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Weak subalgebra lattices of monounary partial algebras*

WIKTOR BARTOL

Abstract. Necessary and sufficient conditions are found for a monounary partial algebra to be uniquely determined (up to isomorphism) by its weak subalgebra lattice.

Keywords: Weak subalgebra lattice, monounary algebra, functional graph

Classification: 08A55, 08A60, 05C75

Lattices of weak subalgebras of partial algebras have been characterized in [**Ba**]. It has been shown that though such lattices carry little information on the original algebra in the general case, they still give some insight into the structure of unary partial algebras. In this paper we characterize all those monounary partial algebras which are uniquely determined by their weak subalgebra lattice.

For notions and results concerning partial algebras see [**Bur**].

Recall that a partial algebra $\mathfrak{B} = (B, (f^{\mathfrak{B}})_{f\in F})$ of type (F, n) (where n is the arity function on the set F of operation symbols) is a *weak subalgebra* of a similar partial algebra $\mathfrak{A} = (A, (f^{\mathfrak{A}})_{f\in F})$ iff $B \subseteq A$ and $f^{\mathfrak{B}} \subseteq f^{\mathfrak{A}}$ for each operation symbol $f \in F$. The set S_w of all weak subalgebras of $\mathfrak{A}$ is an algebraic lattice under (weak subalgebra) inclusion and this lattice will be denoted by $\mathfrak{S}_w(\mathfrak{A})$ (for a characterization see [**Ba**]). When $\mathfrak{A}$ is unary (i.e. all operations are unary), we can associate with it an undirected graph $\mathbf{G}'(\mathfrak{A})$ defined as follows: consider first a directed graph $\mathbf{G}(\mathfrak{A}) = (V_{\mathfrak{A}}, E_{\mathfrak{A}})$ with

$$V_{\mathfrak{A}} = A \text{ and } E_{\mathfrak{A}} = \cup\{E_{ab} : (a,b) \in A^2\},$$

where $E_{ab} \subseteq \{a\} \times F \times \{b\}$ and $(a, f, b) \in E_{ab}$ iff $(a,b) \in f^{\mathfrak{A}}$ (E_{ab} is the set of edges directed from a to b); then $\mathbf{G}'(\mathfrak{A})$ is the corresponding undirected graph obtained from $\mathbf{G}(\mathfrak{A})$ by omitting the orientations of edges (but not edges). In [**Ba**] we prove the following

Theorem 1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be partial algebras of arbitrary unary types. Then*

$$\mathfrak{S}_w(\mathfrak{A}) \cong \mathfrak{S}_w(\mathfrak{B}) \textit{ iff } \mathbf{G}'(\mathfrak{A}) \cong \mathbf{G}'(\mathfrak{B})$$

Let $\mathcal{K}$ be a class of partial algebras. We shall say that a partial algebra $\mathfrak{A} \in \mathcal{K}$ is *uniquely determined in $\mathcal{K}$ by its weak subalgebra lattice* iff for any $\mathfrak{B} \in \mathcal{K}$, $\mathfrak{S}_w(\mathfrak{A}) \cong \mathfrak{S}_w(\mathfrak{B})$ implies $\mathfrak{A} \cong \mathfrak{B}$. When $\mathcal{K}$ is a class of unary partial algebras, then the above theorem reduces the problem of unique determination to that of a unique reconstruction of an algebra from its undirected graph. In particular, take $\mathcal{K}$ as the class of all monounary algebras. Then

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Corollary. *For any $\mathfrak{A}$ in $\mathcal{K}$, $\mathfrak{A}$ is uniquely determined in $\mathcal{K}$ by its weak subalgebra lattice iff any two directed functional graphs obtained from* $\mathbf{G}'(\mathfrak{A})$ *by directing its edges are isomorphic.*

PROOF : is immediate. ∎

We shall base on this observation in the proof of Theorem 4.

Now let $\mathfrak{A} = (A, f^{\mathfrak{A}})$ be a monounary partial algebra. Recall that a *connected component* of $\mathfrak{A}$ is any equivalence class of the following relation $\sim$ in A:

$$a \sim b \text{ iff } (f^{\mathfrak{A}})^n(a) = (f^{\mathfrak{A}})^m(b) \text{ for some } n, m \in N$$

(the equality here is to be understood in a strong sense, i.e. if one side is defined, then both are and then they are equal). Any monounary partial algebra is obviously a disjoint sum of its connected components, which permits to reduce many problems to the case of connected algebras only. It is also evident that the connected components of $\mathfrak{A}$ are in a natural correspondence with the connected components (maximal connected subgraphs) of both $\mathbf{G}(\mathfrak{A})$ and $\mathbf{G}'(\mathfrak{A})$.

