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Uniform bounds for solutions of a degenerate diffusion equation with nonlinear boundary conditions

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Dedicated to the memory of Svatopluk Fučík

Abstract. This paper deals with solutions $u(x,t)$ of the degenerate parabolic equation $(\beta(u))_t = \Delta u$ in the cylinder $D \times (0,T), D \subset R^N$ bounded, $\beta(u) = |u|^m \operatorname{sign} u$ under the assumption, that on the lateral boundary nonlinear boundary conditions of the form $\partial u/\partial \nu = f(u)$, $f(u)u \le L(|u|^{\alpha+1} + 1)$, $\alpha \ge 1$, $L > 0$, are imposed. It is shown that the value of the integral

$$\sup_{0\le t\le T} \oint_\Gamma |u(s,t)|^{(N-1)(\alpha-1)+\varepsilon}\, ds$$

for positive ε is crucial for obtaining the L^∞–estimate of the solution.

Keywords: Parabolic equations, nonlinear boundary conditions, L^∞–estimate

Classification: 35K55, 35K60

Let $u(x,t)$ be a smooth function satisfying the heat equation $u_t = u_{xx}$ in the rectangle $0 < x < 1, 0 < t \le T$ and assume that $u(x,0) = u_0(x)\,(0 \le x \le 1)$, $u_x(0,t) = 0$, $u_x(1,t) = f(u,(1,t))\,(0 \le t \le T)$, $f \in C(R)$. Multiplying the equation by u^r for positive odd r and integrating we immediately derive

$$\int_0^1 |u(x,t)|^{r+1}\, dx \le \int_0^1 |u_0(x)|^{r+1}\, dx + (r+1)(\sup_{|z|\le C} |f(z)|)C^r t$$

for $C = \max_{0\le\tau\le T} |u(1,\tau)|$. Taking the $(r+1)$-th root of both sides and passing to the limit as $r \to \infty$ we obtain

$$|u(x,t)| \le \max_{0\le x\le 1} |u_0(x)| + \max_{0\le\tau\le T} |u(1,\tau)| \tag{1}$$

for all $(x,t) \in [0,1] \times [0,T]$, i.e., the solution u can be pointwise estimated by its maximum value at the beginning $(t = 0)$ and on the boundary $(x = 1)$.

In this note we shall prove a result similar to (1) for more general parabolic equations in several space variables. To begin with, let us consider the problem

$$\begin{aligned} u_t &= \Delta u && \text{for } x \in D, \quad t > 0 \\ \frac{\partial u}{\partial \nu} &= f(u) && \text{for } x \in \Gamma, \quad t > 0 \\ u(x,0) &= u_0(x), && u_0 \in C^2(\overline{D}), \end{aligned} \tag{2}$$

where $D \subset R^N$ is a bounded domain with a smooth boundary Γ, $\partial u/\partial\nu$ denotes the outward directed normal derivative of u on Γ and let f be a smooth function satisfying

$$f(u)u \leq L(|u|^{\alpha+1} + 1)$$

for fixed constants $\alpha \geq 1$, $L > 0$. For u_0 satisfying the compatibility conditions $\partial u_0/\partial\nu = f(u_0)$ on Γ there exists a unique classical solution $u(x,t) \in C^{2,1}(\overline{D} \times [0,T])$ for some positive T (see, e.g. [2], [7]).

We prove that for

$$p > (N-1)(\alpha-1)$$

there exist positive constants M, ν, independent of T, such that

$$|u(x,t)| \leq M(1 + \sup_{x\in D}|u_0(x)|)(1 + \sup_{0\leq\tau\leq T} \oint_\Gamma |u(s,\tau)|^p\, ds)^\nu \tag{3}$$

for all $(x,t) \in \overline{D} \times [0,T]$. The constants M, ν depend solely on the data D, f and on p.

