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NP-hardness results for intersection graphs

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Abstract. Intersection graphs of segments (the class SEG) and of other simple geometric objects in the plane are considered. We show the NP-hardness of membership problems for several such classes of graphs (e.g. string graphs, intersection graphs of segments, intersection graphs of convex sets) in an unified way.

Keywords: intersection graph, string graph, NP-hard problem

Classification: 05C99

1.Introduction.

In this paper we will consider the intersection graphs of segments and of other simple geometric objects in the plane. We concentrate on the lower bound on the complexity of the membership problem for such classes of intersection graphs among all graphs or among the graphs of some wider class. For more background information and related results see the paper [**KM**].

The study of intersection graphs with geometric flavor began probably by the investigation of interval graphs, which turned out to have a nice structure ([**FG**], [**LB**], [**GH**]), are easily recognized ([**BL**]) and have many applications (in biology etc.). Other examples of polynomially recognizable classes are circle graphs (intersection graphs of chords in a circle - [**Bou**], [**Fou**]), path-graphs (intersection graphs of paths in tress, [**Gav**]), circular-arc graphs ([**Tuc**]) and intersection graphs of subtrees in trees (which are just the chordal graphs, [**Gav**]). Much more complicated is the structure of string graphs (the intersection graphs of simple curves in the plane). These were introduced by Sinden [**Sin**] in connection with thin film RC-circuits. Tarjan et al. [**EET**] studied them independently from a theoretical point of view; for other results see [**KGK**]. Until recently, it has been an open problem whether the recognition of string graphs is NP-hard (solved affirmatively in [**Kra1**]). On the other hand, it is not known whether this problem is even decidable. All the graphs we consider are special cases of string graphs.

2.Statement of results.

By a graph we will mean a simple undirected graph (without loops and multiple edges). The vertex set of a graph G is denoted by $V(G)$ and the edge set of G by $E(G)$. An edge with vertices u and v will be denoted by uv.

Let us define the classes of graphs we will consider. If **C** is a class of sets (geometric objects in our case), then the class of **intersection graphs of C**, denoted by IG(**C**), will be the class of all simple undirected graphs, isomorphic to graphs of the form $G = (V, E)$, where $V \subseteq \mathbf{C}$ and $e = uv \in E$ iff $u \cap v \neq \emptyset$. We call this V **a representation** of (the isomorphism class of) G. Usually we will treat a representation of a given graph as a mapping from $V(G)$ into $\mathbf{C}$.

We put

STRING = IG({all simple curves in the plane}),

CONV = IG({all convex sets in the plane}),

k-SEG = IG({all piecewise linear curves consisting of $\leq k$ segments}), $k \geq 1$,

SEG = 1-SEG,

k-DIR$(d_1, \ldots, d_k)$ = IG({all segments with slopes among $d_1, \ldots, d_k$}), $d_1, \ldots, d_k$ real numbers,

k-DIR = $\cup\{k$-DIR$(d_1, \ldots, d_k); d_1, \ldots d_k$ real numbers$\}$.

We will use the term "SEG-representation of G" instead of "the representation of G by segments", and similarly in all other cases.

Other interesting classes of intersection graphs arise when we put some additional restrictions on the relative positions of the objects representing the graphs. We express it by the notation IG$(\mathbf{C})|R$, which will be the class of all graphs of IG$(\mathbf{C})$, such that they have a representation $V \subseteq \mathbf{C}$ satisfying the restriction R. We put

k-CROSS = STRING|(every two curves meet in $\leq k$ points, and they cross at the intersections),

PURE-k-DIR$(d_1, \ldots, d_k)$ = k-DIR$(d_1, \ldots, d_k)|$ (no two segments of the same direction intersect each other),

PURE-k-DIR = $\cup\{$ PURE-k-DIR $(d_1, \ldots, d_k); d_1, \ldots d_k$ real numbers$\}$.

The algorithmic problems concerning intersection graphs of geometric objects we will deal with have features rather unusual for graph-theoretic problems: in many cases, they do not belong to NP or their membership in NP presents an open problem. Here we concentrate on the NP-hardness results. If A, B are classes of intersection graphs, $A \subseteq B$, we denote by **RECOG** $(A|B)$ the following decision problem:

INPUT: AB-representation of a graph G.
QUESTION: Does G belong to the class A?