Undirected graphs corresponding to total monounary algebras have been characterized by Ore [**Ore**]. A simple modification of his argument yields a similar characterization for the partial case.

Theorem 2 ([Ore]). *An undirected graph* $\mathbf{G}$ *is isomorphic to* $\mathbf{G}'(\mathfrak{A})$ *for some total monounary algebra $\mathfrak{A}$ iff each connected component of* $\mathbf{G}$ *either contains exactly one cycle or is a tree with an infinite path.*

Theorem 3. *An undirected graph* $\mathbf{G}$ *is isomorphic to* $\mathbf{G}'(\mathfrak{A})$ *for some partial monounary algebra $\mathfrak{A}$ iff each connected component of* $\mathbf{G}$ *contains at most one cycle.*

Consider any cycle in $\mathbf{G}'(\mathfrak{A})$ (or simply in $\mathfrak{A}$) for a monounary partial algebra$\mathfrak{A}$. It follows from Theorem 3 that for any element a in this cycle the subgraph spanned on the set $\{b \in V_{\mathfrak{A}} : b = a \textit{ or there is a path from } b \textit{ to } a\textit{, disjoint with the cycle}\}$ is a tree rooted in a, which we shall denote by T_a. Define a function t on the set of all cyclic elements of $\mathbf{G}'(\mathfrak{A})$ (or of $\mathfrak{A}$) into some set I so that for any cyclic $a, b \in A, t(a) = t(b)$ iff $T_a \cong T_b$.

A sequence $x_0, \ldots, x_n$ is a *palindrome* iff $x_i = x_{n-i}$ for all $i = 0, \ldots, n$. A cycle is *symmetric* iff for some numbering $c_0, \ldots, c_n$ of its consecutive elements the sequence $t(c_0), \ldots, t(c_n)$ or the sequence $t(c_1), \ldots, t(c_n)$ is a palindrome.

We can now state the main theorem.

Theorem 4. *Let $\mathfrak{A}$ be a partial monounary algebra. Then $\mathfrak{A}$ is uniquely determined in the class of all partial monounary algebras by its weak subalgebra lattice iff each connected component of $\mathfrak{A}$ (or, equivalently, of* $\mathbf{G}'(\mathfrak{A})$*) with more than two elements contains a symmetric cycle.*

PROOF : Let $\mathfrak{A}$ be a partial monounary algebra satisfying the condition of symmetric cycles.Following the Corollary to Theorem 1 we shall prove that each connected component of $\mathfrak{A}$ can be directed (so as to yield a directed graph of a function) in exactly one way (up to isomorphism of directed graphs).

If a component contains no cycle, then by assumption it is a tree with either two vertices and one edge connecting them or with one vertex and no edge. In the former case there are two possible ways of directing the tree and they are obviously isomorphic, the latter case is trivial.

Now let $c_0, \dots, c_n$ be consecutive elements of a cycle in $\mathbf{G}'(\mathfrak{A})$ such that $t(c_0), \dots, t(c_n)$ (or $t(c_1), \dots, t(c_n)$) is a palindrome. Observe that the edges in the cycle have to be directed either from c_i to c_{i+1} for $i = 0, \dots, n-1$ and from c_n to c_0 or from c_{i+1} to c_i for $i = 0, \dots, n-1$ and from c_0 to c_n; moreover, for any c_i the edges in T_{c_i} have to be directed towards the vertex c_i, otherwise the directed graph would not correspond to any function. Thus there are at most two possible ways of directing the component containing a cycle $c_0, \dots, c_n$. An isomorphism h between the directed (connected) graphs corresponding to these two ways is obtained by setting $h(c_i) = c_{n-i}$ for $i = 0, \dots, n$ if the sequence $t(c_0), \dots, t(c_n)$ is a palindrome or $h(c_i) = c_{n-i+1}$ for $i = 1, \dots, n$ and $h(c_0) = c_0$ if the sequence $t(c_1), \dots, t(c_n)$ is a palindrome, and then by taking arbitrary isomorphisms between (isomorphic by assumption) trees T_{c_i} and $T_{c_{n-i}}$ for $i = 0, \dots, n$ in the first case or between T_{c_i} and $T_{c_{n-i+1}}$ for $i = 1, \dots, n$ and the identity on T_{c_0} in the second case.

