-\varepsilon^2R^2)^+}^{t_0} \eta_R^2 \left|u-c_R\right|^2 \left|DG_\theta\right|^2 G_\theta^{-1}.$$

We begin with II_ε.

Now let us check that

(3.17)
$$R \le |x-x_0| \le 2R \text{ and } (t_0+\theta-\varepsilon^2R^2)^+ \le t \le t_0$$
$$\text{implies } G_\theta(x,t) \le \delta(\varepsilon)R^{-n}$$
where $\delta(\varepsilon)$ does not depend on R, nor θ and tends to 0 as $\varepsilon \to 0$.

Indeed from (3.2)

$$G_\theta(x,t) \le k_2(t_0+\theta-t)^{-n/2} \exp\frac{-\delta_2 R^2}{t_0+\theta-t}.$$

Now the function $s^{-n/2}\exp -\beta/s$ attains its maximum for $s > 0$, at $\hat{s} = 2\beta/n$, hence in our case with $\beta = \delta_2 R^2$, at $2\delta_2\frac{R^2}{n}$. Since $t_0+\theta-t < \varepsilon^2R^2$, for $\varepsilon^2 < \frac{2\delta_2}{n}$, the function is on its increasing side and thus is majorized by the value taken at $s = \varepsilon^2R^2$, therefore

$$G_\theta(x,t) \le k_2\varepsilon^{-n}R^{-n}\exp -\frac{\delta_2}{\varepsilon^2} = R^{-n}\delta(\varepsilon).$$

Therefore we can estimate

$$II_\varepsilon \le \delta^{1/2}(\varepsilon)R^{-\frac{n}{2}} \int_{\tilde{B}_{2R}-\tilde{B}_R} \int_{t_0\wedge(t_0+\theta-\varepsilon^2R^2)^+}^{t_0} \eta_R^2 \left|u-c_R\right|^2 \left|DG_\theta\right|^2 G_\theta^{-3/2}$$

$$II_\varepsilon \le R^{-\frac{n}{2}}\delta^{1/2}(\varepsilon)Z$$

and from (3.15) we deduce

$$
\begin{aligned}
II_\varepsilon \leq & c \int_{(t_0-4R^2)^+}^{t_0} \int_{\tilde{B}_{2R}-\tilde{B}_{R/2}} \left(\frac{|u-c_R|^2}{R^2} + |Du|^2 \right) G_\theta \\
& + cR^{-n}\delta(\varepsilon) \int_{(t_0-4R^2)^+}^{t_0} \int_{\tilde{B}_{2R}-\tilde{B}_{R/2}} \left(\frac{|u-c_R|^2}{R^2} + |Du|^2 \right) \\
& + c \left(\int_{\tilde{B}_{2R}} G_\theta(x,0)\, dx + \delta(\varepsilon) \right) |c_R|^2 + cR^\beta, \\
& \text{if } t_0 < 4R^2.
\end{aligned} \tag{3.18}
$$

Consider now I_ε.

Then we use the following property

$$
\begin{aligned}
& \frac{R}{2} \leq |x-x_0| \leq 2R \text{ and } (t_0-4R^2)^+ \leq t \leq t_0+\theta-\varepsilon^2 R^2 \\
& \text{implies} \\
& k_1(4R^2+\theta)^{-\frac{n}{2}} \exp -\frac{\delta_1}{\varepsilon^2} \leq G_\theta(x,t) \leq cR^{-n}.
\end{aligned} \tag{3.19}
$$

Therefore using the right hand side estimate we have

$$
I_\varepsilon \leq cR^{-\frac{n}{2}} \int_{\tilde{B}_{2R}-\tilde{B}_R} \int_{(t_0-4R^2)^+}^{t_0\wedge(t_0+\theta-\varepsilon^2R^2)^+} \eta_R^2 |u-c_R|^2 |DG_\theta|^2 G_\theta^{-3/2}.
$$

