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# A note on flat modules

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*Abstract.* The Chase's theorem on flatness of direct product of flat modules is generalized to the class of modules possessing a set of generators every element of which is anihilated by a given right ideal $I$ such that the factormodule $R/I$ is flat.

*Keywords:* flat module, finite $I$ - presentation

*Classification:* 16A50

Throughout this note $R$ stands for associative ring with identity and all modules are unitary left or right modules. The terminology and notations will be as in [1] or [2]. The properties of flat modules presented there are used without references.

**1.Definition.** *Let $I$ be a right ideal of a ring $R$ and ${}_RL$ be a submodule of a free left module $R^m$. We say that $L$ is finitely $I$ - presented if there is an exact sequence*

(1) $$O \to U \to R^p \xrightarrow{f} L \to O$$

*of left modules such that the inverse image $f^{-1}(I^m)$ of the subgroup $I^m \cap L$ of $L$ is of the form ${}_RZ + I^p$, where ${}_RZ$ is a finitely generated submodule of $U$. In this case the sequence (1) is said to be a finite $I$ - presentation of $L$.*

2.Remark For $I = 0$ we clearly get the ordinary notion of a finitely presented module. It is well known that any rank finite free presentation of a finitely presented module is a finite presentation and so we are going to prove similar result for finitely $I$ - presented modules.

**3.Lemma.** *Let $I$ be a right ideal of $R$ and ${}_RL$ be a finitely $I$ - presented submodule of $R^m$. Then every rank finite free presentation of $L$ is a finite $I$ - presentation of $L$.*

Proof : Let (1) be a finite $I$ - presentation of $L$ and $O \to V \to R^q \xrightarrow{g} L \to O$ be another free presentation of $L$. For $R^p = \bigoplus_{i=1}^{p} Rx_i$ and $R^q = \bigoplus_{j=1}^{q} Ry_j$ define the homomorphism $\psi : R^p \to R^q$ by setting $\psi(x_i) = w_i$ where $f(x_i) = g(w_i), i = 1,\dots,p$. Choosing elements $\tilde{z}_j \in R^p$ such that $f(\tilde{z}_j) = g(y_j), j = 1,\dots,q$, we have $g(\psi(\tilde{z}_j) - y_j) = f(\tilde{z}_j) - g(y_j) = 0$ and consequently $\psi(\tilde{z}_j) = y_j + v_j$ for some $v_j \in V$. By the hypothesis $f^{-1}(I^m) = {}_RZ + I^p$ where ${}_RZ = \langle z_1,\dots,z_n\rangle$. Setting ${}_RW = \langle \psi(z_1),\dots,\psi(z_n), v_1,\dots,v_q\rangle$ we obviously have $W \subseteq V$ and so $W + I^q \subseteq g^{-1}(I^m)$. On the other hand, for $x \in g^{-1}(I^m), x = \sum_{j=1}^q r_j y_j$, it is $f\left(\sum_{j=1}^q r_j\tilde{z}_j\right) = g(x) \in I^m$ and so $y = \sum_{j=1}^q r_j\tilde{z}_j = \sum_{i=1}^n s_i z_i + d, d \in I^p$. Summarizing we have $x = \psi(y) - \sum_{j=1}^q r_j v_j = \sum_{i=1}^n s_i\psi(z_i) - \sum_{j=1}^q r_j v_j + \psi(d) \in W + I^q$ as desired. ■

**4.Definition.** *For a right ideal $I$ of $R$ define $\mathcal{M}(I)$ to be the class of all right $R$ - modules $M$ having a set of generators $\{m_\alpha | \alpha \in A\}$ such that $I \subseteq (O : m_\alpha)$ for each $\alpha \in A$.*

**5.Theorem.** *The following conditions are equivalent for a right ideal $I$ of $R$:*

(a) *$R/I$ is flat and if $\{M_c | c \in C\} \subseteq \mathcal{M}(I)$ is arbitrary then $\prod_{c\in C} M_c$ is flat;*

(b) *For any collection $\{B_c | c \in C\}$ of sets the module $((R/I)^{(B_c)})^C$ is flat;*

(c) *For any set $C$ the cartesian power $(R/I)^C$ is flat;*

(d) *Every finitely generated left submodule of a free module of finite rank is finitely $I$ - presented;*

(e) *Every finitely generated left ideal of $R$ is finitely $I$ - presented.*

PROOF : The implications (a)$\Rightarrow$(b)$\Rightarrow$(c) and (d)$\Rightarrow$(e) are obvious.

