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Asymptotics for robust MOSUM

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Abstract. Asymptotic distribution under the null hypothesis and some local alternatives are derived for test statistics corresponding to robust MOSUM test procedures. These procedures were proposed for testing the constancy of the regression relationship over time.

Keywords: robust MOSUM test procedures, testing the constancy of the regression relationship over time

Classification: 62F10,62E20,62G05,62J05

1. Introduction.

Consider the following linear model:

$$X_i = \mathbf{c}_i'\boldsymbol{\Theta}_i + e_i\ , \quad i = 1, \ldots, n, \tag{1.1}$$

where $\mathbf{c}_i' = (c_{i1}, \ldots, c_{ip})'$, $i = 1, \ldots, n$, are known regression vectors, $\boldsymbol{\Theta}_i$, $i = 1, \ldots, n$, are unknown parameters, $e_1, \ldots, e_n$ are i.i.d. random variables, e_i distributed according to the distribution function (d.f.) F fulfilling certain regularity conditions and unknown otherwise. The problem of testing the constancy of the regression relationships over time is formulated as:

$$H_0 : \ \boldsymbol{\Theta}_1 = \cdots = \boldsymbol{\Theta}_n = \boldsymbol{\Theta}_0 \text{ (known or unknown)} \tag{1.2}$$

against

$$\begin{aligned} &H_n(\mathbf{q}_n) : \text{there exists } 1 \le m < n \text{ such that} \\ &\qquad \boldsymbol{\Theta}_1 = \cdots = \boldsymbol{\Theta}_m = \boldsymbol{\Theta}_0;\ \boldsymbol{\Theta}_{m+1} = \cdots = \boldsymbol{\Theta}_n = \boldsymbol{\Theta}_0 + \mathbf{q}_n, \quad \mathbf{q}_n \neq 0. \end{aligned} \tag{1.3}$$

A variety of test procedures were proposed and studied for this problem (for further information see survey papers, e.g., Hackl (1980), Zacks (1983), Csörgő and Horváth (1988), Krishnaiah and Miao (1988), Hušková and Sen (1989), Hušková (1989a), Antoch and Hušková (1989)). Certain survey of recursive M–tests including CUSUM and MOSUM M–tests together with some resulsts of simulations can be found in Hušková (1989b).

Here we shall concentrate on the robust recursive test procedures called MOSUM M–tests. Classical MOSUM tests for F normal were deeply studied by Hackl (1980). They are based on the moving sums of the properly standardized recursive residuals

$$X_i - \mathbf{c}_i'\widetilde{\boldsymbol{\Theta}}_{i-1}, \quad i = p+1, \ldots, n, \tag{1.4}$$

where $\widetilde{\Theta}_{i-1}$ is the least squares estimator of Θ_0 based on $X_1,\dots,X_{i-1}$. The robust MOSUM M–tests are robust modifications of the classical MOSUM, where the least squares estimators and the recursive residuals (1.4) are replaced by the M–estimators and M–recursive residuals

$$W_i = \psi(X_i - \mathbf{c}_i'\widehat{\Theta}_{i-1}) \quad i = p+1,\dots,n, \tag{1.5}$$

where ψ is a score function from $\mathbf{R}^1$ to $\mathbf{R}^1$ (satisfying $\int \psi(x)dF(x) = 0$ and usually monotone), Θ_{i-1} an M–estimator of Θ_0 (or an estimator related to it), generated by a function ψ^* (which can generally differ from ψ) and based on $X_1,\dots,X_{i-1}$. Notice that for $\psi(x) = \psi^*(x) = x,\ x \in \mathbf{R}^1$, one obtains recursive residuals (1.4).

Typically, the critical region of the MOSUM M–tests is of the form:

$$\bigcup_{k=p+G+1}^{n} \left\{G^{-\frac{1}{2}}\Big|\sum_{j=k-G+1}^{k} W_j\Big|\sigma_k^{-1} > u(\alpha,G,n)\right\}, \tag{1.6}$$

where σ_k^2 is a d_n–consistent estimator of $\int \psi^2(x)dF(x)$, where d_n fulfills (2.2) below, and $u(\alpha,G,n)$ is chosen in such a way that the asymptotic level is $\alpha(\le \alpha)$. The test has a sequential nature: after the k–th observation ($p < k < n$) one either rejects H_0 and stops observations (if $\Big|\sum_{j=k-G+1}^{k} W_j\Big|\sigma_k^{-1} > u(\alpha,G,n)^{\frac{1}{2}}G^{\frac{1}{2}}$) or continues with observations otherwise. The decision for one of the hypotheses is made no later than after the n–th observation.

