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Dedicated to the memory of Svatopluk Fučík

Abstract. We extend some regularity results of Giaquinta–Modica obtained for weak solutions of certain equations of the type of the stationary Navier–Stokes system to local minimizers of quadratic variational integrals in a class of functions with prescribed divergence.

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0. Introduction.

In [GM] Giaquinta–Modica study nonlinear equations of the type of the stationary Navier–Stokes system

$$(0.1)\qquad \begin{cases} a) & \operatorname{div} u = g \quad \text{and} \\ b) & \int_\Omega A^i_\alpha(\cdot,u,Du)\cdot D^i_\alpha\zeta\,dx = \int_\Omega B^i(\cdot,u,Du)\zeta^i\cdot dx \\ & \text{for all solenoidal vector–fields } \zeta\in \overset{\circ}{H}{}^{1,2}(\Omega,\mathbf{R}^n) \end{cases}$$

and prove (partial) regularity theorems imposing natural structure conditions on g, A^i_α and B^i. Especially the growth of B^i in Du is subquadratic;
hence $-D_\alpha(A_\alpha(\cdot,u,Du)) - B(\cdot,u,Du)$ is in the dual space $H^{-1}(\Omega)$ vanishing on solenoidal test–vectorfields and a well–known decomposition theorem (see [A]) shows that

$$(0.2)\qquad -D_\alpha(A_\alpha(\cdot,u,Du)) - B(\cdot,u,Du)) = \operatorname{grad} p$$

holds in the weak sense for a suitable pressure function $p\in L^2(\Omega)$. Since the pressure p is a controllable term, Giaquinta–Modica replace (0.1) b) by (0.2) and apply the methods developed in the study of (nonlinear) elliptic systems (compare [G] for a survey) to prove their theorems.

On the other hand systems of the form (0.1) with $B^i(\cdot,u,Du)$ of quadratic growth naturally arise minimizing quadratic functionals

$$F(u) := \int_\Omega f(\cdot,u,Du)\,dx$$

in the class of admissible functions

$$\mathsf{K} := \{w \in H^{1,2}(\Omega, \mathsf{R}^n) : w = u_0 \quad \text{on } \partial\Omega, \text{div } w = g\}.$$

The purpose of this note is to prove a partial regularity theorem for F–minimizers in the class K concentrating on the quasilinear model case

$$F(u) = \int_\Omega A^{ij}_{\alpha\beta}(\cdot, u) D_\alpha u^i D_\beta u^j \, dx$$

We then show that $\mathsf{H}^{n-2}(\text{Sing } u) = 0$ holds for the interior singular set of a minimizer u.

1. Notations and statement of the result.

Let Ω be a bounded domain in $\mathsf{R}^n, n \geq 2$, and suppose that we are given a function $g : \Omega \to \mathsf{R}$ with $g \in L^s(\Omega)$ for some $s > n$. On the Sobolev space $H^{1,2}(\Omega, \mathsf{R}^n)$ we define the functional

$$F(u, \Omega) := \int_\Omega A^{ij}_{\alpha\beta}(\cdot, u) D_\alpha u^i D_\beta u^j \, dx$$

(indices repeated twice are summed from 1 to n) with uniformly continuous coefficients

$$A^{ij}_{\alpha\beta} : \Omega \times \mathsf{R}^n \to \mathsf{R}, A^{ij}_{\alpha\beta} = A^{ij}_{\beta\alpha},$$

satisfying

$$(1.1) \qquad \begin{cases} |A^{ij}_{\alpha\beta}(x, y)| \leq L \\ A^{ij}_{\alpha\beta}(x, y) Q^i_\alpha Q^j_\beta \geq \lambda |Q|^2 \end{cases}$$

for all $x \in \overline{\Omega}, y \in \mathsf{R}^n$, $Q \in \mathsf{R}^{n \times n}$ with positive constants L, λ. For $u \in H^{1,2}(\Omega, \mathsf{R}^n)$ let

$$\text{Reg}(u) = \{x \in \Omega \mid u \text{ is continuous in a neighborhood of } x\},$$
$$\text{Sing}(u) = \Omega - \text{Reg}(u)$$

denote the interior regular and singular set.

Theorem. *Suppose* $u \in \mathcal{C} := H^{1,2}(\Omega, \mathsf{R}^n) \cap \{w : \text{div} w = g\}$ *has the property* $F(u, \Omega) \leq F(v, \Omega)$ *for all* $v \in \mathcal{C}$ *such that* $\text{spt}(u - v) \subset\subset \Omega$. *Then* $\mathsf{H}^{n-2}(\text{Sing } u) = 0$.

Remarks.

1) As we shall see below a point $x \in \Omega$ is regular for the minimizer iff there is a ball $B_r(x) \subset \Omega$ such that

$$r^{2-n} \int_{B_r(x)} |Du|^2 \, dz < \varepsilon_0$$

holds, ε_0 denoting an absolute constant depending on the data.

