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Differentiability of Musielak–Orlicz sequence spaces

RYSZARD PŁUCIENNIK AND YINING YE

Abstract. In this paper a sufficient and necessary condition of differentiability of Musielak–Orlicz sequence spaces and the expression of gradient are obtained. These results are non-trivial and important generalization of previous results from paper [9] written in Chinese by Yining Ye.

Keywords: Musielak–Orlicz sequence space, Gateaux differentiability, Gateaux differentiable norm, δ_2^0 –condition

Classification: 46E30

1.Preliminaries.

Let X be a Banach space equipped with the norm $\|\cdot\|$ and $S(X)$ be the unite sphere of the space X i.e. $S(X)=\{x\in X:\|x\|=1\}$.

1.1. Definition. The Banach space X is said to have a Gateaux differentiable (or shortly, differentiable) norm at $x_0\in S(X)$ whenever for given $y\in S(X)$

$$\operatorname{grad}(x_0,y)=\lim_{\lambda\to 0}\frac{\|x_0+\lambda y\|-\|x_0\|}{\lambda}$$

exists. If the norm of X is differentiable at each point of $S(X)$ then we say that X is Gateaux differentiable (shortly differentiable) space.

The notion of differentiability of the space X is equivalent to the smoothness of X. It follows immediately from Th.2.1.1 in [1]. We can consider differentiability of Musielak–Orlicz sequence spaces. To this end, denote by $\mathbf{N}$ the set of positive integers and by $\mathbf{R}$ the set if real numbers. The brackets $(\cdot)$, $\{\cdot\}$ we will use for denotation of sequence and set, respectively. Let $\varphi=(\varphi_n)$ be a sequence of Young's functions, i.e. for every $n\in\mathbf{N}$ $\varphi_n:\mathbf{R}\to[0,\infty]$ is a convex, even, not identically equal to zero function vanishing at zero and the function $t\to\varphi_n(tu)$ is left continuous for fixed $u>0$. We define a modular on the family of all sequences $x=(x_n)$ of real numbers by the following formula

$$I_\varphi(x)=\sum_{n=1}^{\infty}\varphi_n(x_n).$$

1.2. Definition. The linear set

$$l_\varphi=\{x=(x_n):\exists_{a>0}I_\varphi(ax)<\infty\}$$

equipped with so called Luxemburg norm

$$\|x\|_\varphi=\inf\{k>0:I_\varphi(k^{-1}x)\le 1\}$$

is said to be Musielak–Orlicz sequence space.

δ_2^0–condition. We say that $\varphi = (\varphi_n)$ satisfies the δ_2^0–condition if there are constants a,k, an integer m and a sequence (c_n) of non-negative real numbers such that

$$\sum_{n=1}^{\infty} c_n < \infty \quad \text{and} \quad \varphi_n(2u) \le k\varphi_n(u) + c_n$$

for all $n \ge m$ and $u \in \mathbf{R}$ with $\varphi_n(u) \le a$.

Define

$$p_i(u) = \begin{cases} 0 & \text{if } \varphi_i(u) = 0 \\ \infty & \text{if } \varphi_i(u) = \infty \\ \text{left derivative of } \varphi_i(u), & \text{otherwise.} \end{cases}$$

It is easy to notice that for every $i \in \mathbf{N}$ $p_i(u)$ is nondecreasing and

$$\varphi_i(u) = \int_0^{|u|} p_i(t)\, dt.$$

Put

$$a_i = \sup\{u > 0 : \varphi_i(u) \le 1\} \quad (i = 1, 2, \dots).$$

1.3. Lemma. *If the function $\varphi = (\varphi_n)$ does not satisfy the δ_2^0-condition, then an element $x \in S(l_\varphi)$ can be found such that for every $\varepsilon \in (0, \frac{1}{2})$ we have*

$$I_\varphi[(1+\varepsilon)x] = \infty \quad \text{and} \quad I_\varphi[(1-\varepsilon)x] \le \frac{1}{2}.$$

PROOF : We will construct $x \in S(l_\varphi)$ with desirable properties. Analyzing the proof of Th. 1.1 from [**2**], we conclude that if φ does not satisfy δ_2^0–condition then there is a sequence $y = (y_i) \in S(l_\varphi)$ such that $\varphi_i(2y_i) < \infty$ $(i = 1, 2, \dots)$, $I_\varphi(y) \le 1$ and $I_\varphi(2y) = \infty$. Put

$$k_0 = \sup\{k : I_\varphi(ky) < \infty\}.$$

Obviously, $1 \le k_0 < 2$. Denote $z = k_0 y$. If for every $k < 1$ we have $I_\varphi(kz) \le \frac{1}{2}$ then we can put $x = z$ and such element x has properties from the thesis of the lemma. Otherwise, there is a number $k_1 < 1$ such that

$$I_\varphi(k_1 z^{(1)}) = \sum_{i=N_1}^{\infty} \varphi_i(k_1 z_i) \le \frac{1}{4},$$

where $z^{(1)} = (0, 0, \dots, 0, \check{z}_{N_1+1}, \dots)$.