Thus any two directed graphs corresponding to the undirected graph $\mathbf{G}'(\mathfrak{A})$ are isomorphic, which proves the "if" part of the theorem.

Now assume that $\mathbf{G}'(\mathfrak{A})$ has a unique (up to isomorphism of graphs) functional orientation of its edges. Suppose a connected component $\mathbf{C}$ of $\mathbf{G}'(\mathfrak{A})$ contains at least three elements and no cycle. Then $\mathbf{C}$ is a tree with at least one vertex of degree not less than 2 (let a be such a vertex). If there is a vertex of degree 1 (let it be b) then transform $\mathbf{C}$ first into a tree with root a and then into a tree with root b, directing the edges towards the root in both cases. Clearly these two directed graphs are not isomorphic (the in-degrees of the roots are different), contrary to our assumption.

If there is no vertex of degree 1, then $\mathbf{C}$ must contain an infinite path. By Ore's theorem it corresponds to a total function, i.e. it can be directed so as to become the directed graph of a total function. Now take any vertex a in $\mathbf{C}$ and transform $\mathbf{C}$ into a tree rooted in a, again directing its edges towards the root. This directed graph corresponds clearly to a partial function, undefined at a. Thus we obtain two non-isomorphic directed graphs, contrary to the assumptions.

This proves that no component of $\mathbf{G}'(\mathfrak{A})$ without cycles can have more than two elements.

Suppose now $\mathbf{C}$ is a connected component of $\mathbf{G}'(\mathfrak{A})$ which contains a cycle. Let h be an isomorphism of one of the directed (functional) graphs obtained from the component $\mathbf{C}$ onto the directed (functional) graph with an opposite orientation of the cycle. Clearly this isomorphism maps the cycle onto itself.

If h has a fix-point in the cycle, call it (any of them, if there are more than one) c_0. Let the consecutive vertices of the cycle in the domain of h be $c_1, \dots, c_n$ respectively. Since c_1 follows c_0 in the first graph, c_n will follow c_0 in the other; thus $h(c_1) = c_n$. More generally, $h(c_i) = c_{n-i+1}$ for $1 \leqslant i \leqslant \frac{n}{2}$ if n is even and for $1 \leqslant i \leqslant [\frac{n}{2}]+1$ (here $[x]$ is the greatest integer not greater than x) if n is odd (observe that in the latter case also $c_{[\frac{n}{2}]+1}$ is a fix-point of h). Since h is a graph isomorphism, this implies that

$t(c_i) = t(c_{n-i+1})$ for all i as above. Thus the sequence $t(c_1), \dots, t(c_{\frac{n}{2}}), \dots, t(c_n)$ (if n is even) or the sequence $t(c_1), \dots, t(c_{[\frac{n}{2}]+1}), \dots, t(c_n)$ (if n is odd) is a palindrome which proves the symmetry of the cycle.

If h has no fix-point in the cycle, then there exist two adjacent vertices in the cycle mapped by h onto each other. Indeed, if $c_0, \dots, c_n$ is any numbering of consecutive vertices of the cycle and $h(c_0) = c_k$ for some $0 < k \leqslant n$, then also $h(c_1) = c_{k-1}$, $h(c_2) = c_{k-2}$ etc., until $h(c_{[\frac{k}{2}]}) = c_{[\frac{k}{2}]+1}$ (observe that k must be odd, otherwise $c_{[\frac{k}{2}]}$ would be a fix-point of h). Now call the second of these last two vertices (in the order determined by the direction of the edge connecting them in the domain of h) d_0 and the vertices following it $d_1, \dots, d_n$, respectively. Thus we have $h(d_n) = d_0$, which implies $h(d_{n-i}) = d_i$ for $0 \leqslant i \leqslant [\frac{n}{2}]$ (now n must be odd). Consequently, $t(d_{n-i}) = t(d_i)$ for all i as above and the sequence $t(d_0), \dots, t(d_{[\frac{n}{2}]}), t(d_{[\frac{n}{2}]+1}), \dots, t(d_n)$ is a palindrome; hence the cycle is symmetric.

This completes the proof of the "only if" part of the theorem. ■

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