This integral is similar to Z except for the upper level of integration in t. Checking the steps leading to the estimate (3.15), one can see that one can state analogously

$$
\begin{aligned}
& \int_{\tilde{B}_{2R}-\tilde{B}_R} \int_{(t_0-4R^2)^+}^{t_0\wedge(t_0+\theta-\varepsilon^2R^2)^+} \eta_R^2 |u-c_R|^2 |DG_\theta|^2 G_\theta^{-3/2} \\
& \leq \int_{\tilde{B}_{2R}-\tilde{B}_R} \int_{(t_0-4R^2)^+}^{t_0\wedge(t_0+\theta-\varepsilon^2R^2)^+} \left(\frac{|u-c_R|^2}{R^2} + |Du|^2 \right) G_\theta^{1/2} \\
& + c \int_{\tilde{B}_{2R}-\tilde{B}_{R/2}} G_\theta^{1/2}(x,0)\, dx\, |c_R|^2 + cR^{\frac{n}{2}+\beta}, \\
& \text{if } t_0 < 4R^2.
\end{aligned} \tag{3.20}
$$

Therefore

$$
\begin{aligned}
I_\varepsilon \leq & cR^{-\frac{n}{2}} \int\limits_{\tilde{B}_{2R}-\tilde{B}_{R/2}} \int\limits_{(t_0-4R^2)^+}^{t_0\wedge(t_0+\theta-\varepsilon^2R^2)^+} \left(\frac{|u-c_R|^2}{R^2}+|Du|^2\right) G_\theta^{1/2} \\
& + cR^{-n/2} \int\limits_{\tilde{B}_{2R}-\tilde{B}_{R/2}} G_\theta^{1/2}(x,0)\,dx\,|c_R|^2 + cR^\beta, \\
& \text{if } t_0 < 4R^2.
\end{aligned}
$$

In order to use the left estimate in (3.19), we shall from now on consider only θ such that

$$\theta < m^2R^2 \quad , \quad m > 1 \text{ arbitrary fixed.} \tag{3.21}$$

Then from (3.18) we have

$$G_\theta(x,t) \geq k_1(4+m^2)^{-n/2}R^{-n}\exp -\frac{\delta_1}{\varepsilon}$$

and thus we can estimate I_ε as follows

$$
\begin{aligned}
I_\varepsilon \leq & K(\varepsilon) \int\limits_{\tilde{B}_{2R}-\tilde{B}_{R/2}} \int\limits_{(t_0-4R^2)^+}^{t_0\wedge(t_0+\theta-\varepsilon^2R^2)^+} \left(\frac{|u-c_R|^2}{R^2}+|Du|^2\right) G_\theta \\
& + K(\varepsilon) \int\limits_{\tilde{B}_{2R}} G_\theta^{1/2}(x,0)\,dx\,|c_R|^2 + cR^\beta, \\
& \text{if } t_0 < 4R^2.
\end{aligned} \tag{3.22}
$$

where $K(\varepsilon)$ is a constant, independent of R, and θ such that (3.21) holds, and which tends to $+\infty$ as $\varepsilon \to 0$.

Collecting these results in (3.8) we can state the following

Lemma 3.3. *For all θ such that* (3.20) *holds, one has the estimate*

(3.23)
$$k_0 \int\limits_{\tilde{Q}_R(z_0)} |Du|^2 G_\theta \le \left(K(\varepsilon) \int\limits_{\tilde{B}_{2R}} G_\theta^{1/2}(x,0)\,dx + \delta(\varepsilon) \right) |c_R|^2$$
$$+ \int\limits_{\tilde{Q}_{2R}-\tilde{Q}_{R/2}} |Du|^2 \left(K(\varepsilon) G_\theta + \delta(\varepsilon) R^{-n}\right)$$
$$+ c \int\limits_{\tilde{Q}_{2R}-\tilde{Q}_{R/2}} \frac{|u-c_R|^2}{R^2} \left(G_\theta + \delta(\varepsilon) R^{-n}\right)$$
$$+ K(\varepsilon) \int\limits_{\tilde{B}_{2R}-\tilde{B}_{R/2}} \int\limits_{(t_0-4R^2)^+}^{t_0 \wedge (t_0+\theta-\varepsilon^2 R^2)^+} \frac{|u-c_R|^2}{R^2} G_\theta + cR^\beta,$$

if $t_0 < 4R^2$.