Concerning the critical value $u(\alpha,G,n)$ for the classical MOSUM test and F normal $N(0,\sigma^2)$ Hackl (1980) recommended approximations based on either the Bonferroni inequality or the Šidák one or the Hunter one leading to the conservative test. The same approximation can be used also in our case, e.g., the Bonferroni inequality leads to the critical value

$$u(\alpha,G,n) = \Phi^{-1}\left(1 - \frac{\alpha}{2(n-G-p)}\right). \tag{1.7}$$

The results of the present paper (Theorem 2.1 below) imply that the test with the critical value

$$\begin{aligned} u(\alpha,G,n) = {} & (2\log(n/G))^{\frac{1}{2}} + (\log\log(n/G) + \log(4/\pi) - \\ & -2\log\log(1-\alpha)^{-1})(8\log(n/G))^{-\frac{1}{2}} \end{aligned} \tag{1.8}$$

has asymptotic level α which gives asymptotically certain improvement even for the classical MOSUM for F normal.

Actually, if we assume that G depends on n (letting $G_n = G$) in a way that $\liminf_{n\to\infty} G_n n^{-\beta} > 0$ for some $\beta \in (0,1)$ and $\lim_{n\to\infty} G_n/n = 0$ then

$$\lim_{n\to\infty} \frac{\Phi^{-1}\left(1 - \frac{\alpha}{2(n-G_n-p)}\right)}{(2\log(2(n-G_n-p)/\alpha))^{\frac{1}{2}}} = 1 \tag{1.9}$$

and hence

$$(1.10)\qquad \liminf_{n\longrightarrow\infty} \frac{\Phi^{-1}\left(1-\frac{\alpha}{2(n-G_n-p)}\right)}{(2\log(n/G_n))^{\frac{1}{2}}} \geq \left(\frac{1}{1-\beta}\right)^{\frac{1}{2}} > 1.$$

The main aim of this paper is to study asymptotic behvior of $\max_{p+G_n<k\leq n}\left\{G_n^{-\frac{1}{2}}\left|\sum_{i=k-G_n+1}^{k} W_i\right|\right\}$ under the null hypothesis H_0 and certain local alternatives (Theorem 2.1 and Theorem 2.2). The critical value $u(\alpha, G, n)$ defined by (1.8) is then a consequence of Theorem 2.1. Towards this one has to extend the results of Deheuvels and Révész (1987) (for completness they are stated in Theorem 2.3 below). As an auxiliary result we prove that the asymptotic distribution of $\max_{p+G_n<k\leq n}\left\{G_n^{-\frac{1}{2}}\left|\sum_{i=k-G_n}^{k} W_i\right|\right\}$ does not change if we replace $\widehat{\Theta}_{i-1}$ in (1.5) by Θ_0, i.e., the estimator of Θ_0 by its true value.

The main assertions are formulated in Section 2, their proofs are contained in Sections 3.

The present paper contains only theoretical results. Some simulation results can be found in Hušková (1989b) and more extensive simulation study will be published elsewhere by Antoch.

2. Main results.

Here the following assumptions will be imposed on ψ, F, G and regression vectors $\mathbf{c}_1, \mathbf{c}_2, \ldots$:

A$_1$: ψ is bounded nondecreasing, there exist positive constans D_1, D_2 such that

$$\int (\psi(x-a)-\psi(x-b))^2\, dF(x) \leq D_2\,|a-b|^2 \text{ for } |a|\leq D_1, |b|\leq D_1.$$

A$_2$: The function $\lambda(a) = -\int \psi(x-a)dF(x)$, $a\in R^1$, fulfills: $\lambda(0)=0$, there exists the first derivate in some neighborhood of 0 continuous at $a=0$ and $\lambda'(0)>0$.
A$_2^*$: There exist constants $D_3>0$, $D_4>0$ and $r>0$ such that

$$|\lambda'(a)-\lambda'(b)| \leq D_3|a-b|^r \text{ for } |a|\leq D_4, \quad |b|\leq D_4.$$

B: The number of summands G_n fulfills:

$$G_n/n \longrightarrow 0\ , \quad G_n \log^{-3} n \longrightarrow \infty \qquad \text{as } n\to\infty.$$

C: The regression vectors $\mathbf{c}_i = (c_{i1},\ldots,c_{ip})'$, $i=1,\ldots,n$, fulfill:

$$n^{-1}\sum_{i=1}^{[nt]} \mathbf{c}_i\mathbf{c}_i' \longrightarrow t\mathbf{C} \quad \text{as } n\longrightarrow\infty \text{ for } t\in\langle 0,1\rangle,$$

$$\limsup_{n\longrightarrow\infty}\max_{1\leq i\leq n}\left\{c_{ij}^2 n^{-1}\log^3 n\right\} < +\infty,$$

$$\limsup_{n\to\infty} n^{-1}\sum_{i=1}^{n} c_{ij}^4 < +\infty,$$

$j = 1, \dots, p$, where $[a]$ denotes the integer part of a and $\mathbf{C}$ is a positive definite matrix.