In particular, RECOG (A) will be RECOG $(A|$ {all graphs}).

The results are summarized in the following theorem:

Theorem 1. *The problems RECOG(STRING), RECOG(CONV) and RECOG (k-SEG), for every $k \geq 1$, are NP-hard.*

The problems RECOG(k-CROSS), for every $k \geq 1$, and RECOG(k-DIR) and RECOG(PURE-k-DIR), for every $k \geq 2$, are NP-complete.

RECOG(1-SEG|2-SEG) is NP-hard and RECOG(1-CROSS|2-CROSS) is NP-complete.

Remark. The NP-membership for RECOG(PURE-k-DIR) and RECOG(k-DIR) for every fixed k is non-trivial and it is proved in [KM].

3.Proofs.

We divide the proof of Theorem 1 into several steps.

Proposition 2. The problems RECOG(STRING), RECOG(1-CROSS), RECOG(SEG) and RECOG(CONV) are NP-hard.

Remarks. The NP-hardness of RECOG(STRING) and RECOG(1-CROSS) is proved in [**Kral**]. Here we are going to provide a reduction which will prove the NP-hardness of all the four problems at once. Hence it follows that for every $k \geq 2$, RECOG(k-SEG) is NP-hard and RECOG(k-CROSS) is NP-complete. We are using the following problem, which is shown to be NP-complete in [**Kra2**].

P3C3SAT (Planar 3-connected 3-satisfiability)
INSTANCE: A formula Φ with a set of clauses C and a set of variables X satisfying

(i) every clause contains exactly 3 variables,
(ii) the graph $G_\Phi = (X \cup C, \{xc | x \in c \in C \text{ or } \neg x \in c \in C\})$ is planar and (vertex) 3-connected.

QUESTION: Is Φ satisfiable?

PROOF of Proposition 2: Take an instance Φ of P3C3SAT and fix a planar drawing D_Φ of G_Φ in which edges are drawn straight (i.e. a Fáry embedding of G_Φ. Such a drawing can be constructed in polynomial time [**FPP**]). We are going to construct a graph $G(\Phi)$ such that every STRING representation of $G(\Phi)$ (if it exists) resembles the drawing D_Φ. To guarantee this, we use two auxiliary graphs called variable- and clause-gadgets. The graph $G(\Phi)$ arises from G_Φ by replacing every variable by a copy of the variable-gadget and every clause by a clause-gadget. Every edge xc of G_Φ is replaced by two vertices adjacent to certain vertices in the corresponding variable- and clause-gadgets. In a STRING representation of $G(\Phi)$ these vertices are represented by two parallel curves, and the order in which these curves meet the boundary of the clause-gadget indicates whether x is TRUE or FALSE in c.

Let us describe the construction more precisely. For every edge e of G_Φ we insert two vertices, $L(e)$ and $R(e)$. These vertices will be also involved in the gadgets. Let x be a variable and let $c_1, c_2, \ldots, c_k, c_{k+1}, \ldots, c_r$ be the clauses containing x (or $\neg x$) labeled in the clock-wise order as the edges xc_i leave the vertex x in D_Φ, so that both the directed angles c_1xc_k and $c_{k+1}xc_r$ are convex. The variable-gadget G_x is depicted in fig. 1, where for $i = 1, 2, \ldots, r(x)$

$$A_i = \begin{cases} L(xc_i) \\ R(xc_i) \end{cases} \quad \text{and} \quad B_i = \begin{cases} R(xc_i) \\ L(xc_i) \end{cases} \quad \text{if} \quad \begin{cases} \neg x \in c_i \\ x \in c_i \end{cases}$$

Let c be a clause and let x_1, x_2, x_3 be the variables occurring in c ordered as the edges x_ic leave the vertex c in D_Φ (again in the clock-wise orientation). The clause gadget G_c is depicted on fig 2.