2) If g and the coefficients of the functional are sufficiently smooth it is not hard to see that higher regularity theorems hold on $\Omega - \text{Sing}(u)$. We refer to [G] and [GM], the details are left to the reader.

2. Proof of the theorem.

The main ingredient is a Caccioppoli–type inequality.

Lemma 1. *Suppose that* $u \in C$ *is a local minimizer under the side condition* $\operatorname{div} u = g$. *Then for any ball* $B_\delta(x) \subset \Omega$

$$
\tag{2.1}
\begin{aligned}
\fint_{B_{\delta/2}(x)} |Du|^2\,dz \le \frac{1}{2}\fint_{B_\delta(x)} |Du|^2\,dz + \\
c_1\Big[\fint_{B_\delta(x)} g^2\,dz + \delta^{-2}\fint_{B_\delta(x)} |u-(u)_\delta|^2\,dz\Big],
\end{aligned}
$$

c_1 *being an absolute constant. Here we use* $(u)_\delta$ *to denote the mean value* $\fint_{B_{\delta(x)}} u\,dz$ *of* u *on the ball* $B_\delta(x)$.

PROOF of Lemma 1: Let $a := (u)_\delta$ and suppose that x is the origin. By Fubini's theorem u, $Du \in L^2(S_R^{n-1})$ for almost all $R \in (\delta/2, \delta)$ and we may choose a radius R such that

$$
\tag{2.2}
\begin{cases}
E(u, S_R^{n-1}) \le c_2\delta^{-1}E(u, B_\delta),\\
W(u, S_R^{n-1}) \le c_2\delta^{-1}W(u, B_\delta),
\end{cases}
$$

where we have abbreviated $E(f,\cdot) = \int |Df|^2, W(f,\cdot) = \int |f-a|^2$.

Let $\overline{u}$ denote the solution of the auxiliary variational problem

$$
\begin{cases}
\int_{B_R} |Dw|^2\,dx \to \text{Min} \qquad \text{in}\\
\{v \in H^{1,2}(B_R, \mathbb{R}^n) : v-u \in \overset{\circ}{H}{}^{1,2}(B_R, \mathbb{R}^n), \operatorname{div} v = g\}.
\end{cases}
$$

Then

$$
\int_{B_R} D\overline{u}\cdot D\zeta\,dx = 0
$$

for all $\zeta \in \overset{\circ}{H}{}^{1,2}(B_R, \mathbb{R}^n)$ $\operatorname{div}\zeta = 0$, and (compare [GM], Theorem 0.1.) there is a function $p \in L^2(B_R)$ such that

$$
\tag{2.3}
-\Delta\overline{u} = \operatorname{grad} p
$$

in the sense of distributions on the ball B_R and

$$
\tag{2.4}
\|p-(p)_R\|_{L^2(B_R)} \le c_3\|-\Delta\overline{u}\|_{H^{-1}(B_R)}
$$

with c_3 independent of B_R. Identifying $H^{-1}(B_R, \mathbb{R}^n)$ with $\overset{\circ}{H}{}^{1,2}(B_R, \mathbb{R}^n)$ via the isomorphism

$$
\Delta : \overset{\circ}{H}{}^{1,2}(B_R, \mathbb{R}^n) \to H^{-1}(B_R, \mathbb{R}^n),
$$

we see

$$
\|-\Delta\overline{u}\|_{H^{-1}(B_R)} = \|Dv\|_{L^2(B_R)},
$$

v being the unique element of $\overset{\circ}{H}{}^{1,2}(B_R, \mathbb{R}^n)$ representing $-\Delta\overline{u}$:

$$\langle -\Delta\overline{u}, \zeta\rangle = \int_{B_R} Dv \cdot D\zeta \, dx.$$

Clearly $v = \overline{u} - h$, h the harmonic extension of $\overline{u}$, hence

$$\| -\Delta\overline{u}\|_{H^{-1}(B_R)} = \|D\overline{u} - Dh\|_{L^2(B_R)} \leq 2\|D\overline{u}\|_{L^2(B_R)},$$

and (2.4) gives

$$\int_{B_R} |p - (p)_R|^2 \, dx \leq c_4 \int_{B_R} |D\overline{u}|^2 \, dx. \tag{2.5}$$

For $r \in [1/2, 1)$ let

$$\eta_r(t) := \begin{cases} 0, & 0 \leq t \leq \frac{1}{2}(3r-1)R \\ 1, & t \geq rR \\ \text{linear}, & \frac{1}{2}(3r-1)R \leq t \leq rR \end{cases}$$

and

$$v_r(x) := a + \eta_r(|x|)\left(u(R\frac{x}{|x|}) - a\right); \quad x \in B_R.$$