3.5. Choice of the constant c_R.

We begin with the simplest case

(a)
$$B_{2R}(x_0) \cap (R^n - \Omega) \neq \emptyset.$$

In this case $c_R = 0$. Hence we deduce from (3.22)

(3.24)
$$k_0 \int\limits_{\tilde{Q}_R(z_0)} |Du|^2 G_\theta \le \int\limits_{\tilde{Q}_{2R}-\tilde{Q}_{R/2}} |Du|^2 \left(K(\varepsilon) G_\theta + \delta(\varepsilon) R^{-n}\right)$$
$$+ c \int\limits_{\tilde{Q}_{2R}-\tilde{Q}_{R/2}} \frac{|u|^2}{R^2} \left(G_\theta + \delta(\varepsilon) R^{-n}\right)$$
$$+ K(\varepsilon) \int\limits_{\tilde{B}_{2R}-\tilde{B}_{R/2}} \int\limits_{(t_0-4R^2)^+}^{t_0 \wedge (t_0+\theta-\varepsilon^2 R^2)^+} \frac{|u|^2}{R^2} G_\theta + cR^\beta.$$

We have from (3.19)

(3.25)
$$\int\limits_{\tilde{B}_{2R}-\tilde{B}_{R/2}} \int\limits_{(t_0-4R^2)^+}^{t_0 \wedge (t_0+\theta-\varepsilon^2 R^2)^+} \frac{|u|^2}{R^2} G_\theta$$
$$\le cR^{-n} \int\limits_{\tilde{B}_{2R}-\tilde{B}_{R/2}} \int\limits_{(t_0-4R^2)^+}^{t_0 \wedge (t_0+\theta-\varepsilon^2 R^2)^+} \frac{|u|^2}{R^2}$$

and by Poincaré's inequality

$$\leq cR^{-n} \int\limits_{\tilde{B}_{2R}-\tilde{B}_{R/2}} \int\limits_{(t_0-4R^2)^+}^{t_0\wedge(t_0+\theta-\varepsilon^2R^2)^+} |Du|^2$$

and by (3.19) and (3.21)

$$\begin{aligned} &\leq cK(\varepsilon) \int\limits_{\tilde{B}_{2R}-\tilde{B}_{R/2}} \int\limits_{(t_0-4R^2)^+}^{t_0} |Du|^2\, G_\theta \\ &\leq cK(\varepsilon) \int\limits_{\tilde{Q}_{2R}-\tilde{Q}_{R/2}} |Du|^2\, G_\theta. \end{aligned}$$

Next one has

$$\begin{aligned} \int\limits_{\tilde{Q}_{2R}-\tilde{Q}_{R/2}} \frac{|u|^2}{R^2} G_\theta &= \int\limits_{\tilde{B}_{2R}-\tilde{B}_{R/2}} \int\limits_{(t_0-4R^2)^+}^{t_0} \frac{|u|^2}{R^2} G_\theta \\ &+ \int\limits_{\tilde{B}_{R/2}} \int\limits_{(t_0-4R^2)^+}^{(t_0-\frac{R^2}{4})^+} \frac{|u|^2}{R^2} G_\theta. \end{aligned} \tag{3.26}$$

In the second integral we have

$$|x - x_0| < R/2 \text{ and } t_0 - 4R^2 < t < t_0 - \frac{R^2}{4}$$

hence

$$cR^{-n} \leq G_\theta(x,t) \leq cR^{-n}$$

therefore using Poincaré we majorize it by

$$\int\limits_{\tilde{B}_{4R}} \int\limits_{(t_0-4R^2)^+}^{t_0-\frac{R^2}{4}} |Du|^2\, G_\theta.$$

In the first integral we split the interval of time into $\left[(t_0-4R^2)^+,\ t_0\wedge(t_0+\theta-\varepsilon^2R^2)^+\right]$ and $\left[t_0\wedge(t_0+\theta-\varepsilon^2R^2)^+, t_0\right]$.