(b)$\Rightarrow$(a). Every $M_c \in \mathcal{M}(I), c \in C$ has a free presentation $O \to K_c \to (R/I)^{(B_c)} \to M_c \to O$ for a suitable set $B_c$. Then we have the exact sequence $O \to \prod_{c\in C} K_c \to ((R/I)^{(B_c)})^C \to \prod_{c\in C} M_c \to O$ and it is easy to see that $\left(\prod_{c\in C} K_c\right) J = \left(\prod_{c\in C} K_c\right) \cap ((R/I)^{(B_c)})^C J$ for every (finitely generated) left ideal $J$ of $R$.

(c)$\Rightarrow$(d). Let ${}_R L = \langle u_1, \ldots, u_p \rangle$ be a finitely generated left submodule of $R^m, u_i = (u_{i1}, \ldots, u_{im}), i = 1, \ldots, p, F = \bigoplus_{i=1}^{p} Rx_i$ be a free left $R$ - module and (1) be the free presentation of $L$ with $f$ given by $f(x_i) = u_i, i = 1, \ldots, p$. Taking $x \in K = f^{-1}(I^m)$ arbitrarily, we have a unique expression $x = \sum_{i=1}^p a_i(x)x_i$ and so $a_i \in R^K, i = 1, \ldots, p$. Further, $f(x) = \sum_{i=1}^p a_i(x)u_i = (\sum_{i=1}^p a_i(x)u_{ij})_{j=1}^m$, which yields $\sum_{i=1}^p a_i(x)u_{ij} \in I$ for each $j = 1, \ldots, m$. Definning $\bar{a}_i \in (R/I)^K$ naturally by $\bar{a}_i(x) = a_i(x) + I$ we have $\sum_{i=1}^p \bar{a}_i(x)u_{ij} = 0$ in $(R/I)^K$ for each $j = 1, \ldots, m$. By flatness there are $\bar{b}_k \in (R/I)^K$ and $r_{ki} \in R$ such that $\bar{a}_i = \sum_{k=1}^n \bar{b}_k r_{ki}$ and $\sum_{i=1}^p r_{ki}u_{ij} = 0$ for all $k = 1, \ldots, n, j = 1, \ldots, m$. This yield $\sum_{i=1}^p r_{ki}u_i = 0$ for each $k = 1, \ldots, n$ and $a_i = \sum_{k=1}^n b_k r_{ki} + c_i, i = 1, \ldots, p$, where $b_k(x)$ is any representative of $\bar{b}_k(x)$ and $c_i \in I^K$. Setting

$$z_k = \sum_{i=1}^{p} r_{ki}x_i \in F \tag{2}$$

and ${}_R Z = \langle z_1, \ldots, z_n \rangle$, we have $Z \subseteq U$ since $f(z_k) = \sum_{i=1}^p r_{ki}u_i = 0$ and consequently $Z + IF \subseteq K$. Conversely, for $x \in K$ we have $x = \sum_{i=1}^p a_i(x)x_i = \sum_{i=1}^p (\sum_{k=1}^n b_k(x)r_{ki} + c_i(x))x_i = \sum_{k=1}^n b_k(x)z_k + \sum_{i=1}^p c_i(x)x_i \in Z + IF$ and (1) is a finite $I$ - presentation of $L$.