Similar results for problems in which the nonlinearity occurs in the equation rather than in the boundary conditions, e.g.

$$\begin{aligned} &u_t = \Delta u + f(u) && (x,t) \in D \times (0,T), \\ &u(x,t) = 0 && (x,t) \in \Gamma \times (0,T), \\ &u(x,0) = u_0(x) && x \in D, \end{aligned} \tag{4}$$

have been obtained using the same Moser type method by Alikakos [1], Rothe [12], Nakao [9], [10], Filo [5] (the list is surely not complete). To point out the difference between the value of the "critical" exponent $(N-1)(\alpha-1)$ for Problem (2) and the analogical one for Problem (4), let us recall the result of [12]. Let

$$\begin{aligned} &r > \frac{N}{2}(\alpha-1) && \text{for} \quad N \geq 3, \\ &r > \alpha - 1 && \text{for} \quad N = 1,2 \end{aligned}$$

and let u be a solution, say classical, to Problem (4). Then there exist positive constants K, ρ, σ, independent of T, such that

$$|u(x,t)| \leq K((1 + \sup_{x\in D}|u_0(x)|) + (1 + \sup_{0\leq t\leq T} (\frac{1}{|D|}\int_D |u(x,t)|^r)^{1/r})^{\rho/\sigma})^\sigma$$

for all $(x,t) \in \overline{D} \times [0,T]$.

As follows from results of Friedman and McLeod [6], this result is sharp (except for $N = 1$) in the following sense. For special choice of the domain D (D being a ball) and initial states it may occur that

$$\sup_{0\leq t\leq T} \int_D |u(x,t)|^r\, dx < \infty \quad \text{for} \quad r < \frac{N}{2}(\alpha-1),$$

but

$$\limsup_{t\to T} \|u(\cdot,t)\|_{L^\infty(D)} = \infty.$$

However, as far as we know, no similar results have been obtained for Problem (2).

In [8] Levine and Payne considered Problem (2) for f satisfying $f(u) = |u|^\alpha h(u)$, h increasing, $\alpha > 1$. They proved that if u_0 is sufficiently large, the corresponding classical solution u breaks down by becoming unbounded in finite time, say T_0. Applying our result (3) we may conclude that also

$$\limsup_{t\to T_0} \oint_\Gamma |u(s,t)|^p \, ds = \infty$$

for all $p > (N-1)(\alpha-1)$, whenever h is bounded.

In [4] it is shown that any global classical solution of Problem (2) with $1 < \alpha < N/(N-2)$ (if $N \geq 3$) is bounded in $H^1(D)$ uniformly with respect to $t \geq 0$. (By a global solution we mean one which exists on $\overline{D} \times [0,\infty)$.) From our result it follows that it is also bounded in $C(\overline{D})$ for $t \geq 0$.

As in the several past years nonlinear diffusion problems have been intensively studied, we shall consider Problem (2) in which the heat equation is replaced by $(\beta(u))_t = \Delta u$ for the exact power law nonlinearity $\beta(u) = |u|^m \operatorname{sign} u$, $m > 0$. This equation is for $0 < m < 1$ well known as the porous medium equation and for $m > 1$ as the fast diffusion equation. (see, e.g. [3] and references therein). Nevertheless, this nonlinearity does not change the value of "critical" exponent and we shall prove the analogy to (3) whenever m is sufficiently small for $N \geq 3$.

The method of our proof consists in modifying suitably the Moser type technic [1], as appearing in [12], [9], [10].

Assumptions and Statement of Results.

We start by introducing some notation. For $0 < T < \infty$ let $Q = D \times (0,T)$, $S = \Gamma \times (0,T)$. The norms in the spaces $L^\infty(D), H^1(D)$ will be denotet by $\|\cdot\|_\infty$, $\|\cdot\|_{1,2}$ and we shall write $u^r := |u|^r \operatorname{sign} u$, $\int_D u(t)\varphi(t) := \int_D u(x,t)\varphi(x,t)\,dx$, $\oint_\Gamma |u(t)|^r := \oint_\Gamma |u(s,t)|^r \, ds$.

Now we consider the initial and boundary value problem

$$(5) \qquad \begin{aligned} (\beta(u))_t &= \Delta u && (x,t) \in Q, \\ \frac{\partial u}{\partial \nu} &= f(u) && (x,t) \in S, \\ u(x,0) &= u_0(x), \quad u_0 \in L^\infty(D) \cap H^1(D). \end{aligned}$$

Throughout the paper we will make the following assumptions, on β and the boundary datum f.

