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Miroslav Krutina

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A note on the relation between asymptotic rates of a flow under a function and its basis–automorphism

MIROSLAV KRUTINA

Abstract. The asymptotic rate of a flow $\{T_t\}_{t\in\mathbf{R}}$ on a countably generated probability space $(\Omega,\mathcal{F},\mu)$ was defined in [6] as $H_\mu(\{T_t\}_{t\in\mathbf{R}}) = H_\mu(T_1)$, where $H_\mu(T_1)$ means the asymptotic rate of the automorphism T_1. In this note, a relation between $H_\mu(\{T_t\}_{t\in\mathbf{R}})$ and the asymptotic rate of the basis–automorphism, which is an analogy of Abramov's formula between entropies, is shown in case of a flow under a function (both relations coincide when the flow is ergodic). An auxiliary assertion that the asymptotic rate of a flow equals the supremum of entropies of its restrictions to flow–invariant (mod 0) subsets, is derived for the proof. The importance of the asymptotic rate consists in the fact that its value determines the minimal generator cardinality of the automorphism in non–ergodic case, see [3] ([9]).

Keywords: flow under a function, asymptotic rate, entropy

Classification: 28D10, 28D20

We begin by recalling some basic notions. Let T be an automorphism of a probability space $(\Omega,\mathcal{F},\mu)$, i.e. a 1:1 bimeasurable measure–preserving transformation of Ω onto itself. By $\mathcal{P}$ we mean the class of all finite $\mathcal{F}$–measurable partitions of Ω. For $\varepsilon > 0$ and $\xi \in \mathcal{P}$, put $L_\mu(\varepsilon,\xi) = \min\{\operatorname{card}\mathcal{X} : \mathcal{X} \subset \xi, \mu(\cup\mathcal{X}) > 1-\varepsilon\}$. Following [9], the asymptotic rate $H_\mu(T)$ of T is a non–negative real number (including $+\infty$) defined by

$$H_\mu(T) = \sup_{\xi\in\mathcal{P}} H_\mu(T,\xi)$$

where, for $\xi \in \mathcal{P}$,

$$H_\mu(T,\xi) = \lim_{\varepsilon\to 0+} \limsup_n \frac{1}{n} \log L_\mu(\varepsilon, \bigvee_{k=0}^{n-1} T^{-k}\xi) \tag{1}$$

($\log = \log_e$ and $\vee$ means the customary operation of the roughest common refinement; the limit (1) always exists and, obviously, $H_\mu(T)$ is a metrical invariant). Let $h_\mu(T)$ be the usual entropy of T (see e.g. [7]). For $E \in \mathcal{F}$ with $\mu(E) > 0$, μ_E means the conditional probability on $(\Omega,\mathcal{F})$, defined by $\mu_E(F) = \mu(E\cap F)/\mu(E)$, $F \in \mathcal{F}$. Clearly, if $E \in \mathcal{I}_T = \{F \in \mathcal{F} : TF = F\}$, μ_E is T–invariant. If $\Omega = \bigcup_n E_n$ is a disjoint (at most countable) union such that $\mu(E_n) > 0$ and $E_n \in \mathcal{I}_T$ for every n, then $H_\mu(T) = \sup_n H_{\mu_{E_n}}(T)$ and

$$h_\mu(T) = \sum_n \mu(E_n)\cdot h_{\mu_{E_n}}(T) \tag{2}$$

by easy computations ([8]).

In what follows, we shall always assume that $(\Omega, \mathcal{F}, \mu)$ is countably generated, i.e. there is a countable collection of measurable sets which generates $\mathcal{F}$ up to symmetric differences of measure zero.

In a special case, when $\mathcal{F}$ is generated by means of countably many sets strictly and when the ergodic decomposition of μ exists (namely, the family $(m_\omega : \omega \in \Omega)$ of regular conditional probabilities induced by $\mathcal{I}_T$ with respect to μ), the relations

$$h_\mu(T) = \int h_{m_\omega}(T) d\mu(\omega) \tag{3}$$

and

$$H_\mu(T) = \operatorname{ess.sup}_{[\mu]} h_{m_\omega}(T) \tag{4}$$

hold (see [6], Lemma 6; $\operatorname{ess.sup}_{[\mu]}$ means the essential supremum modulo μ, almost all measures m_ω are T-invariant and ergodic).