Under appropriate assumptions on F typical ψ–functions fulfill assumptions A_1, A_2, A_2^*, e.g., if the d.f. F has the bounded derivate in a neighborhood of $\pm k$ then for the Huber ψ–function ($\psi(x) = \operatorname{sign} x \min(|x|, k)$, $x \in \mathbf{R}^1$) the mentioned assumptions are satisfied. The assumption of boundedness of ψ is used only to ensure reasonable behavior of $\sum_{i=k-G_n}^{k_n} W_i$, $G_n < k \le k_n$ and can be, of course, replaced by a weaker (but more complicated) assumption.

Assumption B requires the number G_n of summands W_i large, however small w.r.t. n.

Assumption C expresses standard request on $\mathbf{c}_1, \mathbf{c}_2, \dots$.

The main assertions of the present paper are formulated in Theorem 2.1 and Theorem 2.2 below:

Theorem 2.1. *Let assumptions A_1, A_2, B, C be satisfied and let $\widehat{\boldsymbol{\Theta}}_k$ be an estimator of $\boldsymbol{\Theta}_0$ based on $X_1, \dots, X_k$, $p < k \le n$, such that*

$$\max_{k_n \le k \le n} \|\mathbf{C}_k^{\frac{1}{2}}(\widehat{\boldsymbol{\Theta}}_k - \boldsymbol{\Theta}_0)\| = o_p(d_n) \quad \text{as } n \longrightarrow \infty \tag{2.1}$$

for some sequence $\{k_n d_n\}$ satisfying

$$k_n \longrightarrow \infty,\ d_n \nearrow \infty,\ k_n^2 = o(G_n),\ d_n = o((\log(n/G_n))^{\frac{1}{2}}), \tag{2.2}$$

where $\|.\|$ denotes the Euclidean norm and $\mathbf{C}_k = \sum_{i=1}^{k} \mathbf{c}_i \mathbf{c}_i'$.

Then under both the null hypothesis H_0 and the contiguous alternatives $H_n(n^{-\frac{1}{2}}\mathbf{q})$, $\mathbf{q} \ne 0$ (see (1.3))

$$\begin{aligned} P\Big(\max_{p<k\le n}\Big\{G_n^{-\frac{1}{2}}\Big|\sum_{i=k-G_n+1}^{k} W_i\Big|\sigma^{-1}\Big\} \le b(\log(n/G_n), y)\Big) &\longrightarrow \\ \longrightarrow \exp\{-\exp\{-y\}\} \quad \text{as } n \longrightarrow \infty, \quad y \in \mathbf{R}^1,& \end{aligned} \tag{2.3}$$

where W_i is defined by (1.5) and

$$\sigma^2 = \int \psi^2(x)\, dF(x), \tag{2.4}$$

$$b(h, y) = (2h)^{\frac{1}{2}} + (\log h + \log(4/\pi) + 2y)(8h)^{-\frac{1}{2}}. \tag{2.5}$$

Theorem 2.2. *Let assumptions A_1, A_2, A_2^*, B, C, be satisfied. Let $\widehat{\boldsymbol{\Theta}}_k$ be an estimator of $\boldsymbol{\Theta}_0$ based on $X_1, \dots, X_k$ fulfilling:*

$$\begin{aligned} \max_{k_n \le k \le n}\Big\{\Big\|\widehat{\boldsymbol{\Theta}}_{k+1} - \boldsymbol{\Theta}_0 - (\mathbf{I} - \mathbf{C}_{k+1}^{-1}\mathbf{C}_m)\mathbf{q}_n I\{k > m\}\Big\| \min(\|\mathbf{q}_n\|^{-1-v}, d_n^{-1})\Big\} \\ = O_p(1) \quad \text{as } n \longrightarrow \infty \end{aligned} \tag{2.6}$$

for some $v > 0$, where $\{k_n, d_n\}$ satisfies (2.2) and $\{\mathbf{q}_n\}$ does

(2.7)
$$\|\mathbf{q}_n\|^{1+v}(nG_n)^{\frac{1}{4}} + \|\mathbf{q}_n\|^{1+r}G_n^{\frac{1}{2}}(n/G_n))^{(1+r)/4} = \\ = o\left(\log\left(n/G_n\right)\right) \quad \textit{as } n \longrightarrow \infty,$$

(2.8)
$$nG_n^{-1}\|\mathbf{q}_n\|^2 \log(n/G_n) = o(1) \quad \textit{as } n \longrightarrow \infty.$$