(H1) $$\beta(u) = |u|^m \operatorname{sign} u,$$

where m is a positive constant, which may be arbitrary if $N = 1,2$, but must satisfy

$$0 < m < (N+2)/(N-2) \qquad \text{for } N \geq 3.$$

(H2) $f \in C(R)$ is a given function such that

$$f(u)u \leq L(|u|^{\alpha+1} + 1)$$

for some $L > 0$ and $\alpha \geq 1$.

It is known that in general we can not expect of Problem (5) to be solvable in the classical sense even if the data are arbitrarily smooth. Therefore, it is necessary to deal with a suitable class of weak solutions.

Definition. By a weak solution of Problem (5) we mean a function $u \in L^\infty(0,T; H^1(D)) \cap L^\infty(Q)$, such that $(u^{(m+1)/2})_t \in L^2(Q)$, satisfying

$$\int_D \beta(u(\tau))\varphi(\tau) - \int_0^\tau \int_D (\beta(u)\varphi_t - \nabla u \nabla \varphi) = \int_0^\tau \oint_\Gamma f(u)\varphi + \int_D \beta(u_0)\varphi(0) \tag{6}$$

for all $\varphi \in H^1(Q)$ and a.e. $\tau \in (0,T)$.

We note that if u is a weak solution of Problem (5) with $f \in C^1(R)$, then from the results of DiBenedetto [3] it follows that $u \in C(\overline{D} \times (0,T])$. If in addition $u_0(x)$ is continuous in $\overline{D}$, then $u \in C(\overline{Q})$. We can now state our main result.

Theorem 1. *Let u be a weak solution of Problem (5) and assume that (H1), (H2) hold. Let*

$$\mathcal{B}(u) := \sup_{0 \leq t \leq T} \oint_\Gamma |u(t)|^{(N-1)(\alpha-1)+\varepsilon}$$

for some $\varepsilon > 0$.

Then there exist positive constants M, ν depending solely on the data m, D, f and on ε such that

$$\|u(\cdot,t)\|_\infty \leq M(1 + \|u_0\|_\infty)(1 + \mathcal{B}(u))^\nu \quad \textit{for all } 0 \leq t \leq T. \tag{7}$$

Remark 1. One can prove an analogous statement if $\partial u/\partial \nu = f(u)$ holds only over a part Γ_1 of the boundary with positive $(N-1)$ dimensional Lebesgue measure and if we require, e.g. $u \equiv 0$ on $\Gamma_2, \Gamma_2 = \Gamma \setminus \Gamma_1$.

Let us now state a series of assertions, which contain all elements for the proof of Theorem 1 and say more about the dependence of M, ν on the data and on ε.

Proposition 1. *Assume that (H1) holds, and suppose that the (appropriately smooth) function $u(x,t)$ satisfies the inequality*

$$\frac{d}{dt}\int_D |u(t)|^{m+r} + L_0 \|u^{(1+r)/2}(t)\|_{1,2}^2 \leq \leq L_1(m+1)^\xi \Big(\int_D |u(t)|^{m+r}\Big)^{\frac{1+r}{m+r}} + L_2(m+r) \tag{8}$$

for all $r \geq 2$ and a.e. $t \in (0,T)$ with some constants $L_0 > 0$, L_1, L_2, $\xi \geq 0$. Let

$$U_0 := \sup_{0 \leq t \leq T} \Big(\frac{1}{|D|}\int_D |u(t)|^{m+1}\Big)^{1/(m+1)}.$$

Then there exist positive constants C, ϑ depending on D, m, ξ, L_0, L_2, independent of T, such that

$$\sup_{0 \le t \le T} \|u(\cdot,t)\|_\infty \le (1+L_1)^{\vartheta} C \max(1, \|u(\cdot,0)\|_\infty, U_0). \tag{9}$$