By a flow on $(\Omega, \mathcal{F}, \mu)$ we mean any group $\{T_t\}_{t\in\mathbf{R}}$ of its automorphisms which satisfy $T_{s+t} = T_s \circ T_t$ for $s, t \in \mathbf{R}$, and for which $(\omega, t) \to T_t\omega$ is an $\overline{\mathcal{F} \times \mathcal{B}_\mathbf{R}}$ $-$ $\mathcal{F}$ measurable mapping ($\mathcal{B}_\mathbf{R}$ are the Borel sets on the real line $\mathbf{R}$ and $\overline{\mathcal{F} \times \mathcal{B}_\mathbf{R}}$ is the complete product σ-algebra with respect to $\mu \times \lambda$, where λ denotes the usual Lebesgue measure on $\mathcal{B}_\mathbf{R}$). As it is known, such a flow satisfies, for each $E \in \mathcal{F}$, that $\mu(E \Delta T_t E) \to 0$ as $t \to 0$ (Δ means the symmetric difference). Further, $H_\mu(T_t) = |t| \cdot H_\mu(T_1)$ for each $t \neq 0$ ([6]), and the asymptotic rate of the flow is defined by

$$H_\mu(\{T_t\}_{t\in\mathbf{R}}) = H_\mu(T_1).$$

Recall an important case of a flow, namely the flow under a function. Let $(B, \mathcal{B}, \nu)$ be a probability space with an automorphism S. Let f be a real measurable function on B such that, for each $\beta \in B$, $f(\beta) > 0$, $\sum_{j=0}^{\infty} f(S^j\beta) = \sum_{j=0}^{\infty} f(S^{-j}\beta) = \infty$, and $\int f d\nu < \infty$. Put ${}^*B = \{(\beta, s) : \beta \in B, 0 \leq s < f(\beta)\}$, ${}^*\mathcal{B} = {}^*B \cap (\mathcal{B} \times \mathcal{B}_\mathbf{R})$ and ${}^*\nu = c \cdot (\nu \times \lambda) \restriction {}^*\mathcal{B}$ (the restriction of the product-measure to ${}^*\mathcal{B}$ normalized by $c = 1/\int f d\nu$), and define

$$S_t(\beta, s) = (S^i\beta, s + t - \sum_{j=0}^{i-1} f(S^j\beta)) \tag{5}$$

for $t \geq 0$ and $(\beta, s) \in {}^*B$ ($i \in \mathbf{I}$ is there an integer uniquely determined by $\sum_{j=0}^{i-1} f(S^j\beta) \leq s + t < \sum_{j=0}^{i} f(S^j\beta)$; the empty sum is taken as zero). As every S_t ($t \geq 0$) is a 1:1 map of ${}^*B$ onto ${}^*B$, put $S_{-t} = S_t^{-1}$. Clearly, $({}^*B, {}^*\mathcal{B}, {}^*\nu)$ is countably generated if and only if $(B, \mathcal{B}, \nu)$ is. It holds that $\{S_t\}_{t\in\mathbf{R}}$ is a flow on

$({}^*B, {}^*\mathcal{B}, {}^*\nu)$ ([2]); we call it a flow under a function (with a basis-space $(B, \mathcal{B}, \nu)$) and write as $(B, \mathcal{B}, \nu, S, f)$, too.

Our aim is to express the relation between $H_\nu(S)$ and $H_{{}^*\nu}(S_1)$. As it is known, if $(B, \mathcal{B}, \nu)$ is a Lebesgue space, then there holds the Abramov formula

$$h_\nu(S) = \int f d\nu \cdot h_{{}^*\nu}(S_1) \tag{6}$$

between the entropies, see [1]. Below, $E_\nu(f|\mathcal{I}_S)$ means the conditional expectation.