In the first interval we recover (3.25). In the second interval we majorize G_θ by $\delta(\varepsilon)R^{-n}$ according to (3.17), and use Poincaré again.

Collecting results, we have proven the estimate

$$
(3.27)\qquad
\begin{aligned}
k_0 \int_{Q_R(z_0)} |Du|^2 G_\theta &\le K_1(\varepsilon) \int_{\tilde{Q}_{4R}-\tilde{Q}_{R/2}} |Du|^2 G_\theta \\
&+ \delta_1(\varepsilon) R^{-n} \int_{\tilde{Q}_{4R}-\tilde{Q}_{R/2}} |Du|^2 + cR^\beta,
\end{aligned}
$$

with $K_1(\varepsilon) \to \infty$ as $\varepsilon \to 0$, $\delta_1(\varepsilon) \to 0$ as $\varepsilon \to 0$.

(b) $$B_{2R}(x_0) \subset \Omega$$

Note that $B_{2R} = \tilde{B}_{2R}$, $Q_{2R} = \tilde{Q}_{2R}$. We omit the tilde.

We use the notation

$$
u^\rho_{R,x_0,t} = \frac{\int_{B_{2R}} u(x,t)\rho_R(x)\,dx}{\int_{B_{2R}} \rho_R(x)\,dx} = \frac{\int_\Omega u(x,t)\rho_R(x)\,dx}{\int_{B_{2R}} \rho_R(x)\,dx}
$$

where ρ_R has been already defined in (3.11), and

$$
u^\rho_{R,x_0} = u^\rho_{R,x_0,t_0}.
$$

We choose

$$
c_R = u^\varepsilon_{R,z_0} = \frac{\displaystyle\int_{(t_0-4R^2)^+}^{t_0\wedge(t_0+\theta-\varepsilon^2R^2)^+} u^\rho_{R,x_0,t}\,dt}{\displaystyle\int_{(t_0-4R^2)^+}^{t_0\wedge(t_0+\theta-\varepsilon^2R^2)^+} dt}.
$$

This quantity is well defined when $t_0 + \theta - \varepsilon^2 R^2 > 0$. If it is not the case ,then we take $c_R = 0$.

We use the following result

Lemma 3.4. *One has the estimate*

$$
(3.28)\qquad \left| u^\rho_{R,x_0,t} - u^\rho_{R,x_0,s} \right|^2 \le cR^{-n} \int_{t\wedge s}^{t\vee s} \int_{B_{2R}-B_{R/2}} |Du|^2 + cR^{2\beta}.
$$

Proof : We test (1.5) with ρ_R and integrate over x and the interval s,t (assuming to fix the ideas that $s < t$). We deduce easily

$$
\begin{aligned}
&\left| u^\rho_{R,x_0,t} - u^\rho_{R,x_0,s} \right|^2 \\
&\qquad \le cR^{-n} \left| u^\rho_{R,x_0,t} - u^\rho_{R,x_0,s} \right| \int_s^t \int_{B_{2R}-B_{R/2}} \left(|Du|\,|D\rho_R| + |Du|^2 + f \right)
\end{aligned}
$$

and using conveniently Young's inequality the result (3.28) follows. ■

If $t_s + \theta - \varepsilon^2 R^2 \leq 0$, then applying (3.28) with $s = 0$, yields

$$\left|u^\rho_{R,x_0,t}\right|^2 \leq cR^{-n} \int_0^{t_0} \int_{B_{2R}-B_{R/2}} |Du|^2 + cR^{2\beta}. \tag{3.29}$$