Now, if for every $k < 1$, we have $I_\varphi(kz^{(1)}) \le \frac{1}{2}$ then putting $x = z^{(1)}$ we obtain x with desirable properties. Otherwise, there exists a number $k_2 > \frac{k_1+1}{2}$ such that $I_\varphi(k_2 z^{(1)}) > \frac{1}{2}$. Since $I_\varphi(k_2 z^{(1)}) < \infty$, then $N_2 > N_1$ can be found such that

$$I_\varphi(k_2 z^{(2)}) = \sum_{i=N_2}^{\infty} \varphi_i(k_2 z_i) \le 2^{-3},$$

where $z^{(2)} = (0, 0, \dots, 0, z_{N_2}, z_{N_2+1}, \dots)$.

Repeating the above argumentation, we arrive at the conclusion that either there exists a number i such that $I_\varphi(kz^{(i)}) \le \frac{1}{2}$ for every $k < 1$ and then putting $x = z^{(i)}$ we obtain the thesis of the lemma, or otherwise there are two sequences (N_i) and (k_i) such that $N_i \uparrow \infty$ and $k_i \uparrow 1$ as $i \to \infty$. In the second case we define

$$x = (0, 0, \dots, 0, k_1 z_{N_1}, \dots, k_1 z_{N_2-1}, k_2 z_{N_2}, \dots, k_i z_{N_i}, k_i z_{N_i+1}, \dots).$$

Then

$$I_\varphi(x) \le I_\varphi(k_1 z^{(1)}) + I_\varphi(k_2 z^{(2)}) + \dots \le \sum_{i=2}^{\infty} 2^{-i} = \frac{1}{2},$$

So $\|x\|_\varphi \le 1$. On the other hand for any $\varepsilon \in (0, \frac{1}{2})$ there exists i_0 such that $k_{i_0} > \frac{1}{1+\varepsilon}$. Consequently, we have

$$I_\varphi[(1+\varepsilon)x] \ge I_\varphi\left[k_{i_0}(1+\varepsilon)z^{(i_0)}\right] = \infty,$$

so $\|x\|_\varphi$ can not be less then 1. Reassuming, we have that $\|x\|_\varphi = 1$ and for every $\varepsilon \in (0, \frac{1}{2})$ $\quad I_\varphi[(1+\varepsilon)x] = \infty$ and $I_\varphi[(1-\varepsilon)x] \le \frac{1}{2}$, what finishes the proof. ∎

2. Main Result.

2.1. Theorem. *The Musielak–Orlicz sequence space l_φ is differentiable if and only if the following conditions are satisfied:*

(i) *The function $\varphi = (\varphi_n)$ satisfies the δ_2^0-condition.*

(ii) *There do not exist two positive integers n_1 and n_2 such that*

$$\varphi_{n_1}(a_{n_1}) + \varphi_{n_2}(a_{n_2}) \le 1 \quad \textit{and} \quad \varphi_{n_1}(a_{n_1}) > 0,\ \varphi_{n_2}(an_2) > 0$$

(iii) *The left derivative $p_i(u)$ of $\varphi_i(u)$ is continuous for $0 < |u| < a_i$ $\quad (i = 1, 2, \dots)$.*

PROOF of necessity: Suppose the Musielak–Orlicz sequence space l_φ is differentiable and the function $\varphi = (\varphi_i)$ does not satisfy δ_2^0-condition. Then we can divide a sequence (n) of all natural numbers into two subsequences (n_k) and (m_l) possessing the following properties:

a) $\{n_k : k \in \mathbb{N}\} \cap \{m_l : l \in \mathbb{N}\} = \emptyset$

b) $\{n_k : k \in \mathbb{N}\} \cup \{m_l : l \in \mathbb{N}\} = \mathbb{N}$,

c) $\varphi^{(1)} = (\varphi_{n_k})_{k\in\mathbb{N}}$ and $\varphi^{(2)} = (\varphi_{m_l})_{l\in\mathbb{N}}$ do not satisfy δ_2^0-condition.