Theorem. *Let $(B, \mathcal{B}, \nu, S, f)$ be a flow under a function whose basis-space is countably generated. It holds that $H_\nu(S) = 0$ if and only if $H_{{}^*\nu}(\{S_t\}_{t\in\mathbf{R}}) = 0$, and (provided the opposite case $H_\nu(S) > 0$ occurs)*

$$\text{ess.}\inf_{[\nu]} E_\nu(f|\mathcal{I}_S) \cdot H_{{}^*\nu}(\{S_t\}_{t\in\mathbf{R}}) \leq H_\nu(S) \leq \\ \leq \text{ess.}\sup_{[\nu]} E_\nu(f|\mathcal{I}_S) \cdot H_{{}^*\nu}(\{S_t\}_{t\in\mathbf{R}}) \tag{7}$$

(there we admit the value $+\infty$ as well; we put $0 \cdot \infty = 0$ in case when $\text{ess.}\inf_{[\nu]} E_\nu(f|\mathcal{I}_S) = 0$ and $H_{{}^*\nu}(\{S_t\}_{t\in\mathbf{R}}) = +\infty$).

Proof of the theorem.

Here, we give some equivalent expressions for the asymptotic rate at first.

Proposition 1. *Let T be an automorphism of a countably generated $(\Omega, \mathcal{F}, \mu)$. Then $H_\mu(T) = \sup\{h_{\mu_E}(T) : E \in \mathcal{I}_T, \mu(E) > 0\}$.*

PROOF : Recall that μ is said to be T–aperiodic, if for any $E \in \mathcal{F}$ with $\mu(E) > 0$ and for any $n \in \mathbf{N} = \{1, 2, \dots\}$ there exists $F \subset E$, $F \in \mathcal{F}$, such that $\mu(F \cap T^{-n}F) < \mu(F)$. In this case, there is a countable generator ξ for μ and T (see [4], cf. with [6]). It guarantees the existence of a probability measure ϑ on $(\mathbf{N}^\mathbf{I}, \mathcal{C})$ ($\mathcal{C}$ is the σ–algebra in $\mathbf{N}^\mathbf{I}$ generated by cylinders), by which T and the shift S in $\mathbf{N}^\mathbf{I}$ ($(Sx)_i = x_{i+1}$ for $x = (x_j)_{-\infty}^{\infty} \in \mathbf{N}^\mathbf{I}$, $i \in \mathbf{I}$) are conjugated (in the usual sense, see [7]). As $(\mathbf{N}^\mathbf{I}, \mathcal{C})$ is a Polish space (i.e. $\mathcal{C}$ are the Borel sets of a complete separable metric space by a suitable metric), the family of regular conditional probabilities always exists and $H_\vartheta(S) = \sup\{h_{\vartheta_E}(S) : E \in \mathcal{I}_S, \vartheta(E) > 0\}$ by (3) and (4), which implies the assertion due to the conjugacy. In the case when μ is T–purely periodic, i.e. when $\Omega = \bigcup_{n=1}^{\infty} E_n$ is a disjoint union such that for every n, $E_n \in \mathcal{F}$ and $\mu(F \cap T^{-n}F) = \mu(F)$ whenever $F \subset E_n$ ($F \in \mathcal{F}$), then both $H_\mu(T)$ and $\sup\{h_{\mu_E}(T) : E \in \mathcal{I}_T, \mu(E) > 0\}$ equal zero. In other cases, we shall use the fact that there is $\Omega_p \in \mathcal{I}_T$ with $0 < \mu(\Omega_p) < 1$ such that μ_{Ω_p} is T–purely periodic and $\mu_{\Omega\setminus\Omega_p}$ is T–aperiodic anyhow. ∎

Lemma 1. *If $H_\mu(T) > s$, then there is $E \in \mathcal{I}_T$ such that $\mu(E) > 0$ and $h_{\mu_F}(T) > s$ whenever $F \subset E$ with $F \in \mathcal{I}_T$ and $\mu(F) > 0$.*