Now in (3.22) the last integral vanishes. Next

$$\int_{\tilde{Q}_{2R}-\tilde{Q}_{R/2}} \frac{|u|^2}{R^2}(G_\theta + \delta(\varepsilon)R^{-n}) \leq 2\delta(\varepsilon)R^{-n} \int_0^{t_0} \int_{B_{2R}-B_{R/2}} \frac{|u|^2}{R^2} \tag{3.30}$$

and

$$\int_0^{t_0} \int_{B_{2R}-B_{R/2}} \frac{|u|^2}{R^2} \leq 2 \int_0^{t_0} \int_{B_{2R}-B_{R/2}} \frac{\left|u - u^\rho_{R,x_0,t}\right|^2}{R^2} + 2R^{n-2} \int_0^{t_0} \left|u^\rho_{R,x_0,t}\right|^2$$

and by (3.29) and Poincaré

$$\leq c \int_0^{t_0} \int_{B_{2R}-B_{R/2}} |Du|^2 + cR^{n+2\beta}$$

and the right hand side of (3.29) is majorized by

$$R^{-n}\delta(\varepsilon) \int_0^{t_0} \int_{B_{2R}-B_{R/2}} |Du|^2 + cR^{2\beta}$$

and thus (3.27) holds again.

We may therefore assume now that $t_0 + \theta - \varepsilon^2 R^2 > 0$. We first estimate

$$\int_{B_{2R}-B_{R/2}} \int_{(t_0-4R^2)^+}^{t_0\wedge(t_0+\theta-\varepsilon^2R^2)^+} \frac{\left|u - u^\varepsilon_{R,z_0}\right|^2}{R^2} G_\theta$$

$$\leq cR^{-n} \int_{B_{2R}-B_{R/2}} \int_{(t_0-4R^2)^+}^{t_0\wedge(t_0+\theta-\varepsilon^2R^2)^+} \frac{\left|u - u^\varepsilon_{R,z_0}\right|^2}{R^2}$$

$$\leq cR^{-n} \int_{B_{2R}-B_{R/2}} \int_{(t_0-4R^2)^+}^{t_0\wedge(t_0+\theta-\varepsilon^2R^2)^+} \left(\frac{\left|u - u^\rho_{R,x_0,t}\right|^2}{R^2} + \frac{\left|u^\rho_{R,x_0,t} - u^\varepsilon_{R,z_0}\right|^2}{R^2} \right)$$

Using Poincaré and (3.28) yields

$$\leq cR^{-n} \int_{B_{2R}-B_{R/2}} \int_{(t_0-4R^2)^+}^{t_0 \wedge (t_0+\theta-\varepsilon^2 R^2)^+} |Du|^\varepsilon + cR^{2\beta}.$$

Using again (3.19), (3.21) we get

$$\int_{B_{2R}-B_{R/2}} \int_{(t_0-4R^2)^+}^{t_0 \wedge (t_0+\theta-\varepsilon^2 R^2)^+} \frac{\left|u - u^\varepsilon_{R,z_0}\right|^2}{R^2} G_\theta$$

$$\leq cK(\varepsilon) \int_{B_{2R}-B_{R/2}} \int_{(t_0-4R^2)^+}^{t_0 \wedge (t_0+\theta-\varepsilon^2 R^2)^+} |Du|^2 G_\theta + cR^{2\beta}.$$

We proceed then as in the case a) to treat the integral

$$\int_{Q_{2R}-Q_{R/2}} \frac{|u - c_R|^2}{R^2} (G_\theta + \delta(\varepsilon) R^{-n})$$

using Lemma 3.4 adequately.

It remains to evaluate the first term at the right hand side of (3.23). This amounts first to evaluate $|u^\varepsilon_R|^2$ itself. Note that we have to perform this computation only when $t_0 < 4R^2$. We can use (3.28) with $s = 0$, to check easily that

$$\left|u^\varepsilon_{R,z_0}\right|^2 \leq c \left(R^{-n} \int_{B_{2R}-B_{R/2}} \int_{(t_0-4R^2)^+}^{t_0 \wedge (t_0+\theta-\varepsilon^2 R^2)^+} |Du|^2 + R^{2\beta} \right).$$

Since we may assume $\theta + t_0 \geq \varepsilon^2 R^2$, we can use again (3.19) to assert that the first term at the right hand side of (3.23) brings contributions of the same type as before.