Applying Lemma 1.3 we can find $x^{(1)} \in S(l_{\varphi(1)})$ and $x^{(2)} \in S(l_{\varphi(2)})$ such that

$$I_{\varphi(1)}\left[(1+\varepsilon)x^{(1)}\right] = \infty, \quad I_{\varphi(1)}\left[(1+\varepsilon)x^{(2)}\right] = \infty,$$
$$I_{\varphi(1)}\left[(1-\varepsilon)x^{(1)}\right] \le \frac{1}{2}, \quad I_{\varphi(2)}\left[(1-\varepsilon)x^{(2)}\right] \le \frac{1}{2}$$

for any $\varepsilon \in (0, \frac{1}{2})$. Denoting $x^{(1)} = (x_{n_1}, x_{n_2}, \dots)$ and $x^{(2)} = (x_{m_1}, x_{m_2}, \dots)$ we define

$$x = (x_1, x_2, \dots) \quad \text{and} \quad y = (y_1, y_2, \dots),$$

where

$$y_i = \begin{cases} 0 & \text{if} \quad i \in \{m_l : l \in \mathbb{N}\} \\ x_i & \text{if} \quad i \in \{n_k : k \in \mathbb{N}\}. \end{cases}$$

Then we have

$$\begin{aligned} I_\varphi[(1+\varepsilon)x] &\le I_{\varphi(1)}\Big[(1+\varepsilon)x^{(1)}\Big] = \infty, \\ I_\varphi[(1+\varepsilon)y] &= I_{\varphi(1)}\Big[(1+\varepsilon)x^{(1)}\Big] = \infty, \\ I_\varphi[(1-\varepsilon)x] &= I_{\varphi(1)}\Big[(1+\varepsilon)x^{(1)}\Big] + I_{\varphi(2)}\Big[(1-\varepsilon)x^{(2)}\Big] \le \frac{1}{2} + \frac{1}{2} = 1, \\ I_\varphi[(1-\varepsilon)y] &= I_{\varphi(1)}\Big[(1-\varepsilon)x^{(1)}\Big] \le \frac{1}{2} \end{aligned}$$

for every $\varepsilon \in (0, \frac{1}{2})$. Hence $x \in S(l_\varphi)$ and $y \in S(l_\varphi)$. Further, for each $\lambda > 0$ we have

$$I_\varphi\Big(\frac{x+\lambda y}{1+\frac{\lambda}{2}}\Big) \ge I_{\varphi(1)}\Big(\frac{x+\lambda y}{1+\frac{\lambda}{2}}\Big) = I_{\varphi(1)}\Big(\frac{1+\lambda}{1+\frac{\lambda}{2}}x^{(1)}\Big) = \infty,$$

because $(1+\lambda)/(1+\frac{\lambda}{2}) > 1$. This means that $\|x+\lambda y\|_\varphi \ge 1 + \frac{\lambda}{2}$. Therefore

$$\operatorname{grad}(x,y) = \lim_{\lambda \to 0_+} \frac{\|x+\lambda y\|_\varphi - \|x\|_\varphi}{\lambda} \ge \lim_{\lambda \to 0_+} \frac{1+\frac{\lambda}{2}-1}{\lambda} = \frac{1}{2}.$$

On the other hand, for $\lambda < 0$ we have

$$I_\varphi\Big(\frac{x+\lambda y}{1+\frac{\lambda}{3}}\Big) \ge I_{\varphi(2)}\Big(\frac{x+\lambda y}{1+\frac{\lambda}{3}}\Big) = I_{\varphi(2)}\Big(\frac{1}{1+\frac{\lambda}{3}}x^{(2)}\Big) = \infty$$

because $1/(1+\frac{\lambda}{3}) > 1$. Thus $\|x+\lambda y\|_\varphi \ge 1 + \frac{\lambda}{3}$ and

$$\operatorname{grad}(x,y) = \lim_{\lambda \to 0_-} \frac{\|x+\lambda y\|_\varphi - \|x\|_\varphi}{\lambda} \le \lim_{\lambda \to 0_-} \frac{1+\frac{\lambda}{3}-1}{\lambda} = \frac{1}{3}.$$

It proves that the gradient grad (x, y) does not exist what implies that the space l_φ can not be differentiable. This contradiction completes proof of (i).

Now we will prove the necessity of the condition (ii). To this end suppose that the Musielak–Orlicz space l_φ is differentiable and there exist two positive integers n_1 and n_2 ($n_1 < n_2$) such that

$$\varphi_{n_1}(a_{n_1}) + \varphi_{n_2}(a_{n_2}) \le 1 \quad \text{and} \quad \varphi_{n_1}(a_{n_1}) > 0, \quad \varphi_{n_2}(a_{n_2}) > 0.$$

Define

$$x = (0, \dots, 0, a_{n_1}, 0, \dots 0, a_{n_2}, 0, \dots)$$
$$y = (0, \dots, 0, a_{n_1}, 0, \dots).$$

It is easy to verify that $\|x\|_\varphi = 1$ and $\|y\|_\varphi = 1$. For any $\lambda > 0$, we have

$$I_\varphi\left(\frac{x+\lambda y}{1+\lambda}\right) = \varphi_{n_1}(a_{n_1}) + \varphi_{n_2}(\frac{1}{1+\lambda}a_{n_2}) \le 1$$

so $\|x+\lambda y\|_\varphi \le 1+\lambda$. But, for any $0 < k < 1+\lambda$, we get

$$I_\varphi\left(\frac{x+\lambda y}{k}\right) \ge \varphi_{n_1}\left(\frac{1+\lambda}{k}a_{n_1}\right) + \varphi_{n_2}\left(\frac{1}{1+\lambda}a_{n_2}\right) \ge \varphi_{n_1}\left(\frac{1+\lambda}{k}a_{n_1}\right) > 1,$$

i.e. $\|x+\lambda y\|_\varphi \ge 1+\lambda$. Hence, $\|x+\lambda y\|_\varphi = 1+\lambda$. Therefore,

$$\lim_{\lambda\to 0_+} \frac{\|x+\lambda y\|_\varphi - \|x\|_\varphi}{\lambda} = \lim_{\lambda\to 0_+} \frac{1+\lambda-1}{\lambda} = 1.$$