Proposition 2. *Let u be a weak solution of Problem (5) and let (H1), (H2) hold.*
Then there exist positive constants Θ, μ, ξ, depending only on the data D, f, m and on ε such that u satisfies (8) with $L_0 = 1/4m$, $L_2 = |\Gamma|_{N-1}(L + 4L_0)$ and

$$L_1 = \Theta(1 + B(u))^{\mu}. \tag{10}$$

Proposition 3. *Under the hypotheses of Proposition 2 there exist positive constants K, ζ such that*

$$U_0 \le \max(\|u_0\|_\infty, K(1 + B(u))^{\zeta}). \tag{11}$$

The constants K and ζ depend only on the data and not on T.

Let us start with
PROOF of Proposition 2: Putting $\varphi = u^r$ for $r \ge 1$ into (6) we obtain, with the assistance of (H1), (H2),

$$\begin{aligned} \frac{m}{m+r} \frac{d}{dt} \int_D |u(t)|^{m+r} + \frac{4r}{(1+r)^2} \int_D |\nabla u^{(1+r)/2}(t)|^2 \le \\ 2L \oint_\Gamma |u(t)|^{\alpha+r} + L|\Gamma|_{N-1} \end{aligned} \tag{12}$$

for a.e. $t \in (0,T)$. We note that it is possible to take u^r as the test function also in the case of $m > 1$, in which u_t does not always exist. However, in this case $(\beta(u))_t$ exists and (6) yields (12). First of all we estimate the first term on the right hand side of (12). We shall distinguish two case. If $N \ge 3$ then for positive ε we obtain

$$\begin{aligned} \oint_\Gamma |u|^{\alpha+r} \le (\oint_\Gamma |u|^{(1+r)\frac{N-1}{N-2}})^P (\oint_\Gamma |u|^{(N-1)(\alpha-1)+\varepsilon})^Q \cdot \\ \cdot (\oint_\Gamma |u|^{1+r})^R \end{aligned} \tag{13}$$

where

$$P = \frac{(N-2)(\alpha-1)}{(N-1)(\alpha-1)+\varepsilon}, \quad Q = \frac{\alpha-1}{(N-1)(\alpha-1)+\varepsilon}$$

and

$$R = \frac{\varepsilon}{(N-1)(\alpha-1)+\varepsilon}.$$

If $N = 2$ it holds that

$$\begin{aligned} \oint_\Gamma |u|^{\alpha+r} \le (\oint_\Gamma |u|^{2(\alpha-1+\varepsilon)(1+r)/\varepsilon})^P (\oint_\Gamma |u|^{\alpha-1+\varepsilon})^Q \cdot \\ \cdot (\oint_\Gamma |u|^{1+r})^R \end{aligned} \tag{14}$$

where

$$P = \frac{\varepsilon(\alpha-1)}{(2(\alpha-1)+\varepsilon)(\alpha-1+\varepsilon)},$$
$$Q = \frac{\alpha-1}{\alpha-1+\varepsilon} \qquad \text{and} \qquad R = \frac{\varepsilon}{2(\alpha-1)+\varepsilon}.$$

The above inequalities play a key role in our considerations. Now, let us come back to (13). Put

$$p = (N-2)/(N-1)P \quad \text{and} \quad q = 1/R.$$

By the embedding theorem, there exists a $C_\varepsilon > 0$ such that

$$\oint_\Gamma |\varphi|^{2(N-1)/(N-2)} \le (C_\varepsilon \|\varphi\|_{1,2}^2)^{(N-1)/(N-2)}$$
$$\text{for all } \varphi \in H^1(D),$$

and as $p^{-1} + q^{-1} = 1$, we arrive at

$$\oint_\Gamma |u|^{\alpha+r} \le \eta \|u^{(1+r)/2}\|_{1,2}^2 + \tag{15}$$
$$+ (\frac{C_\varepsilon}{\eta})^{(N-1)(\alpha-1)/\varepsilon} (\oint_\Gamma |u|^{(N-1)(\alpha-1)+\varepsilon)})^{(\alpha-1)/\varepsilon} \oint_\Gamma |u|^{1+r}$$