$(e) \Rightarrow (b)$. Let $v_1, \ldots, v_p \in ((R/I)^{(B_c)})^C$ be elements such that $\sum_{i=1}^p v_i u_i = 0, u_i \in R$. Denote $L = \sum_{i=1}^p Ru_i$ the left ideal of $R$ and (1) be its free presentation with $F = \bigoplus_{i=1}^{p} Rx_i$ and $f(x_i) = u_i$. By the hypothesis $L$ is finitely $I$ - presented and so by lemma 3 $f^{-1}(I) \subseteq {}_R Z + IF$, where ${}_R Z = \langle z_1, \ldots, z_n \rangle \subseteq V$ and $z_k$ are of the form (2). Take $c \in C$ arbitrarily. Then $v_i(c)$ lies in $(R/I)^{(B_c)}$ and so

$v_i(c) = (d^\alpha_{ci} + I)$ for some $d^\alpha_{ci} \in R, \alpha \in B_c$, with $d^\alpha_{ci} \notin I$ for a finite number of $\alpha$'s, only. So, let $A \subseteq B_c$ be the finite set of all $\alpha \in B_c$ for which $d^\alpha_{ci} \notin I$ for some $i = 1, \ldots, p$. Now $\sum_{i=1}^p v_i(c)u_i = (\sum_{i=1}^p d^\alpha_{ci}u_i + I)_\alpha = 0$ and so $\sum_{i=1}^p d^\alpha_{ci}u_i \in I$ for each $\alpha \in A$. Consequently $\sum_{i=1}^p d^\alpha_{ci}x_i \in f^{-1}(I)$ for each $\alpha \in A$ and we can write $\sum_{i=1}^p d^\alpha_{ci}x_i = \sum_{k=1}^n q^\alpha_{ck}z_k + \sum_{i=1}^p h^\alpha_{ci}x_i$ with $h^\alpha_{ci} \in I$. Using (2) we get $\sum_{i=1}^p d^\alpha_{ci}x_i = \sum_{k=1}^n \sum_{i=1}^p q^\alpha_{ck}r_{ki}x_i + \sum_{i=1}^p h^\alpha_{ci}x_i$ and consequently $d^\alpha_{ci} = \sum_{k=1}^n q^\alpha_{ck}r_{ki} + h^\alpha_{ci}, \alpha \in A, i = 1, \ldots, p$. For every $k = 1, \ldots, n$ set $w^\alpha_k(c) = q^\alpha_{ck} + I$ if $\alpha \in A$ and $w^\alpha_k(c) = I$ otherwise. Then $w^\alpha_k(c) \in (R/I)^{(B_c)}$ and since $\sum_{k=1}^n w^\alpha_k(c)r_{ki} = \sum_{k=1}^n (q^\alpha_{ck} + I)r_{ki} = d^\alpha_{ci} + I$ for each $\alpha \in A$ and $\sum_{k=1}^n w^\alpha_k(c)r_{ki} = I$ for $\alpha \in B_c \setminus A$, we see that $\sum_{k=1}^n w^\alpha_k(c)r_{ki} = v_i(c)$ and hence $\sum_{k=1}^n w^\alpha_k r_{ki} = v_i, i = 1, \ldots, p$. Moreover, by (2) it is $\sum_{i=1}^p r_{ki}u_i = f(z_k) = 0$ which shows that $((R/I)^{(B_c)})^C$ is flat and the proof is complete. ∎

At the end of this note we list some conditions equivalent to the flatness of a homomorphic image of a given cyclic flat right $R$ - module.

**6.Proposition.** *Let $I \subseteq J$ be right ideals of $R, R/I$ flat. The following conditions are equivalent:*

(a) *$R/J$ is flat;*
(b) *For every left ideal $L$ of $R$ the equality $JL + I = (J \cap L) + I$ holds;*
(c) *For each $v \in J$ there are $y \in J$ and $u \in I$ with $v = yv + u$;*
(d) *For each $v \in J$ there exists a homomorphism $f : R \to J$ with $f(v) = v + u$ for some $u \in I$;*
(e) *For any elements $v_1, \ldots, v_n \in J$ there exists a homomorphism $f : R \to J$ with $f(v_i) = v_i + u_i$ for some $u_i \in I, i = 1, \ldots, n$;*
(f) *For all elements $a_i, q_i \in R, i = 1, \ldots, m$ with $\sum_{i=1}^m a_iq_i \in J$ there exist elements $p_i \in R$ such that $p_i - a_i \in J$ for each $i = 1, \ldots, n$ and $\sum_{i=1}^m p_iq_i \in I$;*
(g) *For all elements $a_i, q_{ij} \in R, i = 1, \ldots, m, j = 1, \ldots, n$ with $\sum_{i=1}^m a_iq_{ij} \in J$ there exist elements $p_i \in R$ such that $p_i - a_i \in J$ for each $i = 1, \ldots, m$ and $\sum_{i=1}^m p_iq_{ij} \in I$ for each $j = 1, \ldots, n$.*