On the other hand, for $\lambda < 0$ we have $1 > 1+\frac{\lambda}{2} > 1+\lambda$ and

$$I_\varphi\left(\frac{x+\lambda y}{1+\frac{\lambda}{2}}\right) = \varphi_{n_1}\left(\frac{1+\lambda}{1+\frac{\lambda}{2}}a_{n_1}\right) + \varphi_{n_2}\left(\frac{1}{1+\frac{\lambda}{2}}a_{n_2}\right) \ge \varphi_{n_2}\left(\frac{1}{1+\frac{\lambda}{2}}a_{n_2}\right) > 1$$

so $\|x+\lambda y\|_\varphi > 1+\frac{\lambda}{2}$. Consequently,

$$\lim_{\lambda\to 0_-} \frac{\|x+\lambda y\|_\varphi - \|x\|_\varphi}{\lambda} \le \lim_{\lambda\to 0_-} \frac{1+\frac{\lambda}{2}-1}{\lambda} = \frac{1}{2}.$$

Thus,

$$\lim_{\lambda\to 0} \frac{\|x+\lambda y\|_\varphi - \|x\|_\varphi}{\lambda}$$

does not exist. This contradiction proves the necessity of (ii).

For the proof of necessity of (iii) let us assume that the space l_φ is differentiable and that exist a natural number N and a real number u such that $0 < u < a_N$ and $p_N(.)$ is not continuous at the point u. We can choose a sequence of real numbers (u_n) such that $0 < u_i < a_i$ for $i \ne N$ and

$$I_\varphi(x) = \sum_{i\ne N} \varphi_i(u_i) + \varphi_N(u) = 1, \tag{1}$$

where $x = (u_1, u_2, \dots, u_{N-1}, u, u_{N+1}, \dots)$. Then, by (i), $x \in S(l_\varphi)$.

Let $y = (0, \dots, 0, a_N, 0, \dots)$. Obviously, $I_\varphi(y) \le 1$ and $\|y\|_\varphi = 1$.

Denote $k_\lambda = \|x + \lambda y\|_\varphi$. It is easy to notice, that $k_\lambda > 1$ for $\lambda > 0$. First, we will prove without δ_2^0-condition that

$$I_\varphi(\frac{x+\lambda y}{k_\lambda}) = 1. \tag{2}$$

To this end suppose $I_\varphi(\frac{x+\lambda y}{k_\lambda}) < 1$ and denote $\varepsilon = 1 - I_\varphi(\frac{x+\lambda y}{k_\lambda})$. Since

$$\sum_{i\neq N} \varphi_i(\frac{u_i}{k_\lambda}) \leq \sum_{i\neq N} \varphi_i(u_i) < I_\varphi(x) \leq 1,$$

so there is a natural number $N_0 > N$ such that for every $k_\lambda > k > 1$ we have

$$\sum_{i=N_0+1}^{\infty} \varphi_i(\frac{u_i}{k}) < \frac{\varepsilon}{3}.$$

Further

$$\sum_{i=1,i\neq N}^{N_0} \varphi_i(\frac{u_i}{k_\lambda}) + \varphi_N(\frac{u+\lambda a_N}{k_\lambda}) \leq 1 - \varepsilon.$$

Since $k_\lambda > 1$ and $u/a_N < 1$, then $\lambda > 0$ can be found such that $\lambda < k_\lambda - u/a_N$. For λ defined in this manner, we have $(u + \lambda a_N)/k_\lambda < a_N$. By the continuity of φ_i $(i = 1, 2, \dots)$ on the interval $(0, a_i)$, there is $k - \lambda > k_\varepsilon > 1$ such that

$$\sum_{i=1,i\neq N}^{N_0} \varphi_i(\frac{u_i}{k_\varepsilon}) + \varphi_N(\frac{u+\lambda a_N}{k_\varepsilon}) < 1 - \frac{2}{3}\varepsilon.$$