We put

$$G(\Phi) = \bigcup_{x \in X} G_x \cup \bigcup_{c \in C} G_c.$$

We claim that Φ is satisfiable if and only if $G(\Phi) \in$ STRING, and in that case even $G(\Phi) \in$ SEG. Suppose $G(\Phi)$ is representable by curves, and let R be such a representation. Call $C(x)$ the cycle $C_1C_2 \ldots C_{4r}$ in G_x. The curves which represent the vertices of $C(x)$ form a cycle in the plane, which divides the plane into two regions. We will call the inner region Ω_1 and the outer one Ω_2. Since the graph $G_\Phi - x$ is connected, the curves representing the vertices of $G(\Phi) - G_x$ lie either all inside Ω_1 or all inside Ω_2. Without loss of generality we may assume that they lie in Ω_2 (otherwise we apply a homeomorphism similar to the circle inversion).

Analogously, we may suppose that R is such that for every clause c, all the curves representing the vertices of $G(\Phi) - G_c$ lie outside the cycle $r_{v_1} r_{v_2} \dots r_{v_6}$. (Here and later on, r_u is the curve in R which represents a vertex $u \in V(G(\Phi))$. Thus there exist pairwise disjoint regions $\Omega_x, x \in X$ and $\Omega_c, c \in C$ such that all the curves representing the vertices of $G_x - \{A_1, B_1, A_2, B_2, \dots, A_{r(x)}, B_{r(x)}\}$ lie inside Ω_x, and all the curves representing the vertices of $G_c - \{L(x_1c), R(x_1c), \dots, R(x_3c)\}$ lie inside Ω_c. Then $\Omega(\Phi) = (\{\Omega_x : x \in X\} \cup \{\Omega_c : c \in C\}, \{r_{L(xc)}, r_{R(xc)} : x \in c \in C$ or $\neg x \in c \in C\})$ (the drawing obtained from R by contracting the regions Ω_x and Ω_c into single points)is a planar drawing of G_Φ (only the edges are doubled). Since G_Φ is 3-connected, it has a unique (from the topological point of view) planar drawing, and $\Omega(\Phi)$ is homeomorphic to D_Φ. Thus we may suppose without loss of generality that R is such that the regions Ω_x and Ω_c are placed around the corresponding vertices x and c in D_Φ, and the curves $r_{L(xc)}$ and $r_{R(xc)}$ follow the drawings of the corresponding edges xc. Consider a variable x. The pairs of curves (A_i, B_i) leave the boundary of Ω_x in this (clock-wise) order, the two curves r_{A_i}, r_{B_i} being always close to each other. It follows that the variable gadget G_x may be represented in exactly two essentially different ways, depicted in fig.3. We put $F(x)=$ TRUE if G_x is represented as in fig. 3 left and $F(x)=$ FALSE if it is represented as in fig. 3 right. Note that in the former case the curves r_{A_i}, r_{B_i} leave the boundary of Ω_x in the order $A_1, B_1, A_2, B_2, \dots, A_r, B_r$, while in the latter case they leave it in the order $B_1, A_1, B_2, A_2, \dots, B_r, A_r$. It follows that the curves $r_{L(xc)}, r_{R(xc)}$ arrive to the boundary of Ω_c in this order (with respect to the clock-wise orientation), if and only if x receives the value TRUE in c (cf. fig.4).

Now consider a clause c and the realization of the clause gadget G_c in R. To obtain all the necessary intersections of the curves $r_{w_1}, r_{w_2}, \dots, r_{w_6}$, these curves must lie entirely inside the cycle formed by the curves $r_{v_1}, r_{v_2}, \dots, r_{v_6}$ (in fact this is the boundary of Ω_c). If we extend the curves r_{w_i}along the corresponding $r_{L(xc)}$ (or $r_{R(xc)}$) to the boundary of Ω_c, we obtain a representation of the complement of the cycle of length 6 inside Ω_c with the curves meeting the boundary in the order determined by the truth values of the variables occurring in c. In particular, if all the three variables receive the value FALSE in c, the order is $w_1, w_2, \dots, w_6$ and it is known, that then such a representation does not exist ([Sin], [**Kral**]). On the other hand, if the order of at least one pair w_{2i-1}, w_{2i} is permuted,the representation exists and is realizable by straight line segments (see fig. 5).

Therefore if $G(\Phi)$ is representable by curves,the truth valuation $F(x)$ defined above satisfies Φ. On the other hand, if Φ is satisfiable, a representation of $G(\Phi)$ by segments is constructed in an obvious way (using fig. 5).This completes the proof. ∎

Using different clause gadgets in the above described construction of the graph $G(\Phi)$, we can prove

Proposition 3. RECOG(SEG|2-SEG) is NP-hard.

