Boundedness of global solutions for the heat equation with nonlinear boundary conditions

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Dedicated to the memory of Svatopluk Fučík

Abstract. Global solutions of the heat equation with nonlinear boundary conditions (which describe an absorption law) are shown to be bounded in $H^1(D)$ and in $C(\overline{D})$ uniformly for $t > 0$.

Keywords: global solutions, heat equation, nonlinear boundary conditions

Classification: 35K60,35B40

In this paper we study the problem

$$(1) \qquad u_t = \Delta u \qquad \text{for} \quad x \in D, t > 0,$$

$$(2) \qquad \frac{\partial u}{\partial \nu} = f(u) \qquad \text{for} \quad x \in \partial D, t > 0,$$

$$(3) \qquad u(\cdot, 0) = u_0 \in C^2(\overline{D}),$$

where D is a smoothly domain in R^N and f is superlinear. As an example we may consider $f(u) = |u|^{p-1}u$, $p > 1$.

For this problem the blow up phenomenon may occur (cf. [LP]).

Our main aim is to show that any global classical solution is bounded in $H^1(D)$ and in $C(\overline{D})$ (uniformly for $t > 0$), provided

$$p < \frac{N}{N-2} \qquad \text{if } N > 2.$$

By a global solution we mean a solution which exists on $\overline{D} \times [0, \infty)$.

Similar results for problems like

$$u_t = \Delta u + f(u) \qquad \text{for} \quad x \in D, t > 0,$$
$$u = 0 \qquad \text{for} \quad x \in \partial D, t > 0,$$

were established in [NST], [CL], [G], [F1], [F2]. The (sharp) condition on p in [CL], [G], [F1] was : $p < (N+2)/(N-2)$ if $N > 2$. In [F1] degenerate problems and problems with rapidly growing nonlinearities were treated. In [F2] also an equation with a gradient term was considered.

The proof of the present result is a new illustration of the main idea from [F1]. We shall proceed by contradiction. There are two possible types of behaviour of a global solution $u(t, u_0)$ which is not bounded in $H^1(D)$. Either

$$\|u(t,u_0)\|_{H^1(D)} \to \infty \quad \text{as} \quad t \to \infty \tag{4}$$

or

$$\begin{aligned} &\limsup_{t\to\infty} \|u(t,u_0)\|_{H^1(D)} = \infty, \\ &\liminf_{t\to\infty} \|u(t,u_0)\|_{H^1(D)} = k < \infty. \end{aligned} \tag{5}$$

(4) can be excluded using an appropriate modification of the classical concavity method. (5) leads to a contradiction with an a priori bound of every equilibrium lying in the ω–limit set of $u(t, u_0)$.

Our assumptions on f will be

$$|f(u) - f(v)| \le C(|u|^{p-1} + |v|^{p-1} + 1)\,|u - v| \tag{H1}$$

for $u, v \in R$ and some $C > 0$, $p > 1$, $p < N/(N-2)$ if $N > 2$.

$$uf(u) \ge (q+1)\int_0^u f(v)\,dv - C_1 \ge C_2|u|^{q+1} - C_3 \tag{H2}$$

for $u \in R$ and some $q > 1$, $C_i > 0$, $C_3 \ge C_1$.

It is known (cf. e.g. [A1, Theorem 6.1]) that Problem (1)-(3) possesses a unique maximal classical solution $u(t, u_0)$ provided $\partial u_0/\partial\nu = f(u_0)$ on ∂D and f is regular enough.

Let $t_{\max}(u_0)$ denote the existence time of the maximal solution emanating from u_0. The following known energy equality will play an important role in our considerations.

$$\int_0^t \int_D (u_t)^2 + V(u(t)) = V(u_0) \qquad \text{for } 0 \le t < t_{\max}(u_0), \tag{6}$$

where

$$V(u) := \frac{1}{2}\int_D |\nabla u|^2 - \int_{\partial D} F(u),\ F(u) := \int_0^u f(v)\,dv.$$

Lemma 1. *Let (H2) hold. If $\|u(t,u_0)\|_{H^1(D)} \to \infty$ as $t \to t_{\max}(u_0)$, then $t_{\max}(u_0) < \infty$.*

PROOF : We shall use the classical concavity method (see e.g. [PS], [LP]) similarly as in the proofs of corresponding results in [F1], [F2].

Suppose $t_{\max} = \infty$ and denote $M(t) := \int_0^t \int_D u^2$. Then

$$M'(t) = \int_D u^2 = \int_0^t \int_D (u^2)_t + \int_D u_0^2$$

and if we choose $0 < \varepsilon < q-1$, we obtain from (H2), (6)

$$\frac{1}{2}M''(t) = -\int_D |\nabla u|^2 + \int_{\partial D} uf(u) =$$

$$= -(2+\varepsilon)V(u) + \frac{\varepsilon}{2}\int_D |\nabla u|^2 + \int_{\partial D}(uf(u) - (q+1)F(u)) + (q-1-\varepsilon)\int_{\partial D} F(u)$$

$$\geq (2+\varepsilon)\int_0^t \int_D (u_t)^2 + \frac{\varepsilon}{2}\int_D |\nabla u|^2 + k_1 \int_{\partial D} |u|^{q+1} - k_2. \tag{7}$$