Hence

$$\sum_{i\neq N} \varphi_i(\frac{u_i}{k_\varepsilon}) + \varphi_N(\frac{u+\lambda a_N}{k_\varepsilon}) < 1 - \frac{2}{3} + \sum_{i=N_0+1}^{\infty} \varphi_i(\frac{u_i}{k_\varepsilon}) < 1 - \frac{1}{3}\varepsilon < 1,$$

which contradicts the definition of k_λ. This finishes the proof of equality (2). From (2) we obtain

$$\sum_{i\neq N} \varphi_i(\frac{u_i}{k_\lambda}) + \varphi_N(\frac{u}{k_\lambda} + \frac{\lambda a_N}{k_\lambda}) = 1. \tag{3}$$

Hence

$$\varphi_N(\frac{u}{k_\lambda} + \frac{\lambda a_N}{k_\lambda}) = 1 - \sum_{i\neq N} \varphi_i(\frac{u_i}{k_\lambda}) > 1 - \sum_{i\neq N} \varphi_i(u_i) = \varphi_N(u).$$

Thus, by the monotonity of φ_N, we have

$$\frac{u}{k_\lambda} + \frac{\lambda a_N}{k_\lambda} > u.$$

Therefore, applying (1) and (3), we get

$$\sum_{i\neq N}\varphi_i(u_i)-\sum_{i\neq N}\varphi_i(\frac{u_i}{k_\lambda})=\varphi_N(\frac{u}{k_\lambda}+\frac{\lambda a_N}{k_\lambda})-\varphi_N(u).$$

This gives

$$\sum_{i\neq N}\int_{u_i/k_\lambda}^{u_i} p_i(t)\,dt=\int_u^{u/k_\lambda+(\lambda/k_\lambda)a_N} p_N(t)\,dt.$$

Denote

$$\lim_{t\to v_-} p_i(t)=P_i^-(v),\quad \lim_{t\to v_+} p_i(t)=P_i^+(v)\quad (i=1,2,\dots).$$

Since $p_i(\cdot)$ is non–decreasing function, we have

$$\int_{\frac{u_i}{k_\lambda}}^{u_i} p_i(t)\,dt\le P_i^-(u_i)(u_i-\frac{u_i}{k_\lambda})$$

and

$$\int_u^{(u+\lambda a_N)/k_\lambda} p_N(t)\,dt\ge P_N^+(u)\left[\frac{1}{k_\lambda}(u+\lambda a_N)-u\right].$$

Consequently,

$$\sum_{i\neq N}P_i^-(u_i)(u_i-\frac{u_i}{k_\lambda})\ge P_N^+(u)\left[\frac{1}{k_\lambda}(u+\lambda a_N)-u\right].$$

The above inequality is equivalent to the following one

$$(k_\lambda-1)\left[\sum_{i\neq N}P_i^-(u_i)u_i+P_N^+(u)u\right]\ge\lambda a_N P_N^+(u),$$

so

(4) $$\frac{k_\lambda-1}{\lambda}\ge\frac{a_N P_N^+(u)}{\sum_{i\neq N}u_iP_i^-(u_i)+uP_N^+(u)}$$

for $\lambda>0$.

Now, we will consider the case of $\lambda<0$. Then we have $k_\lambda=\|x+\lambda y\|_\varphi\le 1$. Repeating this same argumentation as above we obtain the equality

$$\sum_{i\neq N}\int_{u_i}^{\frac{u_i}{k_\lambda}} p_i(t)\,dt=\int_{(u+\lambda a_N)/k_\lambda}^{u} p_N(t)\,dt.$$

Since $k_\lambda \leq 1$ and $p_i(\cdot)$ is non-decreasing, so

$$\sum_{i \neq N} \int_{u_i}^{\frac{u_i}{k_\lambda}} p_i(t)\,dt \geq \sum_{i=N} P_i^+(u_i)\left[\frac{u_i}{k_\lambda} - u_i\right]$$

and

$$\int_{\frac{1}{k_\lambda}(u+\lambda a_N)}^{u} p_N(t)\,dt \leq P_N^-(u)\left[u - \frac{1}{k_\lambda}(u+\lambda a_N)\right].$$

Thus

$$\sum_{i \neq N} P_i^+(u_i)\left[\frac{u_i}{k_\lambda} - u_i\right] \leq P_N^-(u)\left[u - \frac{1}{k_\lambda}(u+\lambda a_N)\right].$$

Hence

$$\frac{k_\lambda - 1}{\lambda} \leq \frac{a_N P_N^-(u)}{\sum_{i\neq N} u_i P_i^+(u_i) + u P_N^-(u)} \tag{5}$$

Since $p_N(.)$ is not continuous at u, then $P_N^-(u) < P_N^+(u)$. It implies that

$$\frac{a_N P_N^-(u)}{\sum_{i\neq N} u_i P_i^+(u_i) + u P_N^-(u)} < \frac{a_N P_N^+(u)}{\sum_{i\neq N} u_i P_i^-(u_i) + u P_N^+(u)},$$

so, by (4) and (5),

$$\lim_{\lambda\to 0} \frac{\|x+\lambda y\|_\varphi - \|x\|_\varphi}{\lambda} = \lim_{\lambda\to 0} \frac{k_\lambda - 1}{\lambda}$$

does not exist. This contradiction completes the proof of necessity of (iii).