PROOF : There is $E' \in \mathcal{I}_T$ with $\mu(E') > 0$ and $h_{\mu_{E'}}(T) > s$ by Proposition 1. Let $\mathcal{E} = \{F : F \subset E', F \in \mathcal{I}_T, \mu(F) > 0, h_{\mu_F}(T) \leq s\}$ and let $F \in \mathcal{E}$. In the case when $E = E' \setminus F$ does not satisfy the above assertion, there is $F' \subset E' \setminus F$ such that $F' \in \mathcal{E}$. Thus $F' \cup F \in \mathcal{E}$ by (2) and, of course, $\mu(F) < \mu(F' \cup F)$. Hence, after at most countably many steps, we obtain the desired E because $\sup\{\mu(F) : F \in \mathcal{E}\} < \mu(E')$. ∎

Proposition 2. *Let $\{T_t\}_{t\in\mathbf{R}}$ be a flow on a countably generated $(\Omega,\mathcal{F},\mu)$, let $\mathcal{I} = \mathcal{I}(\{T_t\}_{t\in\mathbf{R}},\mu) = \bigcap_{t\in\mathbf{R}}\{F\in\mathcal{F}:\mu(F\Delta T_tF)=0\}$. Then $H_\mu(\{T_t\}_{t\in\mathbf{R}}) = \sup\{h_{\mu_E}(T_1): E\in\mathcal{I},\mu(E)>0\}$.*

PROOF : Write $s=\sup\{h_{\mu_E}(T_1): E\in\mathcal{I},\mu(E)>0\}$. $H_\mu(\{T_t\}_{t\in\mathbf{R}})\geq s$ by Proposition 1 because for any $E\in\mathcal{I}$ there is $E'\in\mathcal{I}_{T_1}$ with $\mu(E'\Delta E)=0$. On the other hand, let us suppose that $H_\mu(\{T_t\}_{t\in\mathbf{R}})>s$. By Lemma 1, there is $E\in\mathcal{I}_{T_1}$ such that $\mu(E)>0$ and that $h_{\mu_F}(T_1)>s$ for any $F\subset E$ with $F\in\mathcal{I}_{T_1}$ and $\mu(F)>0$. If we put $E^*=\bigcup_{t\in Q}T_tE$ (where Q means the rationals), then $E^*\in\mathcal{I}$ holds and, for some (finite or infinite) sequence (t_n) in Q, it is $\mu(E^*\setminus\bigcup_n T_{t_n}E)=0$ and $\mu(T_{t_n}E\setminus\bigcup_{k=1}^{n-1}T_{t_k}E)>0$ for every n. Every set $E_n=T_{t_n}E\setminus\bigcup_{k=1}^{n-1}T_{t_k}E$ belongs to $\mathcal{I}_{T_1}$ because E does, and $h_{\mu_{E_n}}(T_1)=h_{\mu_{E_n'}}(T_1)>s$ for $E_n'=T_{-t_n}E_n$ because $E_n'\subset E$, $E_n'\in\mathcal{I}_{T_1}$ and $\mu(E_n')>0$. By (2), we obtain a contradiction $h_{\mu_{E^*}}(T_1)>s$. ■

Lemma 2. *For any flow under a function $(B,\mathcal{B},\nu,S,f)$ whose basis-space is countably generated, there exists a flow under a function $(B',\mathcal{B}',\nu',S',f')$ such that*

(i) *$(B',\mathcal{B}')$ is a Polish space, $\int f d\nu=\int f'd\nu'$,*
(ii) *the basis-automorphisms S and S' are conjugated,*
(iii) *the automorphisms S_1 and S_1' (defined as in (5) by $t=1$) are conjugated.*