Then under the alternative hypothesis $H_n(\mathbf{q}_n)$ the asymptotic distribution of

$$(2\log(n/G_n))^{\frac{1}{2}}\Big(\max_{p+G_n<k\le n}\Big\{G_n^{-\frac{1}{2}}\Big|\sum_{i=k-G_n+1}^{k} W_i\Big|\sigma^{-1}\Big\} - \\ -2\log(n/G_n) - \log\log(n/G_n) - \log(4/\pi)\Big)$$

is the same as that of

$$(2\log(n/G_n))^{\frac{1}{2}}\Big(\max_{p+G_n<k\le n}\Big\{G_n^{-\frac{1}{2}}\Big|\sum_{i=k-G_n+1}^{k} (\psi(e_i) + \lambda'(0)\omega(i,m))\Big|\sigma^{-1}\Big\} - \\ -2\log(n/G_n) - \log\log(n/G_n) - \log(4/\pi)\Big),$$

where $e_i = X_i - \mathbf{c}_i'(\Theta_0 + \mathbf{q}_n)$, $i = 1, \ldots, n$, and

(2.9)
$$\begin{aligned} \omega(i,m) &= 0 && i < m \\ &= \mathbf{c}_i'\mathbf{C}_{i-1}^{-1}\mathbf{C}_m\mathbf{q}_n && m \le i \le n. \end{aligned}$$

Remark. Reasonable candidates for estimators $\widehat{\Theta}_k$ are the usual M–estimators, the recursive M–estimators and the stochastic approximation type estimators all generated by a function ψ^* (which can differ from ψ). For the definition and properties of the recursive M–estimators and the stochastic approximation type estimators see Poljak and Tsypkin (1979) and Hušková (1989c) resp. If the absolute moment of a proper order is finite then also least squares estimators can be used.

Theorem 2.1 covers the behavior of $\max_{p+G_n<k\le n}\Big\{G_n^{-\frac{1}{2}}\Big|\sum_{i=k-G_n+1}^{k} W_i\Big|\sigma^{-1}\Big\}$ under the null hypothesis and the contiguous alternative and, as a consequence, it gives the critical value $u(\alpha, G, n) = b(\log n/G_n, y)$ in the critical region (1.6) with asymptotic level α.

According to Theorem 2.2 this test usually cannot distinguish the alternatives $H_{n1}(\mathbf{q}_n)$ with $\|\mathbf{q}_n\| = o\left(G_n^{-\frac{1}{2}}(n/G_n)\right)$.

It should be remarked that the conditions (2.7), (2.8) give also certain restrictions on relations between G_n and n. The assumptions imposed on G_n, q_n, k_n, Θ_k and $\mathbf{c}_i'$ in Theorem 2.2 constitutes one of possible sets of assumptions, e.g., one can weaken the assumptions on $\widehat{\Theta}_k\big(\max_{k_n\le k\le n}\|\widehat{\Theta}_k-\Theta_0\|=O(\|\mathbf{q}_n\|)\big)$ then, however, the assumption on G_n must be strenghtened.

Theorem 2.1 follows from Theorems 2.3 and 2.4 below which are of their own importance. Theorem 2.3 was proved by Deheuvels and Révész (for the moving sums of i.i.d. random variables). Theorem 2.4 says that the asymptotic distribution of $\max_{G_n+p<k\le n}\Big\{G_n^{-\frac12}\Big|\sum_{i=k-G_n+1}^{k}\psi(X_i-\mathbf{c}_i'\Theta_0)\Big|\Big\}$ does not change if we replace Θ_0 (which we usually do not know) by its estimator $\widehat{\Theta}_{i-1}$ based on $X_1,\dots,X_{i-1}$.

Theorem 2.3. *Let $Y_1,\dots,Y_n$ be i.i.d. random variables with zero mean, unit variance and finite generating function $E\ \exp\{tY_1\}$ for all $|t|\le t_0$ for some $t_0>0$ and assumption B be satisfied. Then*

$$P\Big(\max_{G_n<k\le n}\Big\{G_n^{-\frac12}\Big|\sum_{i=k-G_n+1}^{k}Y_i\Big|\Big\}\le b(\log(n/G_n),y)\Big)\longrightarrow \tag{2.10}$$

$$\longrightarrow\exp\{-\exp\{-y\}\}\quad \textit{as } n\longrightarrow\infty,\quad y\in\mathbf{R}^1,$$

where $b(h,y)$ is defined by (2.5).

PROOF : See Deheuvels and Révész (1987). ■

Theorem 2.4. *Let the assumptions of Theorem 2.1 be satisfied. Then under the null hypothesis*

$$\max_{G_n+p<k\le n_c}\Big\{G_n^{-\frac12}\Big|\sum_{i=k-G_n+1}^{k}\big(\psi(X_i-\mathbf{c}_i'\widehat{\Theta}_{i-1})-\psi(X_i-\mathbf{c}_i'\Theta_0)\big)\Big|\Big\}= \tag{2.11}$$

$$=o_p\big((\log(n/G_n))^{\frac12}\big)\quad \textit{as } n\longrightarrow\infty$$

and

$$\max_{n_c<k\le n}\Big\{G_n^{-\frac12}\Big|\sum_{i=k-G_n+1}^{k}\big(\psi(X_i-\mathbf{c}_i'\widehat{\Theta}_{i-1})-\psi(X_i-\mathbf{c}_i'\Theta_0)\big)\Big|\Big\}= \tag{2.12}$$

$$=o_p\big((\log(n/G_n))^{-\frac12}\big)\quad \textit{as } n\longrightarrow\infty,$$

where $n_c=n^cG_n^{1-c}$, $c\in(0,1)$ arbitrary.