Proof of sufficiency: Let $x \in S(l_\varphi)$. By the assumption (ii), at most one i-th coordinate can be equal to a_i or $-a_i$. Consider two cases:

I. We will show differentiability of the norm at x with exactly one (say N-th) coordinate equal to a_N or $-a_N$, i.e.

$$x = (u_1, u_2, \dots), \quad \text{where} \quad |u_N| = a_N \quad \text{and} \quad |u_i| < a_i \text{ for } i \neq N.$$

II. We will prove differentiability of the norm at other points x from $S(l_\varphi)$, i.e.

$$x = (u_1, u_2, \dots) \text{ and } |u_i| < a_i \text{ for every } i \in \mathbb{N}.$$

I. Let $y = (y_1, y_2, \dots) \in S(l_\varphi)$. First we will consider the case $\lambda u_N y_N < 0$. For

$$0 < K < 1 + \lambda\frac{y_N}{u_N}$$

we have

$$|\frac{u_N + \lambda y_N}{K}| > a_N.$$

Therefore

$$I_\varphi(\frac{x+\lambda y}{K}) = \sum_{i\neq N} \varphi_i(\frac{u_i+\lambda y_i}{K}) + \varphi_N(\frac{u_N+\lambda y_N}{K}) \geq \varphi_N(\frac{u_N+\lambda y_N}{K} \geq 1$$

i.e. $\|x+\lambda y\|_\varphi \geq 1 + \lambda y_N/u_N$.

Now we will give an upper estimation of the norm of element $x + \lambda y$. To this end, let $1 > M > 1 + \lambda y_N/u_N$.By (i), there exist constants a, k an integer m and a sequence (c_n) non-negative real numbers such that

$$\varphi_n(2u) \leq k\varphi_n(u) + c_n \text{ and } \sum_{i=1}^{\infty} c_i < \infty$$

for all $n \geq m$ and $u \in \mathbf{R}$, provided $\varphi_n(u) \leq a$. Fix an $\varepsilon > 0$. let N_1, N_2 and N_3 be natural numbers greater then N such that

$$\sum_{i=N_1}^{\infty} c_i < min\{\frac{a}{2}, \frac{\varepsilon}{k+1}\}, \tag{6}$$

$$\sum_{i=N_2}^{\infty} \varphi_i(u_i) < \min\{\frac{1}{k}(a - \sum_{j=N_1}^{\infty} c_j), \frac{\varepsilon}{4k^2}\} \tag{7}$$

and

$$\sum_{i=N_3}^{\infty} \varphi_i(y_i) < \min\{\frac{1}{k}(a - \sum_{j=N_1}^{\infty} c_j), \frac{\varepsilon}{4k^2}\}. \tag{8}$$

Using δ_2^0-condition, we have

$$\sum_{i=N_2}^{\infty} \varphi_i(2u_i) < a \text{ and } \sum_{i=N_3}^{\infty} \varphi_i(2y_i) < a.$$

Moreover, we will show that there is a natural number $N_4 > N$ such that

$$|\frac{u_i+\lambda y_i}{M}| < a_i \tag{9}$$

for $i \geq N_4$ and every $|\lambda| < \frac{1}{4}$ with $\lambda u_N y_N < 0$.

Since $M > 1 + \lambda y_N/u_N > 1 - |\lambda|$ and $|u_i + \lambda y_i| \leq |u_i| + \lambda|a_i|$, then inequality (9) is true provided there is $N_4 > N$ such that

$$|u_i| < a_i(1 - 2|\lambda|)$$

for $i \geq N_4$ and $|\lambda| < \frac{1}{4}$ with $\lambda u_N y_N < 0$. Further, note that $\varphi_i(a_i)$ can be equal to zero only for finite number of a_i. Indeed, if $i > m$ and $\varphi_i(a_i) = 0 < a$ then, by δ_2^0-condition, we get

$$\varphi_i(2a_i) \leq k\varphi_i(a_i) + c_i = c_i.$$

But (c_i) is convergent to zero, so without loss of generality we can assume that $c_i < 1$ for $i > m$. Thus $\varphi_i(2a_i) < 1$, what contradicts the definition of a_i. Therefore, we can assume that $\varphi_i(a_i) > 0$ for $i > m$. Moreover, by the assumption (ii), we have $\varphi_i(a_i) \geq \frac{1}{2}$ $(i > m)$ except at most one integer, say $i = n_0$. Further, by δ_2^0–condition $I_\varphi(2x) < \infty$. Hence there is an integer $N_4 > n_0$ such that