for any $\eta > 0$, where Young's inequality has been used. Let L_0 be a constant such that $0 < L_0 \le r(m+r)/m(1+r)^2$ for all $r \ge 1$. Specifying η as $mL_0/2L(m+r)$, (12) and (15) then yield

$$\frac{d}{dt}\int_D |u(t)|^{m+r} + 3L_0 \|u^{(1+r)/2}(t)\|_{1,2}^2 \le \tag{16}$$
$$\le C_1(B(u))^{(\alpha-1)/\varepsilon}(m+r)^\sigma \oint_\Gamma |u(t)|^{1+r} + L_2(m+r)$$

where

$$\sigma = 1 + \frac{(N-1)(\alpha-1)}{\varepsilon}$$

and the nonnegative constant C_1 depends only on the data D, m, f and on ε.

Next, the following inequality is very useful

$$\oint_\Gamma |\varphi|^2 \le \delta \int_D |\nabla\varphi|^2 + \frac{C}{\delta}\int_D |\varphi|^2 \tag{17}$$

for all $\varphi \in H^1(D)$ and all sufficiently small δ, say $0 < \delta \le \delta_0$, δ_0 being given, where the positive constant C does not depend on δ (see, for example, [11, page 15]). Now, with the assistance of (17), (16) gives

$$\frac{d}{dt}\int_D |u(t)|^{m+r} + 2L_0 \|u^{(1+r)/2}(t)\|_{1,2}^2 \le \tag{18}$$
$$C_2(B(u))^{2(\alpha-1)/\varepsilon}(m+r)^{2\sigma} \int_D |u(t)|^{1+r} + L_2(m+r)$$

for a.e. $t \in (0,T)$ and all $r \geq 1$ ($C_2 = CC_1^2/L_0$). If $m \geq 1$ (8) and (10) follow easily.

Thus, let $0 < m < 1$. In this case, Hölder's inequality and Sobolev embedding theorem immediately yield

$$\int_D |u|^{1+r} \leq (C_s \|u^{(1+r)/2}\|_{1,2}^2)^P (\int_D |u|^{m+r})^Q$$

for

$$P = \frac{N(1-m)}{N(1-m)+2(m+r)} \quad \text{and} \quad Q = \frac{2(1+r)}{N(1-m)+2(m+r)}.$$

Now, applying Young's inequality, we arrive at

$$\int_D |u|^{1+r} \leq \eta \|u^{(1+r)/2}\|_{1,2}^2 + (\frac{C_s}{\eta})^{\frac{N(1-m)}{2(m+r)}} (\int_D |u|^{m+r})^{\frac{1+r}{m+r}}$$

for $\eta > 0$ and (8) follows.

If $N = 2$, considering (14), the proof is essentially the same. ■

PROOF of Proposition 1: Many, but not all, of the technical details used in our proof were established by Alikakos in [1]. However, to make our work selfcontained, we include the proof of Proposition 1 for $0 < m \leq 1$ here. We note that the case $m > 1$ can be proved using the same procedure. To simplify the notation, put

$$r_k = 2^k \text{ and } q_k = L_1(m + r_k)^\xi \quad \text{for } k = 0, 1, 2, \ldots.$$

Consider first the case $N \geq 3$. Using Hölder's inequality and Sobolev embedding theorem, the integral in the first term on the right hand side of (8) can be estimated as follows,

$$\int_D |u|^{m+r_k} \leq (C_s \|u^{(1+r_k)/2}\|_{1,2}^2)^P (\int_D |u|^{m+r_{k-1}})^Q \tag{19}$$

where

$$P = \frac{N r_{k-1}}{N(1-m) + 2(m + r_{k-1}) + N r_{k-1}}$$

and

$$Q = \frac{N(1-m) + 2(m + r_k)}{N(1-m) + 2(m + r_{k-1}) + N r_{k-1}}.$$

Now, by Young's inequality, (19) yields

$$(\int_D |u|^{m+r_k})^{\frac{1+r_k}{m+r_k}} \leq \varepsilon_k \|u^{(1+r_k)/2}\|_{1,2}^2 + \delta_k (\int_D |u|^{m+r_{k-1}})^{s_k} \tag{20}$$

where

$$s_k = \frac{1 + r_k}{m + r_{k-1}} \quad \text{and} \quad \delta_k = (\frac{C_s}{\varepsilon_k})^{P s_k / Q}.$$