PROOF : $(a) \Rightarrow (b)$. Obvious.
$(b) \Rightarrow (c)$. Setting $L = Rv$ for $v \in J$ we have $v \in J \cap L \subseteq JL + I$, so that $v = \sum_{k=1}^n j_kr_kv + u$ where $u \in I$ and $y = \sum_{k=1}^n j_kr_k \in J$.
$(c) \Rightarrow (d)$. Definning $f : R \to J$ by $f(1) = y$ we have $f(v) = yv = v - u$.
$(d) \Rightarrow (e)$. The case $m = 1$ is clear and we shall induct on $m$. Taking $g : R \to J$ with $g(v_{m+1}) = v_{m+1} - s_{m+1}, s_{m+1} \in I$, we have $g(v_i) = v_i - s_i, s_i \in J, i = 1, \ldots, m$. There is $t : R \to I$ with $t(s_{m+1}) = s_{m+1}$, $R/I$ being flat. Setting $z_i = (1-t)(s_i)$, the induction hypothesis gives the existence of $h : R \to J$ with $h(z_i) - z_i \in I$. An easy computation now shows that $f = 1 - (1-h)(1-t)(1-g)$ has all desired properties.
$(e) \Rightarrow (g)$. There is $f : R \to J$ with $f(\sum_{i=1}^m a_iq_{ij}) = \sum_{i=1}^m a_iq_{ij} - u_j, u_j \in I, j = 1, \ldots, n$. Now the elements $p_i = a_i - f(a_i), i = 1, \ldots, m$, have desired properties.
$(g) \Rightarrow (f)$. Obvious.
$(f) \Rightarrow (b)$. For $1.v \in J \cap L$ there is $p \in R$ with $1 - p \in J$ and $pv \in I$ showing that $v = (1-p)v + pv \in JL + I$.

$(b) \Rightarrow (a)$. Every element $v \in J \cap L \subseteq JL + I$ can be written in the form $v = x + i, x \in JL, i \in I$. But then $i = v - x \in J \cap L \cap I = I \cap L = IL \subseteq JL$ gives $v \in JL$. ■

**7.Corollary.** *Let $I$ be a two-sided ideal of $R, R/I$ right flat. The following conditions for a right ideal $J$ of $R$ containing $I$ are equivalent:*

(a) *$R/J$ is a flat $R$-module;*
(b) *For each $v \in J$ there is a homomorphism $\bar{f} : R/I \to J/I$ with $\bar{f}(v+I) = v+I$;*
(c) *$R/J$ is a flat $R/I$ - module;*
(d) *For each left ideal $L$ of $R$ containing $I$ it holds $J \cap L = JL + I$.*

PROOF : $(a) \Rightarrow (b)$. By proposition 6 there is $f : R \to J$ with $f(v) = v + u$ for some $u \in I$. Since $f(I) \subseteq I$, $f$ induces naturally $\bar{f} : R/I \to J/I$ which has the desidered property.
$(b) \Rightarrow (a)$. Let $\bar{f}(1 + I) = y + I$. Definning $f : R \to J$ by $f(1) = y$ we have $f(v) = v + u, u \in I$, and proposition 6 applies.
The equivalence of the conditions (b) and (c) is well-known. Assuming (d) we have $J/I.L/I = (JL + I)/I = (J \cap L)/I = (J/I) \cap (L/I)$ which is equivalent to (c) while the converse implication follows by proposition 6.(b). ■

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