3. Proofs of Theorems.

In this section we shall write G instead of G_n; Q_v, $v=1,2,\dots$ denote generic constants.

PROOF of Theorem 2.1: Without loss of generality one may put $\sigma^2=1$. Let us start with the null hypothesis. Applying Theorem 2.2 with $Y_i=\psi(X_i-\mathbf{c}_i'\Theta_0)$, $1\le i\le n$, one obtains

$$P\Big(\max_{p+G<k\le n}\Big\{G^{-\frac12}\Big|\sum_{i=k-G+1}^{k}\psi(X_i-\mathbf{c}_i'\Theta_0)\Big|\Big\}\le b(\log(n/G),y)\Big)\longrightarrow \tag{3.1}$$

$$\longrightarrow\exp\{-\exp\{-y\}\}\quad \text{as } n\longrightarrow\infty,\quad y\in\mathbf{R}^1.$$

Moreover, for $n_c = n^c G^{1-c}$, $c \in (0,1)$ arbitrary the following is true

$$\frac{b\left(\log(n/G), y\right)}{b\left(\log(n_c/G), y\right)} \longrightarrow \sqrt{c} \quad \text{as } n \longrightarrow \infty,\ y \in \mathbf{R}^1, \tag{3.2}$$

which together with (3.1) for $n = n_c$ yields that

$$\begin{aligned} &P\Big(\max_{p+G<k\le n_c}\Big\{G^{-\frac{1}{2}}\Big|\sum_{i=k-G+1}^{k}\psi(X_i - \mathbf{c}_i'\Theta_0)\Big|\Big\} \ge \\ &\qquad \ge (2c^*\log(n/G))^{\frac{1}{2}}\Big) = o(1) \quad \text{as } n \longrightarrow \infty \end{aligned} \tag{3.3}$$

for any $c^* \in (c, 1\rangle$.

Hence the assertion of our Theorem under H_0 follows from (3.1–3.3), (2.11) and (2.12).

As for the contiguous alternatives $H_n(n^{-\frac{1}{2}}\mathbf{q}_n)$, $\mathbf{q}_n \ne$), the relations (3.3), (2.11) and (2.12) remain true even in this situation and hence asymptotic behavior of

$$\max_{p+G<k\le n}\Big\{G^{-\frac{1}{2}}\Big|\sum_{i=k-G+1}^{k}\psi(X_i - \mathbf{c}_i'\widehat{\Theta}_{i-1})\Big|\Big\}$$

is the same as that of

$$\max_{p+G<k\le n}\Big\{G^{-\frac{1}{2}}\Big|\sum_{i=k-G+1}^{k}\psi(X_i - \mathbf{c}_i'\Theta_0)\Big|\Big\}.$$

Since

$$\begin{aligned} &\max_{G+p<k\le n}\Big\{G^{-\frac{1}{2}}\Big|\sum_{i=k-G+1}^{k}E\psi(X_i - \mathbf{c}_i'\Theta_0)\Big|\Big\} = \\ &= \max_{m\le k\le n}\Big\{G^{-\frac{1}{2}}\Big|\sum_{i=\max(k-G+1,m)}^{k}\lambda\left(-(\mathbf{c}_i'\mathbf{q}n^{-\frac{1}{2}})\right)\Big|\Big\} = \\ &= O\Big(\max_{m<k\le n}\Big\{G^{-\frac{1}{2}}\sum_{i=\max(k-G+1,m)}^{k}\|\mathbf{c}_i\|\,\|\mathbf{q}\|n^{-\frac{1}{2}}\Big\}\Big) = \\ &= O((G/n)^{\frac{1}{4}}), \end{aligned} \tag{3.4}$$

the assertion under the contiguous alternatives follows. ∎

Proof of Theorem 2.4: Define $\{\Theta_k^*\}$ as follows:

$$\begin{aligned} \Theta_k^* &= \widehat{\Theta}_k && p<k\le k_n \\ &= \widehat{\Theta}_k && k_n<k\le n,\quad \|\mathbf{C}_k^{\frac{1}{2}}(\widehat{\Theta}_k - \Theta_0)\| \le d_n \\ &= \widehat{\Theta}_k^0 && k_n<k\le n,\quad \|\mathbf{C}_k^{\frac{1}{2}}(\widehat{\Theta}_k - \Theta_0)\| > d_n, \end{aligned} \tag{3.5}$$

where $\widehat{\Theta}_k^0$ is an arbitrary point from $\{\Theta; \|C_k^{\frac{1}{2}}(\Theta - \Theta_0)\| \le d_n\}$.