The positive number ε_k will be determined later. Next, due to (20) and (8), we arrive at

$$\frac{d}{dt}\int_D |u(t)|^{m+r_k} + (L_0 - q_k\varepsilon_k - \varepsilon_k^2)\|u^{(1+r_k)/2}(t)\|_{1,2}^2 \le$$
$$-\varepsilon_k\Big(\int_D |u(t)|^{m+r_k}\Big)^{(1+r_k)/(m+r_k)} + L_2(m+r_k) +$$
$$+\varepsilon_k\delta_k\Big(1+\frac{q_k}{\varepsilon_k}\Big)\Big(\int_D |u(t)|^{m+r_{k-1}}\Big)^{s_k}$$

for all $k = 1, 2, \ldots$ and a.e. $t \in (0,T)$.

Now, if we choose

$$\varepsilon_k = \varepsilon_0/(m+r_k)^{\xi}(1+L_1)$$

for ε_0 sufficiently small we obtain $L_0 - q_k\varepsilon_k - \varepsilon_k^2 \ge 0$ for all $k = 1, 2, \ldots$. Solving the differential inequality

$$y' + \varepsilon y^{\nu} \le \varepsilon P \qquad \text{a.e. on } (0,T),$$

for $\varepsilon > 0$, $\nu \ge 1$, we obtain

$$y(t) \le \max(y(0),\ P^{1/\nu}) \quad \text{for all } t \in [0,T],$$

and thus

$$y_k(t) \le \max\Big(y_k(0),\ \Big(\delta_k\Big(1+\frac{q_k}{\varepsilon_k}\Big)\Big)^{\frac{m+r_k}{1+r_k}} |D|^{\frac{-r_k-1}{m+r_{k-1}}} U_{k-1}^{m+r_k} + \tag{21}$$
$$+\Big(\frac{L_2}{\varepsilon_k}(m+r_k)\Big)^{\frac{m+r_k}{1+r_k}}/|D|\Big)$$

for all $t \in [0,T]$ and $k = 1, 2, \ldots$, where

$$y_k(t) = \frac{1}{|D|}\int_D |u(t)|^{m+r_k}$$

and

$$U_k = \sup_{0\le t\le T}\Big(\frac{1}{|D|}\int_D |u(t)|^{m+r_k}\Big)^{1/(m+r_k)}.$$

Taking ε_0 small, it is not difficult to verify that

$$1 < d_k := \max\Big(\Big(\delta_k\Big(1+\frac{q_k}{\varepsilon_k}\Big)\Big)^{\frac{m+r_k}{1+r_k}} |D|^{\frac{-r_k-1}{m+r_{k-1}}}, \Big(\frac{L_2}{\varepsilon_k}(m+r_k)\Big)^{\frac{m+r_k}{1+r_k}}\frac{1}{|D|}\Big) \le \tag{22}$$
$$\le (1+L_1)^{\kappa}\, a\, r_k^{\sigma}$$

for some positive κ, a, σ (independent of k) and all $k = 1, 2, \ldots$. We note that the constants κ, σ depend only on N, m, ξ and a on L_0, L_2, D, m. Therefore, (21) implies

$$y_k(t) \le \max(\|u_0\|_{\infty}^{m+r_k},\ d_k(U_{k-1}^{m+r_k}+1)) \tag{23}$$

for all $t \in [0,T]$. Put $K = \max(1, \|u_0\|_\infty, U_0)$, then (23) yields

$$U_i \leq (\prod_{j=1}^{i} (2d_j)^{1/(m+r_j)})K \tag{24}$$

for all $i = 1,2,\ldots$. Now, with the assistance of (24) and (22), (23) gives

$$(\int_D |u(t)|^{m+r_k})^{1/(m+r_k)} \leq |D|^{1/(m+r_k)}(2(1+L_1)^\kappa a)^{S_1} 2^{\sigma S_2} K$$

for all $k = 1,2,\ldots$ and $t \in [0,T]$, where

$$S_1 = \sum_{i=1}^{\infty} \frac{1}{m+r_i}, \quad S_2 = \sum_{i=1}^{\infty} \frac{i}{m+r_i}.$$

Finally, passing to the limit as $k \to \infty$ we obtain (9).