PROOF : Use the clause gadget depicted in fig. 6. It turns out that it is not representable by convex sets when all three variables receive the value FALSE in the clause. (Its representations in all other cases are in fig. 8.) Suppose for the contrary that G_c is representable by convex sets. Then we have a region Ω_c and convex sets $W_1, W_2, \dots, W_6$ lying inside Ω_c and meeting its boundary in this order, such that $W_i \cap W_j \neq \emptyset$ iff $w_i w_j \in E(G_c)$. Denote by P_i an intersecting point of W_i and the boundary of Ω_c and by P_{ij} an intersecting point of W_i and W_j (if $W_i \cap W_j \neq \emptyset$). Let Ω_{ij} be the region bounded by the arc of the boundary of Ω_c between P_i and P_j and the sets W_j and W_i (see fig.7).

We see that $W_2 \subset \Omega_{13}$, the segment P_4P_{24} crosses the segment P_1P_{13} and so P_4 and P_{24} lie in opposite halfplanes determined by the line P_1P_{13}. We may suppose that $P_{14} = P_4P_{24} \cap P_1P_{13}$. Since $P_5 \in \Omega_{41}$ and $W_5 \cap W_3 \neq \emptyset, W_5$ crosses the segment P_1P_{14}, say in the point P_{15}. Now $P_6 \in \Omega_{51}$, and so we may suppose that $P_{16} = P_1P_{15}$. Also $W_6 \cap W_4 \neq \emptyset$. However, every segment P_6X, with $X \in W_4$, either crosses the segmentP_5P_{35} or leaves the region Ω_c, a contradiction. A 2-SEG representation of this case is in fig. 12. ■

Remark. The theorem says that given a representation formed by piece-wise linear curves, each consisting of at most two linear pieces, it is NP-hard to decide whether the curves of the representation can be stretched so that the intersection graph remains unchanged.[1]

Proposition 4. RECOG(1-CROSS|2-CROSS) is NP-complete.

PROOF : Use the clause gadget depicted in fig. 9. It turns out that the gadget is representable by curves, each pair of them sharing at most one common crossing point, whenever at least one variable receives the value TRUE in the clause. On the other hand, when all three variables receive the value FALSE in the clause, the gadget is only representable by curves,one pair them sharing two intersecting points (see fig. 10,11). ■

Remark. The theorem says that given a system of curves in the plane, each pair of them sharing at most two crossing points, it is NP-complete to decide whether the representation can be rearranged so that any two curves share at most one common point and the intersection graph remains unchanged.

Proposition 5. For every $k \geq 2$, RECOG (k -DIR) and RECOG (PURE-k-DIR) are NP-complete problems.
PROOF : Using the same clause gadgets as in the proof of Proposition 3, but modifying slightly the construction of the graph $G(\Phi)$, one gets the NP-completeness of recognizing k-DIR and PURE-k-DIR graphs for $k \geq 3$. The case of 2-DIR and

[1] **Actually even more is true. It follows from the concept of almost satisfied formulas that given a system of straight line segments and one piece-wise linear curve consisting of two linear pieces, it is NP-hard to decide whether this curve can be stretched (and the others rearranged) so that the intersection graph remains unchanged [KK]. A similar strengthening applies to the remark after Proposition 4.**

PURE-2-DIR graphs requires not only another construction of the clause gadget, but also the reduction starts with a slightly more restrictive satisfiability problem. The proof in detail is given in [Kra2]. ∎