Due to the assumptions one has

$$P\Big(\max_{G+p<k\le n}\{\|\Theta_k^* - \widehat{\Theta}_k\|\} \ne 0\Big) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \tag{3.6}$$

and hence

$$\begin{aligned} &P\Big(\max_{G+p<k\le n}\Big\{G^{-\frac{1}{2}}\Big|\sum_{i=k-G+1}^{k} \psi(X_i - c_i'\widehat{\Theta}_{i-1})\Big|\Big\} \ne \\ &\ne \max_{G+p<k\le n}\Big\{G^{-\frac{1}{2}}\Big|\sum_{i=k-G+1}^{k} \psi(X_i - c_i'\Theta_{i-1}^*)\Big|\Big\}\Big) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{3.7}$$

Next, since ψ is bounded it is easily seen that

$$\max_{G+p<k\le n}\Big\{G^{-\frac{1}{2}}\Big|\sum_{i=k-G+1}^{k-G+k_n} \psi(X_i - c_i'\widehat{\Theta}_{i-1})\Big|\Big\} = O(1). \tag{3.8}$$

Consequently, to prove (2.11) and (2.12) it suffices to show

$$\begin{aligned} &\max_{G+p<k\le n_c}\Big\{G^{-\frac{1}{2}}\Big|\sum_{i=k-G+k_n+1}^{k} \big(\psi(X_i - c_i'\Theta_{i-1}^*) - \psi(X_i - c_i'\Theta_0)\big)\Big|\Big\} = \\ &= o_p\big((\log(n/G))^{\frac{1}{2}}\big) \quad \text{as } n \longrightarrow \infty \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} &\max_{n_c<k\le n}\Big\{G^{-\frac{1}{2}}\Big|\sum_{i=k-G+1}^{k} \big(\psi(X_i - c_i'\Theta_{i-1}^*) - \psi(X_i - c_i'\Theta_0)\big)\Big|\Big\} = \\ &= o_p\big((\log(n/G))^{-\frac{1}{2}}\big) \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{3.10}$$

Letting

$$\begin{aligned} S_k &= \sum_{i=p+1}^{k} \big(\psi(X_i - c_i'\Theta_{i-1}^*) - \psi(X_i - c_i'\Theta_0) + \lambda(c_i'(\Theta_{i-1}^* - \Theta_0))\big), \quad p < k \le n \\ &= 0 \quad , \quad 0 \le k \le p \vee n < k \end{aligned}$$

one observes

$$\begin{aligned} &\max_{G+p<k\le n_c}\Big\{G^{-\frac{1}{2}}\Big|\sum_{i=k-G+k_n+1}^{k} \big(\psi(X_i - c_i'\Theta_{i-1}^*) - \psi(X_i - c_i'\Theta_0)\big)\Big|\Big\} \le \\ &\le \max_{G+p<k\le n_c}\{G^{-\frac{1}{2}}|S_k - S_{k-G+k_n}|\} + \\ &+ \max_{G+p<k\le n_c}\Big\{G^{-\frac{1}{2}}\Big|\sum_{i=k-G+k_n+1}^{k} \lambda(c_i'\Theta_{i-1}^*)\Big|\Big\}. \end{aligned} \tag{3.11}$$

Further,

$$\max_{0\le v\le[\frac{n_c}{G}]+1}\ \max_{(v-1)G<k\le vG}\left\{G^{-\frac{1}{2}}|S_k-S_{k-G+k_n}|\right\}\le$$
$$\le 2\max_{0\le v\le[\frac{n_c}{G}]+1}\ \max_{(v-2)\le k\le vG}\left\{G^{-\frac{1}{2}}|S_k-S_{(v-2)G+k_n}|\right\}. \tag{3.12}$$

Since $\{S_k-S_{(v-2)G+k_n}\ \ ,k=(v-2)G,\dots,vG\}$ is a martingale, the Chow inequality can be applied, which yields

$$P\Big(\max_{(v-2)G<k\le vG}\left\{G^{-\frac{1}{2}}|S_k-S_{(v-2)G+k_n}|\right\}\ge\kappa\Big)\le$$
$$\le Q_1\kappa^{-2}G^{-1}\sum_{k=(v-2)G+1}^{vG}\|\mathbf{c}_k\|^2k^{-1}d_n^2 \tag{3.13}$$

for some $Q_1>0$. Now, (3.12) and (3.13) ensure that

$$P\Big(\max_{0\le v\le[\frac{n_c}{G}]+1}\ \max_{(v-1)G<k\le vG}\left\{G^{-\frac{1}{2}}|S_k-S_{k-G+k_n}|\right\}\ge\kappa\Big)=$$
$$=O\Big(G^{-1}\kappa^{-2}\sum_{k=1}^{n_c}\|\mathbf{c}_k\|^2k^{-1}d_n^2\Big)=O\left(\kappa^{-2}G^{-1}d_n^2\log n_c\right)=o(1) \tag{3.14}$$
$$\text{as } n\longrightarrow\infty$$