PROOF : There is a sequence $(D_n)_{n=1}^\infty$ in $\mathcal{B}$ which generates it up to symmetric differences of measure zero. Put $B'=(\{0,1\}^{\mathbf{N}})^{\mathbf{I}}$ equipped with the usual metric of coordinate-convergence (thus B' is a compact metric space; let $\mathcal{B}'$ be its Borel sets). For $\beta\in B$, write $\psi\beta=(\chi_{D_n}(\beta))_{n=1}^\infty$ (χ_D means the indicator of a set D), and $\varphi\beta=(\psi S^i\beta)_{i=-\infty}^\infty$. φ is a $\mathcal{B}-\mathcal{B}'$ measurable map of B into B'; let us define the measure ν' on $(B',\mathcal{B}')$ by $\nu'=\nu\varphi^{-1}$. Obviously, there is a measure–algebra isomorphism $\Phi:\widetilde{\mathcal{B}}\to\widetilde{\mathcal{B}'}$ between the corresponding measure–algebras $(\widetilde{\mathcal{B}},\widetilde{\nu})$ and $(\widetilde{\mathcal{B}'},\widetilde{\nu'})$. Next, define $S'(\beta_i')=\beta_{i+1}'$ for each $\beta'=(\beta_j')_{j=-\infty}^\infty\in B'$, $i\in\mathbf{I}$. S' is an automorphism of $(B',\mathcal{B}',\nu')$, and $\Phi\circ\widetilde{S}=\widetilde{S'}\circ\Phi$ holds for the induced transformations on $\widetilde{\mathcal{B}}$ and $\widetilde{\mathcal{B}'}$. As f is measurable, $f=\lim_n f_n$ for a non-decreasing sequence of simple measurable functions $f_n=\sum_m c_{n,m}\chi_{C_{n,m}}$. For each n,m, choose an arbitrary $C'_{n,m}\in\Phi\widetilde{C}_{n,m}$ (where $\widetilde{C}_{n,m}\in\widetilde{\mathcal{B}}$ is the equivalence class containing $C_{n,m}$). The limit $\lim_n\sum_m c_{n,m}\chi_{C'_{n,m}}$ exists ν'-a.e., and let us take its arbitrary measurable extension f' to B'. We can easily find out that there is a measure–algebra isomorphism ${}^*\Phi:\widetilde{{}^*\mathcal{B}}\to\widetilde{{}^*\mathcal{B}'}$ which makes S_1 and S_1' conjugated. ■

For a flow under a function $(B,\mathcal{B},\nu,S,f)$, there is a canonical correspondence between $\mathcal{I}_S$ and $\mathcal{I}(\{S_t\}_{t\in\mathbf{R}},{}^*\nu)$. Namely, any strictly flow-invariant set is a tube ${}^*E=\{(\beta,s)\in{}^*B:\beta\in E\}$ over some $E\in\mathcal{I}_S$ and vice versa, and for any $F\in\mathcal{I}(\{S_t\}_{t\in\mathbf{R}},{}^*\nu)$ there is a strictly flow-invariant set ${}^*E$ such that ${}^*\nu(F\Delta{}^*E)=0$. (Let us give a sketch of the second part. Let $\overline{\mathcal{B}},\overline{{}^*\mathcal{B}},\overline{\nu},\overline{{}^*\nu}$ be the completions of the σ–algebras and measures. Of course, there is a strictly flow–invariant set $F' \in {}^*\overline{\mathcal{B}}$ such that ${}^*\overline{\nu}(F \triangle F') = 0$, which is necessarily a tube ${}^*E'$ over an S–invariant set $E' \in \overline{\mathcal{B}}$. Further, there is $E \in \mathcal{I}_S$ with $\overline{\nu}(E' \triangle E) = 0$ and, consequently, the tube ${}^*E$ is strictly flow–invariant and satisfies ${}^*\nu(F \triangle {}^*E) = 0$.) Hence,

$$h_{\nu_E}(S) = \int f d\nu_E \cdot h_{({}^*\nu)_{{}^*E}}(S_1) \tag{8}$$

for every $E \in \mathcal{I}_S$ with $\nu(E) > 0$, provided the basis–space is countably generated. This follows from an application of Lemma 2 to $(B, \mathcal{B}, \nu_E, S, f)$ and from the Abramov formula (6), as $({}^*\nu)_{{}^*E} = {}^*(\nu_E)$, and $(B', \mathcal{B}', \nu')$ becomes a Lebesgue space after completion, when $(B', \mathcal{B}')$ is Polish.

The theorem follows now directly from Proposition 1, Proposition 2 and (8) due to this correspondence between $\mathcal{I}_S$ and $\mathcal{I}(\{S_t\}_{t\in\mathbf{R}}, {}^*\nu)$ (note that $\int f d\nu_E = \int E_\nu(f|\mathcal{I}_S) d\nu_E$ if $E \in \mathcal{I}_S$).