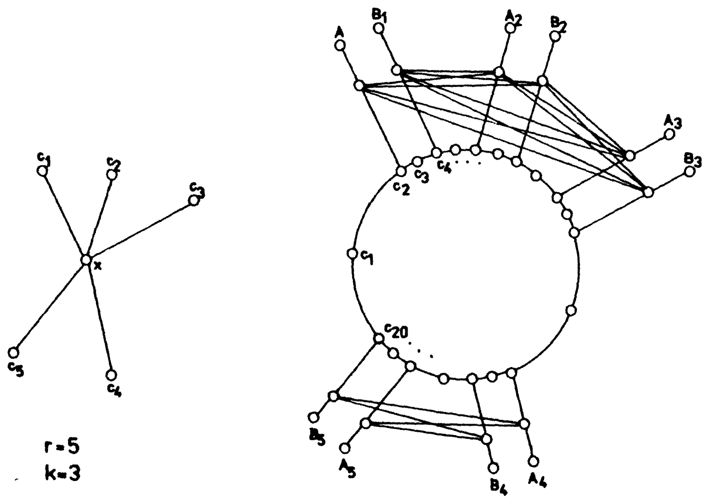


Fig.1

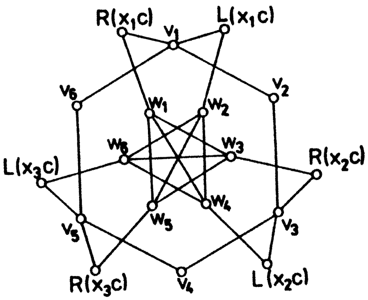


Fig.2

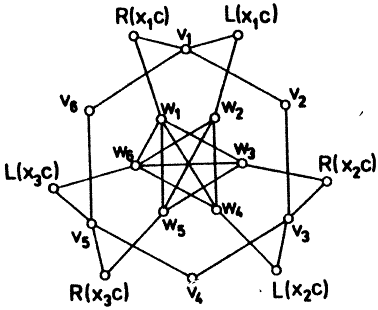


Fig.6

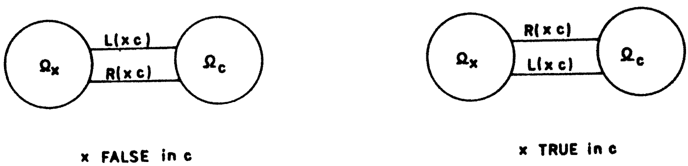
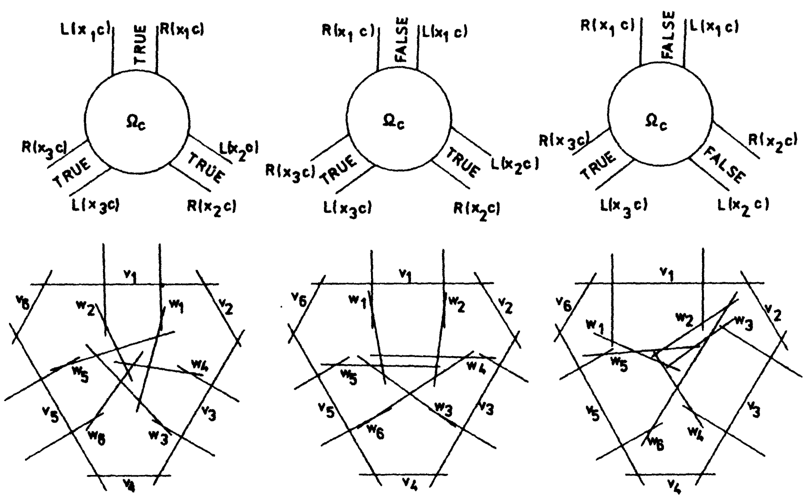


Fig.4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R(x1c)
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R(x3c)
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L(x2c)
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L(x1c)
Ωc
R(x3c)
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R(x1c)
FALSE
L(x1c)
Ωc
R(x3c)
R(x2c)
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FALSE
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v2
v3
v4
v5
v6
w1
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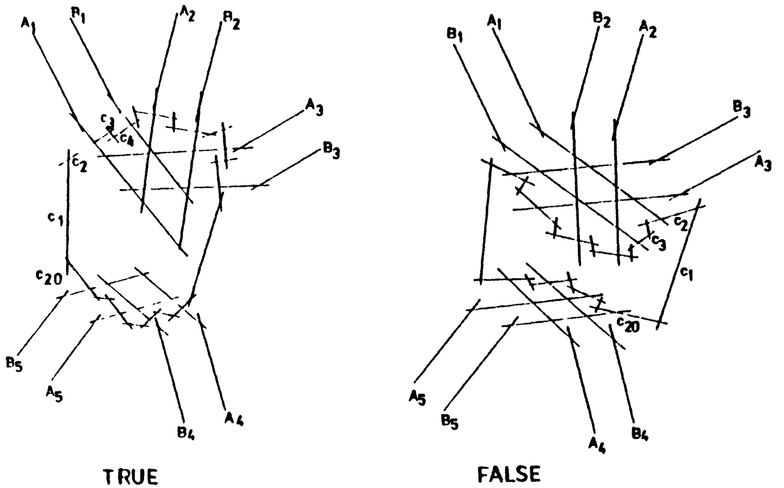
w6