for arbitrary $\kappa>0$, where we used the following simple inequality:

$$\sum_{k=1}^{n_c}\|\mathbf{c}_k\|^2k^{-1}\le Q_2\sum_{k=1}^{n_c}\|\mathbf{c}_k\|^2\Big(\sum_{i=k}^{n_c}i^{-2}+n_c^{-1}\Big)=$$
$$=O\Big(\sum_{i=1}^{n_c}i^{-1}\Big)=O(\log n_c)\quad\text{as } n\longrightarrow\infty \tag{3.15}$$

for some $Q_2>0$.

Now, regarding assumption A_2 and the definition of $\mathbf{\Theta}_{i-1}^*$ one can write:

$$\max_{G+p<k\le n_c}\Big\{G^{-\frac{1}{2}}\Big|\sum_{i=k-G+k_n}^{k}\lambda\left(\mathbf{c}_i'(\mathbf{\Theta}_{i-1}^*-\mathbf{\Theta}_0)\right)\Big|\Big\}=$$
$$=O\Big(\max_{G+p<k\le n_c}\Big\{G^{-\frac{1}{2}}\sum_{i=k-G+1}^{k}\|\mathbf{c}_i\|i^{-\frac{1}{2}}d_n\Big\}\Big)= \tag{3.16}$$
$$=O\Big(\max_{G+p<k\le n_c}\Big\{G^{-\frac{1}{2}}d_n\Big(\sum_{j=k-G+1}^{k}j^{-\frac{1}{2}}+k^{-\frac{1}{2}}\sum_{i=k-G+1}^{k}\|\mathbf{c}_i\|\Big)\Big\}\Big),$$

where we used the inequality similar to the first one in (3.15). Obviously,

$$\max_{G+p<k\le n_c}\Big\{G^{-\frac12}d_n\Big(\sum_{j=k-G+1}^{k} j^{-\frac12}+k^{-\frac12}\sum_{i=k-G+1}^{k}\|\mathbf{c}_i\|\Big)\Big\}= = O(d_n) \quad \text{as } n\longrightarrow\infty \tag{3.17}$$

and

$$\max_{2G\le k\le n_c}\Big\{G^{-\frac12}d_n\Big(\sum_{j=k-G+1}^{k} j^{-\frac12}+k^{-\frac12}\sum_{i=k-G+1}^{k}\|\mathbf{c}_i\|\Big)\Big\}= = O(d_n) \text{ as } n\longrightarrow\infty. \tag{3.18}$$

Assertion (3.9) can be easily concluded from (3.11), (3.12), (3.14) and (3.16–3.18).

Now we turn to (3.10). Using the same arguments as in treating $\max_{G+p<k\le n_c}\{G^{-\frac12}|S_k-S_{k-G}|\}$ one receives

$$\begin{aligned} &P\big(\max_{n_c<k\le n}\{G^{-\frac12}|S_k-S_{k-G}|\}\ge\kappa\big)\le \\ &\le Q_3\kappa^{-2}G^{-1}\sum_{k=n_c}^{n}\|\mathbf{c}_k\|^2d_n^2k^{-1}<Q_4\kappa^{-2}G^{-1}d_n^2\log n \end{aligned} \tag{3.19}$$

for some $Q_3>0$, $Q_4>0$.

Finally proceeding similarly as in (3.16) one arrives at

$$\begin{aligned} &\max_{n_c\le k\le n}\Big\{G^{-\frac12}\Big|\sum_{i=k-G+k_n}^{k}\lambda\left(\mathbf{c}_i'(\Theta_{i-1}^{*}-\Theta_0)\right)\Big|\Big\}= \\ &=O\Big(\max_{n_c<k\le n}\Big\{G^{-\frac12}d_n\Big(\sum_{j=k-G+1}^{k} j^{-\frac12}+k^{-\frac12}\sum_{i=k-G+1}^{k}\|\mathbf{c}_i\|d_n\Big)\Big\}\Big)= \\ &=O\big(d_n(G/n)^{c/4}\big). \end{aligned} \tag{3.20}$$

Applying (3.19) with $\kappa=(\log n)^{-\frac12}d_n^0$, where $\{d_n^0\}$ is a sequence with the properties: $d_n^0\searrow 0$ and $G^{-1}\log^3 d_n^0\longrightarrow 0$ as $n\longrightarrow\infty$ (such a sequence exists by the assumptions) and regarding (3.20) one easily finds that (3.10) holds true. ■