As test vector in (2.3) we use $\zeta := \overline{u} - v_r$ with the result (observe (2.5))

$$\int_{B_R} |D\overline{u}|^2 \, dx \leq c_5 \left[\int_{B_R} |Dv_r|^2 \, dx + \int_{B_R} |\operatorname{div}(\overline{u} - v_r)|^2 \, dx\right]. \tag{2.6}$$

For the energy of v_r we have

$$\int_{B_R} |Dv_r|^2 \, dx \leq c_6 \cdot \left\{R(1-r)E(u, S_R^{n-1}) + \frac{1}{R(1-r)} W(u, S_R^{n-1})\right\},$$

and recalling $\operatorname{div}\overline{u} = g$ we find

$$\int_{B_R} |\operatorname{div}(\overline{u} - v_r)|^2 \, dx \leq \int_{B_R} |g|^2 \, dx + \int_{B_R} |Dv_r|^2 \, dx$$

Combining these results with (2.2) and (2.6) we arrive at

$$\int_{B_R} |D\overline{u}|^2 \, dx \leq c_7 \times \left\{\int_{B_\delta} g^2 \, dx + (1-r)\int_{B_\delta} |Du|^2 \, dx + (1-r)^{-1}\delta^{-2} \cdot \right.$$
$$\left. \cdot \int_{B_\delta} |u - (u)_\delta|^2 \, dx\right\}$$

Now u is locally F-minimizing in the class C so that

$$F(u, B_R) \leq F(\overline{u}, B_R),$$

and from the structure condition (1.1) we deduce

$$\int_{B_{\delta/2}} |Du|^2\,dx \leq c_8 F(u, B_R) \leq c_9 \int_{B_R} |D\overline{u}|^2\,dx \leq$$
$$c_{10} \cdot \left\{ \int_{B_\delta} g^2\,dx + (1-r) \int_{B_\delta} |Du|^2\,dx + (1-r)^{-1} \cdot \delta^{-2} \int_{B_\delta} |u-(u)_\delta|^2\,dx \right\}.$$

Choosing r sufficiently close to 1 inequality (2.1) is established. ∎

From Lemma 1 we immediately deduce higher integrability of the gradient of a minimizer:

Lemma 2. *If $u \in C$ is a local F-minimizer, then Du is locally q-integrable for some $q > 2$ and for $B_R \subset B_{2R} \subset \Omega$*

$$(2.7)\quad \left(\fint_{B_R} |Du|^q\,dx \right)^{1/q} \leq c_{11} \cdot \left\{ \left(\fint_{B_{2R}} |Du|^2\,dx \right)^{1/2} + \left(\fint_{B_{2R}} |g|^q\,dx \right)^{1/q} \right\}.$$

Proof : combine [G], Prop.1.1, Chapter V, and Lemma 1. ∎

If the dimension n is two, then u is locally Hölder continuous: For $n \geq 3$ the proof of the Theorem can be completed following ideas of [GG]:

Fix a ball $B_R = B_R(x_0) \subset \Omega$ and consider the solution v of the problem

$$\int_{B_R(x_0)} A^{ij}_{\alpha\beta}(x_0, u_R) D_\alpha v^i D_\beta v^j =: F_0(v, B_R) \to \text{Min}$$

in the class

$$\{w \in H^{1,2}((B_R, \mathbb{R}^n) : w = u \quad \text{on } \partial B_R, \operatorname{div} w = g\}$$

From [GM], Prop. 1.12, we infer the Campanato–type–estimate

$$(2.8)\qquad \int_{B_r} |Dv|^2\,dx \leq c_{12} \cdot \left[(\frac{r}{R})^n \cdot \int_{B_R} |Dv|^2\,dx + \int_{B_R} |g - g_R|^2\,dx \right]$$

and for energy of $u - v$ we have the bound

$$\lambda \cdot \int_{B_R} |Du - Dv|^2\,dx \leq \int_{B_R} A^{ij}_{\alpha\beta}(x_0, u_R) \cdot D_\alpha(u^i - v^i) D_\beta(u^j - v^j)\,dx =$$
$$= F_0(u, B_R) - F(u, B_R) + F(u, B_R) - F(v, B_R) + F(v, B_R) - F_0(v, B_R) \leq$$
$$\leq F_0(u, B_R) - F(u, B_R) + F(v, B_R) - F_0(v, B_R).$$

The assumptions concerning the coefficients $A^{ij}_{\alpha\beta}$ imply the existence of a continuous, increasing, concave function $\omega : [0,\infty) \to [0,\infty)$ satisfying $\omega(0) = 0, \omega(t) \le L$ such that

$$|A^{ij}_{\alpha\beta}(x,y) - A^{ij}_{\alpha\beta}(\widetilde{x},\widetilde{y})| \le \omega(|x-\widetilde{x}|^2 + |y-\widetilde{y}|^2),$$

hence

$$\int_{B_R} |Du - Dv|^2\,dx \le$$
$$\le c_{13}\cdot \int_{B_R} \omega(R^2 + |u-u_R|^2)|Du|^2\,dx + c_{14}\cdot \int_{B_R} \omega(R^2 + |v-u_R|^2)|Dv|^2\,dx$$