$$\sum_{i=N_4}^{\infty} \varphi_i(2u_i) < \frac{1}{2} \leq \varphi_j(a_j)$$

for $j = N_4, N_4 + 1, \ldots$, so

$$\varphi_i(2u_i) < \varphi_i(a_i) \quad (i = N_4, N_4 + 1, \ldots).$$

Consequently, by the definition of Young's function, we obtain

$$2|u_i| < a_i \quad (i = N_4, N_4 + 1, \ldots).$$

This implies that

$$|u_i| < (1 - 2|\lambda|)a_i$$

for $i \geq N_4$ and $|\lambda| < \frac{1}{4}$ with $\lambda u_N y_N < 0$. Thus (9) holds for every $|\lambda| < \frac{1}{4}$ with $\lambda u_N y_N < 0$ and $i \geq N_4$. Taking $N_0 = \max\{N_1, N_2, N_3, N_4\}$ and using (9), δ_2^0–condition, (6), (7) and (8), we have

$$\begin{aligned}
&\sum_{i=N_0}^{\infty} \varphi_i\left(\frac{u_i + \lambda y_i}{M}\right) < \sum_{i=N_0}^{\infty} \varphi_i(2u_i + 2y_i) \leq \\
&\leq \frac{1}{2}\sum_{i=N_0}^{\infty} \varphi_i(4u_i) + \frac{1}{2}\sum_{i=N_0}^{\infty} \varphi_i(4y_i) \leq \frac{k}{2}\sum_{i=N_0}^{\infty} [\varphi_i(2u_i) + \varphi_i(2y_i)] + \sum_{i=N_0}^{\infty} c_i \leq \\
&\leq \frac{k^2}{2}\sum_{i=N_0}^{\infty} \varphi_i(u_i) + \frac{k}{2}\sum_{i=N_0}^{\infty} c_i + \frac{k^2}{2}\sum_{i=N_0}^{\infty} \varphi_i(y_i) + \frac{k}{2}\sum_{i=N_0}^{\infty} c_i + \sum_{i=N_0}^{\infty} c_i = \\
&= \frac{k^2}{2}\sum_{i=N_0}^{\infty} [\varphi_i(u_i) + \varphi_i(y_i)] + (k+1)\sum_{i=N_0}^{\infty} c_i < \frac{3}{4}\varepsilon.
\end{aligned}$$

Further, for any $i = 1, 2, \ldots, N_0 - 1$ and $i \neq N$, by $|u_i| < a_i$, a real number λ_i can be found such that $\lambda_i u_N y_N < 0$ and

$$\frac{|u_i| + |\lambda_i y_i|}{1 + \lambda_i \frac{y_N}{u_N}} < a_i.$$

Denote $A = \{1, 2, \ldots, N_0 - 1\}/\{N\}$ and $\lambda_0 = \min_{i\in A}\{\lambda_i\}$. Obviously,

$$\lambda_0 u_N y_N < 0 \quad \text{and} \quad P_i^-(\frac{|u_i| + |\lambda_0 y_i|}{1 + \lambda_0 \frac{y_N}{u_N}}) < \infty \text{ for } \quad i \in A.$$

We put

$$P^- = \max_{i\in A}\{P_i^-(\frac{|u_i| + |\lambda_0 y_i|}{1 + \lambda_0 \frac{y_N}{u_N}})\}$$

and

$$\lambda_0' = \min_{i\in A}\left\{\frac{\varepsilon}{2^{i+2}P^-}(|\frac{\lambda_0 y_N}{u_N}| + |y_i|)^{-1}\right\}(-\text{sign } u_N y_N).$$

For $0 < |\lambda| < min\{|\lambda_0|, |\lambda_0'|, \frac{1}{4}\}$ with sign $\lambda = -$sign $u_N y_N$, we have

$$\sum_{i\in A}\left[\varphi_i(\frac{|u_i| + |\lambda y_i|}{M}) - \varphi_i(u_i)\right] = \sum_{i\in A} \int_{|u_i|}^{(|u_i|+|\lambda y_i|)/M} p_i(t)\,dt \leq$$

$$\leq P^- \sum_{i\in A}(\frac{|u_i| + |\lambda y_i|}{M} - |u_i|) \leq P^- \sum_{i\in A}\frac{(1-M)|u_i| + |\lambda y_i|}{M} \leq$$

$$\leq P^- \sum_{i\in A}(|\frac{y_N u_i}{a_N}| + |y_i|)|\lambda| \leq \sum_{i\in A}\frac{\varepsilon}{2^{i+2}} < \frac{1}{4}\varepsilon.$$

Moreover,

$$\varphi_N(\frac{u_N + \lambda y_N}{M}) \leq \varphi_N(a_N),$$

by previous assumptions concerning M. Reassuming, we have

$$I_\varphi(\frac{x + \lambda y}{M}) = \sum_{i\in A}\varphi_i(\frac{u_i + \lambda y_i}{M}) + \varphi_N(\frac{u_N + \lambda y_N}{M}) + \sum_{i=N_0}^{\infty}\varphi_i(\frac{u_i + \lambda y_i}{M}) \leq$$

$$\leq \sum_{i\neq N}\varphi_i(u_i) + \frac{1}{4}\varepsilon + \varphi_N(a_N) + \frac{3}{4}\varepsilon = 1 + \varepsilon$$