Proof of Theorem 2.2: Define $\{\Theta_k^0\}$ as follows:

$$\begin{aligned} \Theta_k^0 &= \widehat{\Theta}_k \qquad p<k\le k_n \\ &= \widehat{\Theta}_k \qquad k_n<k\le n,\quad \|\widehat{\Theta}_k-\Theta_0-(\mathbf{I}-\mathbf{C}_k'\mathbf{C}_m))\mathbf{q}_nI\{k>m\}\|\le \\ &\qquad\qquad\qquad\qquad\qquad \le\max\big(\|\mathbf{q}_n\|^{1+v},k^{-\frac12}d_n\big) \\ &= \widetilde{\Theta}_k \qquad k_n<k\le n,\quad \|\widehat{\Theta}_k-\Theta_0-\left(\mathbf{I}-\mathbf{C}_k^{-1}\mathbf{C}_m\right)\mathbf{q}_nI\{k>m\}\|> \\ &\qquad\qquad\qquad\qquad\qquad >\max\big(\|\mathbf{q}_n\|^{1+v},k^{-\frac12}d_n\big), \end{aligned} \tag{3.21}$$

where $\widetilde{\Theta}_k$ is an arbitrary point from

$$\left\{\Theta; \left\|\Theta - \Theta_0 - (\mathbf{I} - \mathbf{C}_k^{-1}\mathbf{C}_m)\mathbf{q}_n I\{k > m\}\right\| \le \max\left(\|\mathbf{q}_n\|^{1+v}, k^{-\frac{1}{2}} d_n\right)\right\}.$$

Then by the assumption (2.6) it suffices to treat

$$\max_{p+G<k\le n}\left\{G^{-\frac{1}{2}}\left|\sum_{i=k-G+1}^{k} \psi(X_i - \mathbf{c}_i'\Theta_{i-1}^0)\right|\right\}$$

instead of

$$\max_{p+G<k\le n}\left\{G^{-\frac{1}{2}}\left|\sum_{i=k-G+1}^{k} \psi(X_i - \mathbf{c}_i'\widehat{\Theta}_{i-1})\right|\right\}.$$

Similar considerations as in the proof of Theorem 2.4 lead to

$$\begin{aligned}
&P\Big(\max_{p+G<k\le n}\Big\{G^{-\frac{1}{2}}\Big|\sum_{i=k-G+1}^{k} \big(\psi(X_i - \mathbf{c}_i'\Theta_{i-1}^0) - \psi(X_i - \mathbf{c}_i'(\Theta_0 - \\
&-\mathbf{q}_n I\{i > m\})) + \lambda(\mathbf{c}_i'(\Theta_{i-1}^0 - \Theta_0 - \mathbf{q}_n I\{i > m\}))\big)\Big|\Big\} \ge \kappa\Big) \le \\
&\le Q_5 \kappa^{-2} G^{-1} \sum_{k=1}^{n} \|\mathbf{c}_k\|^2 (k^{-1} d_n^2 + \|\mathbf{q}_n\|^2) \le \\
&\le Q_6 \kappa^{-2} G^{-1} \left(d_n^2 \log n + n\|\mathbf{q}_n\|^2\right),
\end{aligned} \tag{3.22}$$

for some $Q_5 > 0$, $Q_6 > 0$, where we used the fact that under the alternative $H_n(\mathbf{q}_n)$:

$$E(\psi(X_i - \mathbf{c}_i'\Theta_{i-1}^0)|\Theta_{i-1}^0) = -\lambda(\mathbf{c}_i'(\Theta_{i-1}^0 - \Theta_0 - \mathbf{q}_n I\{i > m\})). \tag{3.23}$$

Next

$$\begin{aligned}
&\max_{p+G<k\le n}\Big\{G^{-\frac{1}{2}}\Big|\sum_{i=k-G+1}^{k} \big(\lambda(\mathbf{c}_i'(\Theta_{i-1} - \Theta_0 - \mathbf{q}_n I\{i > m\})) + \\
&+\lambda'(0)\mathbf{c}_i'\mathbf{C}_{i-1}^{-1}\mathbf{c}_m \mathbf{q}_n I\{i > m\}\big)\Big|\Big\} = \\
&= O\Big(\max_{p+G<k\le n}\Big\{G^{-\frac{1}{2}} \sum_{i=k-G+1}^{k} \big\{\|\mathbf{c}_i\|^{r+1}\,\|\mathbf{q}_n\|^{r+1}\, I\{i > m\} + \\
&+\|\mathbf{c}_i\|^{r+1}\left(\max(\|\mathbf{q}_n\|^{v+1}, i^{-\frac{1}{2}} d_n\right)^{r+1} + \|\mathbf{c}_i\| \max\left(\|\mathbf{q}_n\|^{1+v}, i^{-\frac{1}{2}} d_n\right)\big\}\Big\}\Big) = \\
&= o\left((\log(n/G))^{-\frac{1}{2}}\right) \text{ as } n \longrightarrow \infty.
\end{aligned} \tag{3.24}$$

The assertion of Theorem 2.2 can be concluded from (3.22) and (3.24). ∎

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