Therefore we have proven

Lemma 3.5. *For all $R < R_1$ and $\theta < m^2 R^2$ (m fixed), the inequality* (3.27) *holds.*

3.6. End of the proof.

We derive from (3.27)

$$\int_{\tilde{Q}_{R/2}} |Du|^2 G_\theta \leq K(\varepsilon) \int_{\tilde{Q}_{4R}-\tilde{Q}_{R/2}} |Du|^2 G_\theta + \delta(\varepsilon) \int_{\tilde{Q}_{4R}-\tilde{Q}_{R/2}} |Du|^2 G_{R^2} + cR^\beta. \tag{3.31}$$

By the hole filling trick, we deduce

$$\sup_{\theta<m^2R^2} \int_{\tilde{Q}_{R/2}} |Du|^2 G_\theta \leq \nu(\varepsilon) \sup_{\theta<64m^2R^2} \int_{\tilde{Q}_{4R}} |Du|^2 G_\theta + cR^\beta$$

with $\nu(\varepsilon) < 1$, for ε conveniently chosen.

Multiply by $R^{-2\alpha}$, $\alpha < \beta$ and $8^{2\alpha}\nu(\varepsilon) < 1$, we deduce

$$R^{-2\alpha} \sup_{\theta<m^2R^2} \int_{\tilde{Q}_{R/2}} |Du|^2 G_\theta \leq \mu(8R)^{-2\alpha} \sup_{\theta<64m^2R^2} \int_{\tilde{Q}_{4R}} |Du|^2 G_\theta + c$$

with $\mu < 1$.

One deduces from this estimate that

$$\sup_{0<R<R_1} R^{-2\alpha} \sup_{\theta<4m^2R^2} \int_{\tilde{Q}_R} |Du|^2 G_\theta \leq c$$

and in particular

$$R^{-2\alpha} \int_{\tilde{Q}_R} |Du|^2 G_{R^2} \leq c$$

hence also

$$R^{-(n+2\alpha)} \int_{\tilde{Q}_R} |Du|^2 \leq c. \tag{3.32}$$

But using the following estimate which can be proved from (3.27) (cf. Struwe [**10**])

$$\int_{\tilde{Q}_R(z_0)} |u - u_{R,z_0}|^2 \leq cR^2 \int_{\tilde{Q}_{4R}(z_0)} |Du|^2 + cR^{n+2+2\beta}$$

one obtains

$$\int_{\tilde{Q}_R(z_0)} |u - u_{R,z_0}|^2 \leq cR^{n+2+2\gamma}$$

for a convenient $\gamma > 0$.

Therefore (1.9) is proved. The proof of theorem 1.1 has been completed.

4. $H^{2,p}$–REGULARITY

For the step from C^α–regularity to $H^{2,p}$–regularity one can use the following theorem concerning solutions of systems of parabolic inequalities

$$\left|u_t^j + Au^j\right| \leq K |\nabla u|^2 + K \quad , \quad j = 1, \ldots, N \tag{4.1}$$

with the initial regularity

$$u^j \in C^\alpha_{loc}(Q) \cap L^2_{loc}(0, T, H^1_{loc}(\Omega)).$$

For simplicity we assume that A is a second order uniformly elliptic linear operator in the space variables with C^1–coefficients.

Theorem 4.1. *Under the above assumptions any local distributional solution of* (4.1) *is contained in* $H^{1,p}_{loc}(Q) \cap L^p_{loc}(0,T,H^{2,p}_{loc}(\Omega))$, *for all* $p < \infty$.

This theorem is considered to be "known" by several authors although we have problems to give an explicit reference. An elegant way to prove is analogue to the elliptic case [**5**].

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