Fig.5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B2
A3
B3
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B4
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B1
A1
B2
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B3
A3
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c3
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c20
A5
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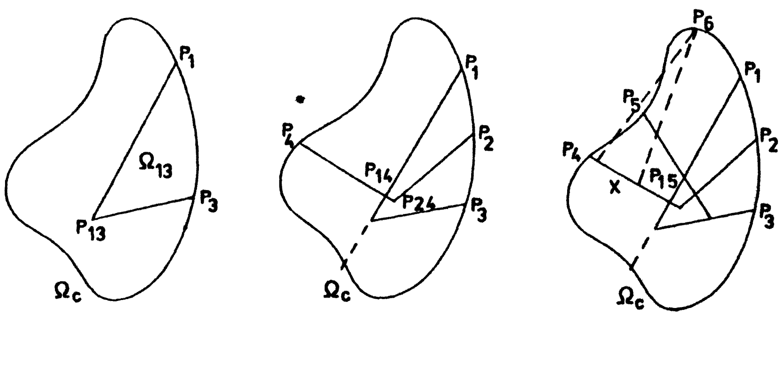
A4
B4
FALSE

Fig.3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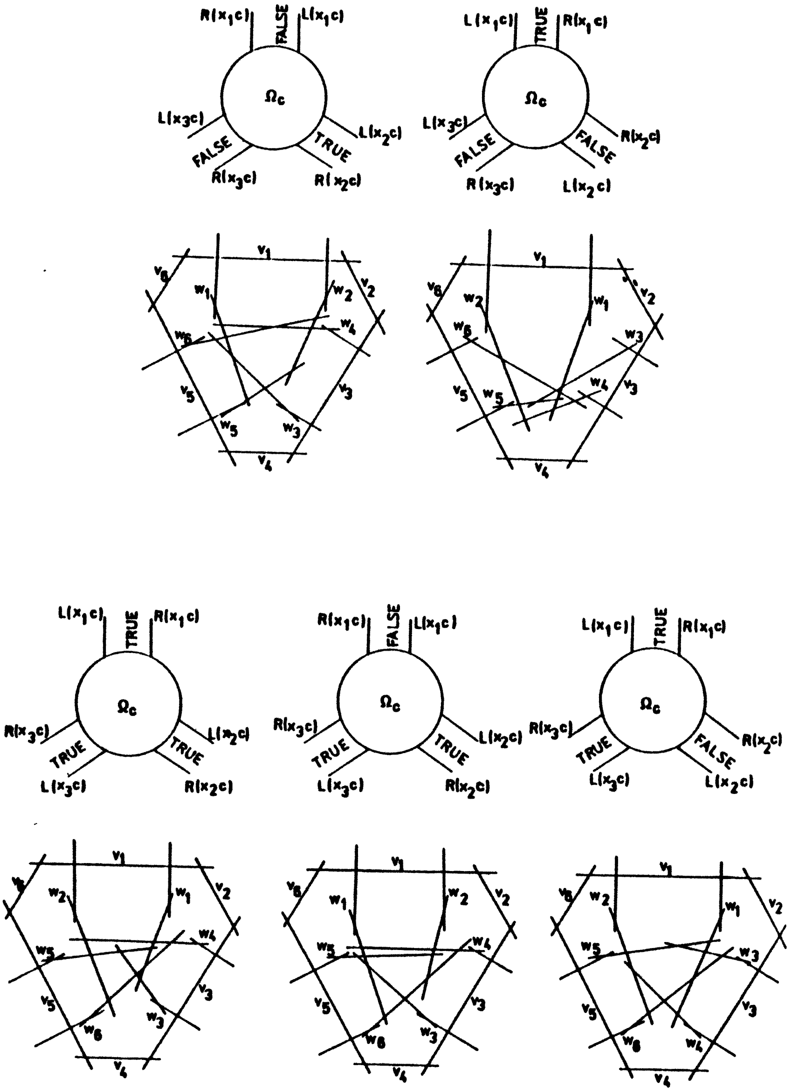
P2
P4
X
P15
P3
Ωc