Remarks.

Following the theorem, if $0 < \text{ess.inf}_{[\nu]} E_\nu(f|\mathcal{I}_S)$ and $\text{ess.sup}_{[\nu]} E_\nu(f|\mathcal{I}_S) < \infty$, then $H_\nu(S)$ is finite if and only if $H_{{}^*\nu}(\{S_t\}_{t\in\mathbf{R}})$ is. Example 1 shows that this is not true in general. Further, $(B, \mathcal{B}, \nu, S, f)$ is ergodic if and only if S is. Then $E_\nu(f|\mathcal{I}_S) = \int f d\nu$ ν–a.e. and, as a consequence of both the propositions, $H_{{}^*\nu}(\{S_t\}_{t\in\mathbf{R}}) = h_{{}^*\nu}(S_1)$ and $H_\nu(S) = h_\nu(S)$. Hence, in this case, (7) and the Abramov formula (6) coincide.

Example 1. Let S be an automorphism of a countably generated probability space $(B, \mathcal{B}, \nu)$ such that $B = \bigcup_{n=1}^{\infty} B_n$ is a disjoint union of sets from $\mathcal{I}_S$ with $\nu(B_n) > 0$ for every n and $\sum_{n=1}^{\infty} \nu(B_n) n^2 < \infty$.

(a) Suppose that $H_{\nu_{B_1}}(S) = H_{\nu_{B_2}}(S) = \cdots = H$, where $0 < H < +\infty$, and put $f = \sum_{n=1}^{\infty} (1/n)\chi_{B_n}$. Due to Proposition 2 and (8), the asymptotic rate $H_{{}^*\nu}(\{S_t\}_{t\in\mathbf{R}})$ of $(B, \mathcal{B}, \nu, S, f)$ is infinite, though $H_\nu(S)$ is not.

(b) Let $H_{\nu_{B_n}}(S) = 1/(\nu(B_n)n^2)$ for each n, and put $f = \sum_{n=1}^{\infty} H_{\nu_{B_n}}(S) \cdot \chi_{B_n}$. $\int f d\nu < \infty$, and due to Proposition 2 and (8) again,

$$H_{{}^*\nu}(\{S_t\}_{t\in\mathbf{R}}) = \sup_n H_{{}^*(\nu_{B_n})}(\{S_t\}_{t\in\mathbf{R}}) = 1,$$

though $H_\nu(S) = \sup_n H_{\nu_{B_n}}(S) = +\infty$.

Example 2. Regardless of the ergodicity, any aperiodic flow can be represented as a flow under a function, the relation (7) of which becomes an equality. More precisely, let $\rho \in (0,1)$ and p, q be two positive real numbers with p/q irrational. Every aperiodic flow on a countably generated probability space can be represented as a flow under a function $(B, \mathcal{B}, \nu, S, p\chi_X + q\chi_{B\setminus X})$, where $X \in \mathcal{B}$ and $\lim_n \frac{1}{n} \sum_{i=0}^{n-1} \chi_X(S^i\beta) = \rho$

ν–a.e. (see [5]). Hence, $E_\nu(p\chi_X + q\chi_{B\setminus X}|\mathcal{I}_S) = p\rho + q(1-\rho)$ ν–a.e. by the ergodic theorem. Thus, from (7) we obtain that

$$(9) \qquad (p\rho + q(1-\rho)) \cdot H_{^*\nu}(\{S_t\}_{t\in\mathbf{R}}) = H_\nu(S).$$

Moreover, due to the propositions and (8), the equality (9) implies that $h_{\nu_E}(S) < H_\nu(S)$ for every $E \in \mathcal{I}_S$ with $\nu(E) > 0$ if and only if $h_{(^*\nu)_F}(S_1) < H_{^*\nu}(\{S_t\}_{t\in\mathbf{R}})$ for every $F \in \mathcal{I}(\{S_t\}_{t\in\mathbf{R}}, {}^*\nu)$ with ${}^*\nu(F) > 0$.

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Math. Institute of Charles University, Sokolovská 83, 186 00 Praha 8, Czechoslovakia

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