If $N = 2$ we obtain (19) with $P = r_{k-1}/(1-m+r_{k-1})$ and (20) with s_k as above. The rest of the proof is then the same. ■

Proof of Proposition 3: To prove Proposition 3, we shall deal with the differential inequality (16) for $r = 1$. Let us suppose that $(N-1)(\alpha-1)+\varepsilon < 2$ as otherwise the proof is straightforward. In this case, using Hölder's inequality, the embedding theorem and Young's inequality, we arrive at

$$\oint_\Gamma |u|^2 \leq \eta \|u\|_{1,2}^2 + (\frac{C_\varepsilon}{\eta})^P (\oint_\Gamma |u|^{(N-1)(\alpha-1)+\varepsilon})^{2/((N-1)(\alpha-1)+\varepsilon)}$$

for any $\eta > 0$, where

$$P = \frac{(N-1)(2-(N-1)(\alpha-1)-\varepsilon)}{(N-1)(\alpha-1)+\varepsilon}$$

for $N \geq 3$,

$$P = \frac{p(3-\alpha-\varepsilon)}{(p-2)(\alpha-1+\varepsilon)}$$

for $N = 2$, $p > 2$ being arbitrary, and $C_\varepsilon > 0$ originates from the embedding theorem. Thus, (16) yields

$$\frac{d}{dt}\int_D |u(t)|^{m+1} + 2L_0\|u(t)\|_{1,2}^2 \leq C_3(1+\mathcal{B}(u))^\omega \text{ a.e. on } (0,T),$$

where the constants C_3, ω depend solely on the data D, m, f and on ε. According to (H1), $L^{m+1}(D)$ is embedded into $H^1(D)$, hence

$$y_0'(t) + C_4 y_0^{2/(m+1)}(t) \leq C_3(1+\mathcal{B}(u))^\omega/|D|,$$

$C_4 = 2L_0|D|^{(1-m)/(m+1)}/C_s$. Solving this differential inequality we obtain

$$U_0 \leq \max(\|u_0\|_\infty, (\frac{C_s C_3}{2L_0|D|^{2(m+1)}})^{1/2}(1+\mathcal{B}(u))^{\omega/2}),$$

hence (11).

The proof of Theorem 1 is completed. ■

Remark 2. The preceding theorem can be extended to problems, where the reaction term occurs also in the equation, i.e.

$$(25)\qquad \begin{aligned} &(\beta(u))_t = \Delta u + g(u) && \text{in } Q, \\ &\partial u/\partial\nu = f(u) \quad \text{on } S,\ u(x,0) = u_0(x) && \text{in } D, \end{aligned}$$

where g is sufficiently smooth, satisfying

$$g(\lambda)\lambda \leq C(|\lambda|^{\gamma m+1} + 1)$$

for some $C > 0$, $\gamma \geq 1$.

Theorem 2. *Let u be a weak solution of Problem (25). Put*

$$\mathcal{F}(u) := \sup_{0\leq t\leq T} \int_D |u(t)|^r$$

for

$$r > \frac{N}{2}(\gamma m - 1) \text{ if } N \geq 3, \quad r > \gamma m - 1 \text{ if } N = 1,2, \qquad r > 0.$$

Under the preceding hypotheses on u_0, β, f and g, there exists a constant M, independent of T, such that

$$\|u(\cdot,t)\|_\infty \leq M$$

for all $t \in [0,T]$. The constant M depends on $D, f, g, \beta, \|u_0\|_\infty, \varepsilon, r, \mathcal{B}(u)$ and $\mathcal{F}(u)$.

Theorem 2 can be proved in a manner similar to that of Theorem 1 and analogical one in [5].

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