Since ε is arbitrary, we have

$$\|x + \lambda y\|_\varphi \leq 1 + \lambda\frac{y_N}{u_N}.$$

Therefore

$$\|x + \lambda y\|_\varphi = 1 + \lambda\frac{y_N}{u_N}.$$

For $\lambda u_N y_N \geq 0$ one can be proved analogously the same equality. Thus, if $x = (u_1, u_2, \dots) \in S(l_\varphi)$, $|u_i| < a_i$ for $i \neq N$ and $|u_N| = a_N$, then

$$\text{grad}\,(x, y) = \lim_{\lambda \to 0} \frac{\|x + \lambda y\|_\varphi - \|x\|_\varphi}{\lambda} = \frac{y_N}{u_N}$$

for every $y \in S(l_\varphi)$. This completes proof of the case I.

II. Let $x = (u_1, u_2, \dots) \in S(l_\varphi)$ with $|u_i| < a_i$ for every $i \in \mathbb{N}$. Since φ satisfies δ_2^0-condition, so, by Theorem 1.1 from [2] $I_\varphi(x) = 1$. Fix $y \in S(l_\varphi)$. Denote $k_\lambda = \|x + \lambda y\|_\varphi$. We will show that there is $0 < \lambda_0 < \frac{1}{4}$ such that

$$|\frac{u_i + \lambda y_i}{k_\lambda}| < a_i \tag{10}$$

for $|\lambda| < \lambda_0$ and every $i \in \mathbb{N}$.

Since $k_\lambda \geq \|x\|_\varphi - \|\lambda y\|_\varphi = 1 - |\lambda|$ and $|u_i + \lambda y_i| \leq |u_i| + |\lambda| a_i$, using this same argumentation as in proof of inequality (9), we conclude that there is a natural number N such that (10) is satisfied for $i \geq N$ and $\lambda_0 = \frac{1}{4}$. Further for every $1 \leq i < N$ there is $\lambda_i > 0$ such that

$$|\frac{u_i + \lambda y_i}{k_\lambda}| < a_i \quad \text{for } |\lambda| < \lambda_i.$$

Thus, putting

$$\lambda_0 = \min\{|\lambda_1|, |\lambda_2|, \dots, |\lambda_N|, \frac{1}{4}\},$$

the inequality (10) is proved.

Moreover,

$$\|\frac{x + \lambda y}{k_\lambda}\|_\varphi = 1$$

then, by δ_2^0 - condition,

$$I_\varphi(\frac{x + \lambda y}{k_\lambda}) = 1$$

(see [2], Th. 1.1). Hence, we have

$$\sum_{i=1}^{\infty} \left[\varphi_i(\frac{u_i + \lambda y_i}{k_\lambda}) - \varphi_i(u_i)\right] = 0,$$

i.e.

$$\sum_{i=1}^{\infty} \int_{u_i}^{\frac{u_i + \lambda y_i}{k_\lambda}} p_i(t)\, dt = 0.$$

In view of (iii) there exists a real number v_i between u_i and $\frac{u_i + \lambda y_i}{k_\lambda}$ such that

$$\sum_{i=1}^{\infty} p_i(v_i)(\frac{u_i + \lambda y_i}{k_\lambda} - u_i) = 0.$$

It follows that

$$\sum_{i=1}^{\infty} p_i(v_i)\frac{1-k_\lambda}{k\lambda}u_i + \sum_{i=1}^{\infty} p_i(v_i)\frac{\lambda}{k_\lambda}y_i = 0.$$

Hence

$$\frac{k_\lambda - 1}{\lambda} = \frac{\sum_{i=1}^{\infty} p_i(v_i)y_i}{\sum_{i=1}^{\infty} p_i(v_i)u_i}$$

for every $|\lambda| < \lambda_0$. Therefore

$$\text{grad }(x,y) = \lim_{\lambda\to 0}\frac{\|x+\lambda y\|_\varphi - \|x\|_\varphi}{\lambda} = \lim_{\lambda\to 0}\frac{k_\lambda - 1}{\lambda} = \frac{\sum_{i=1}^{\infty} p_i(u_i)y_i}{\sum_{i=1}^{\infty} p_i(u_i)u_i},$$

what completes the proof of Theorem 2.1. ∎

Analysing the proof of sufficiency of Theorem 2.1 it is easy to conclude the following:

2.2.Corollary. *If conditions (i),(ii) and (iii) are satisfied then for every $x = (u_1, u_2, \dots)$ and $y = (y_1, y_2, \dots)$ from unite sphere $S(l_\varphi)$ we have*

$$grad\ (x,y) = \frac{\sum_{i=1}^{\infty} p_i(u_i)y_i}{\sum_{i=1}^{\infty} p_i(u_i)u_i}.$$

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