The first integral one the right–hand side can be handled with the help of (2.7):

$$\int_{B_R} \omega(R^2 + |u-u_R|^2)|Du|^2\,dx \le$$
$$\le c_{15}\cdot\omega\left(\fint_{B_R} R^2 + |u-u_R|^2\,dx\right)^{1-2/q} \times$$
$$\times\left[\int_{B_{2R}} |Du|^2\,dx + R^{n(1-2/q)}\left(\int_{B_{2R}} |g|^q\,dx\right)^{2/q}\right]$$

Since v solves a constant–coefficient–problem it is easy to see that v satisfies Caccioppoli–type inequalities up toe the boundary (compare e.g. [GM], Theorem 2.2) which imply global higher integrability of Dv, more precisely:
$Dv \in L^{\overline{q}}(B_R)$ for some exponent $2 < \overline{q} \le q$ and

$$(2.9)\quad \left(\fint_{B_R} |Dv|^{\overline{q}}\,dx\right)^{1/\overline{q}} \le c_{16}\left\{\left(\fint_{B_R} |Dv|^2\,dx\right)^{1/2} + \left(\fint_{B_R} |Du|^{\overline{q}}\,dx\right)^{1/\overline{q}} + \right.$$
$$\left. + \left(\fint_{B_R} |g|^{\overline{q}}\,dx\right)^{1/q}\right\}$$

For simplicity we may assume $\overline{q} = q$. Then, using the minimality of v and estimate (2.7), (2.9) can be rewritten as

$$\left(\fint_{B_R} |Dv|^q\,dx\right)^{1/q} \le c_{17}\cdot\left\{\left(\fint_{B_{2R}} |Du|^2\,dx\right)^{1/2} + \left(\fint_{B_{2R}} |g|^q\,dx\right)^{1/q}\right\}$$

and implies the inequality

$$\int_{B_R} \omega(R^2 + |v-u_R|^2)\cdot|Dv|^2\,dx \le c_{18}\cdot\omega\left(\fint_{B_R} R^2 + |v-u_R|^2\,dx\right)^{1-2/q} \times$$
$$\times\left[\int_{B_{2R}} |Du|^2\,dx + R^{n(1-2/q)}\left(\int_{B_{2R}} |g|^q\,dx\right)^{2/q}\right].$$

Since

$$\fint_{B_R} |v - u_R|^2 \, dx \le c_{19} \cdot R^2 \fint_{B_R} |Du|^2 \, dx,$$

we finally arrive at

$$\int_{B_r} |Du|^2 \, dx \le c_{20} \cdot \left\{ (\frac{r}{R})^n + \psi(x_0, R) \right\} \int_{B_{2R}} |Du|^2 \, dx +$$

$$+ c_{21} \cdot \left\{ \int_{B_{2R}} |g - g_{2R}|^2 \, dx + R^n \left(\fint_{B_{2R}} |g|^q \, dx \right)^{2/q} \right\},$$

$$\psi(x_0, R) := \omega \left(c_{22} \cdot R^{2-n} \int_{B_R(x_0)} (1 + |Du|^2) \, dx \right),$$

where we have used (2.8) and the foregoing estimates for the energy of $u - v$. Recall $g \in L^s(\Omega)$ for some exponent $s > n$, therefore

$$\int_{B_{2R}} |g - g_{2R}|^2 \, dx + R^n \left(\fint_{B_{2R}} |g|^q \, dx \right)^{2/q} \le c_{23} R^{n(1-2/s)} \|g\|^2_{L^s(\Omega)}.$$

We may write $n(1 - 2/s) = n - 2 + 2\alpha$ for some $0 < \alpha < 1$ and end up with the result

$$\int_{B_r(x_0)} |Du|^2 \, dx \le c_{24} \cdot \left[(\frac{r}{R})^n + \psi(x_0, R) \right] \int_{B_{2R}(x_0)} |Du|^2 \, dx + c_{25} \cdot R^{n-2+2\alpha}$$

for all balls $B_r(x_0) \subset B_R(x_0) \subset B_{2R}(x_0) \subset \Omega$. The statement of the Theorem now follows as in [G] or [GG].

Remarks.

1) Since $Du \in L^q_{\text{loc}}(\Omega)$ for some $q > 2$ we have $\mathrm{H}^{n-q}(\operatorname{Sing} u) = 0$.
2) The case of non–uniformly continuous coefficients needs some changes which can be found in [GG].

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