Fig.7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w6
w4
v4

Fig.8

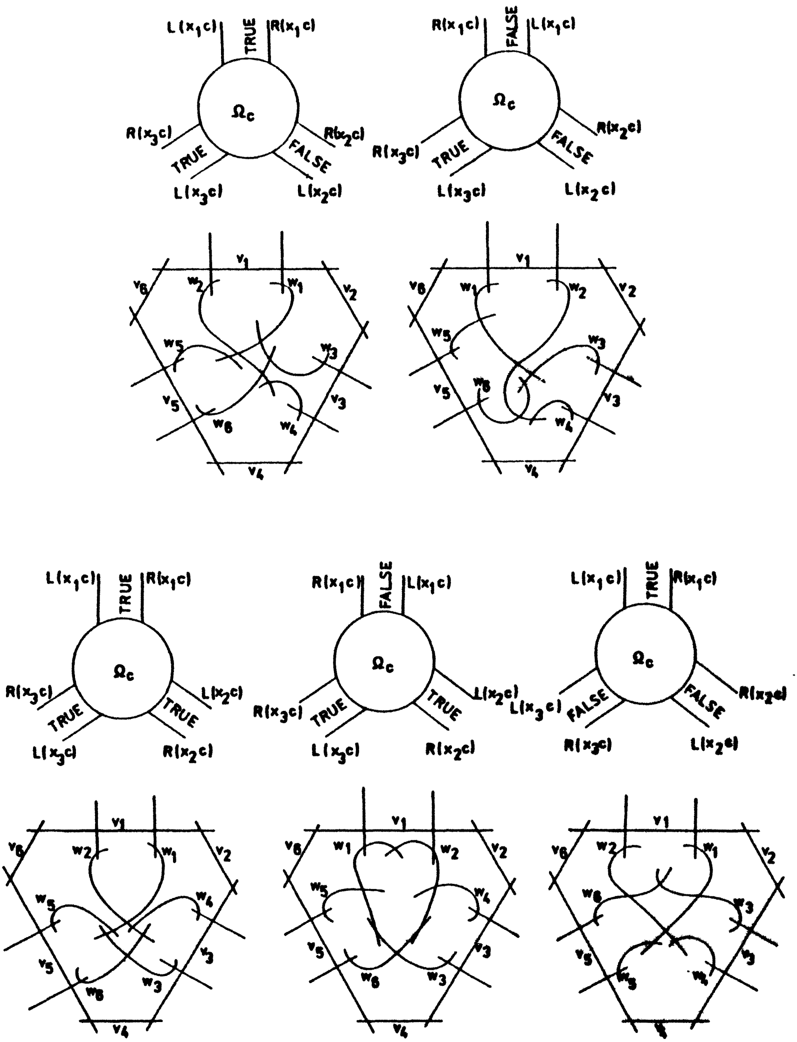
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Fig.10

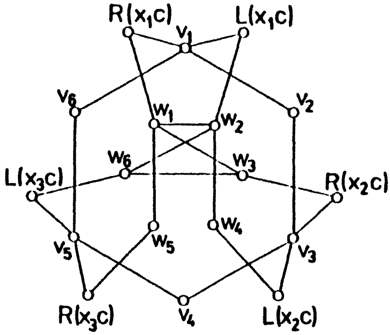

Fig.9

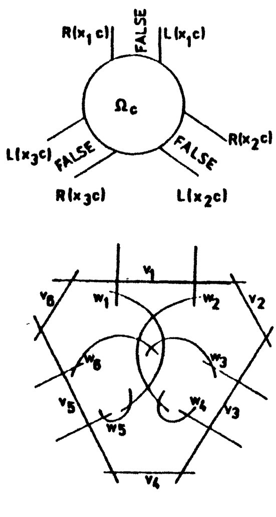

Fig.11

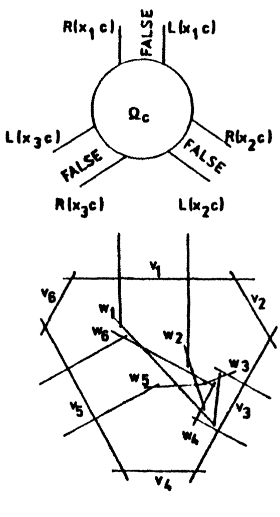

Fig.12

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