Here and in what follows positive constants which depend only on the data f, u_0, D will be denoted by k_i. From (7) it follows

$$M''(t) \geq k_3 \|u(t)\|^2_{H^1(D)} - k_4,$$

hence $M'(t) \to \infty$ as $t \to \infty$. On the other hand, (7) yields

$$M''(t) \geq 2\left((2+\varepsilon)\int_0^t \int_D (u_t)^2 + k_5 M'(t) - k_6\right),$$

therefore

$$M\,M'' - (1+\frac{\varepsilon}{2})(M') \geq$$

$$\geq 2(2+\varepsilon)\left(\int_0^t \int_D u^2 \int_0^t \int_D (u_t)^2 - (\int_0^t \int_D u\, u_t)^2\right) +$$

$$+2M(k_5 M' - k_6) - k_7 M'.$$

The first term on the right hand side is nonnegative according to the Schwarz inequality and the second one tends to infinity as $t \to \infty$. Thus, there is a $t_0 \geq 0$ such that the right hand side is positive for $t \geq t_0$. This implies that $(M^{-\varepsilon/2})'' < 0$ for $t \geq t_0$. Since $M^{-\varepsilon/2}$ is decreasing, it must have a root $t_1 > 0$ – a contradiction. ∎

The next lemma is based on the theory of parabolic equations with nonlinear boundary conditions developed by Amann in [A2]. It follows from this theory that (1), (2) define a local semiflow in $H^1(D)$ (in a way which will be made precise below) if the mapping $u \mapsto f(u)$ is locally Lipschitz from $H^1(D)$ into $\partial W^{-1+2\alpha} := W_2^{2\alpha-3/2}(\partial D)$ where we choose α such that

$$1 < 2\alpha < 1 + \frac{p}{p+1} - \frac{N}{2}\frac{p-1}{p+1} (\leq \frac{3}{2}).$$

This Lipschitz continuity is guaranteed by (H1). Indeed, with our choice of α

$$L^{(p+1)/p}(\partial D) \subset \partial W^{-1+2\alpha}. \tag{8}$$

By this imbedding and the Hölder inequality

$$\|f(u)-f(v)\|_{\partial W^{-1+2\alpha}} \le K\|f(u)-f(v)\|_{L^{(p+1)/p}(\partial D)} \le$$
$$\le K'\|u-v\|_{L^{p+1}(\partial D)}(\|u\|_{L^{p+1}(\partial D)}+\|v\|_{L^{p+1}(\partial D)}+1)^{p-1}.$$

The claim follows, since

$$H^1(D) \subset L^{p+1}(\partial D) \tag{9}$$

under our restriction on p.

Now, if $u(t,u_0)$ is weak solution of (1)-(3) on $[0,T)$, i.e. $u \in C([0,T);H^1(D))$ and

$$\int_0^T \int_D (-\phi_t u + \nabla\phi\nabla u) = \int_0^T \int_{\partial D} \phi f(u) + \int_D \phi(0)\, u_0$$

for all $\phi \in C^1([0,T);(H^1(D))') \cap C([0,T);H^1(D))$ vanishing near T, then $u(\cdot,\cdot)$ is a local semiflow on $H^1(D)$ (cf. [**A2**, Theorem 12.3]). Moreover, $u(t,u_0)$ satisfies certain integral equation – the variation of constants formula. We shall not sate this formula here because its consequence – the inequality (10) below (cf. [**A2**, (9) p.248 and Theorem 8.1]) will be sufficient for our purposes.

Lemma 2. *Let $(H1)$ hold. If $u(t,u_0)$ is a global solution which satisfies (5), then for every number B large enough there is an equilibrium $w \in \omega(u_0)$ (= the ω-limit set of u_0) such that $\|w\|_{H^1(D)} = B$.*

Proof : Similarly as in the proof of Lemma 2.2 in [**F1**], choose a sequence $\{t_n\}, t_n \to \infty$, satisfying the following three conditions:

(a) $\|u(t_n,u_0)\|_{H^1(D)} = B$,
(b) $\|u(t,u_0)\|_{H^1(D)} \le B$ for $t \in [t_{2n},t_{2n+1}]$,
(c) there is a sequence $\{s_n\}$ such that $s_n \in (t_{2n},t_{2n+1})\|u(s_n,u_0)\|_{H^1(D)} \le k+1$.

The variation of constants formula yields

$$\begin{aligned}\|u(t_{2n+1})\|_{H^{2\gamma}(D)} &\le L(t_{2n+1}-\tau_n)^{1/2-\gamma}e^{\sigma(t_{2n+1}-\tau_n)}\|u(\tau_n)\|_{H^1(D)}\\ &+L\int_{\tau_n}^{t_{2n+1}}(t_{2n+1}-\tau)^{\alpha-\gamma-1}e^{\sigma(t_{2n+1}-\tau)}\|f(u(\tau))\|_{\partial W^{-1+2\alpha}}\,d\tau\end{aligned} \tag{10}$$

for $\gamma \in [1/2,\alpha)$, where $H^{2\gamma}(D)$ is the usual Sobolev–Slobodeckii space $W_2^{2\gamma}(D)$, σ is an arbitrary positive number, L is a positive constant depending only on D,σ. Notice that $\|f(u(\tau))\|_{\partial W^{-1+2\alpha}}$ is bounded by a constant depending on B for $\tau \in [s_n,t_{2n+1}]$ according to (8), (H1) and (9). From (10) with $\tau_n = s_n$, $\gamma = 1/2$ we obtain that $t_{2n+1}-s_n \ge \delta > 0$ if we take $B > (k+1)L$.

Now the compact imbedding of $H^{2\gamma}(D)$ into $H^1(D)$ for $\gamma \in (1/2,\alpha)$ and (10) with $\tau_n = t_{2n+1}-\delta$, $\gamma \in (1/2,\alpha)$ imply the existence of a $w \in H^1(D)$, such that $u(t_n,u_0) \to w$ in $H^1(D)$ through a subsequence. Obviously $\|w\|_{H^1(D)} = B$ and standard arguments enable us to conclude that w is an equilibrium since our local semiflow admits a continuous Lyapunov functional (the functional V from (6)). ∎

Lemma 3. *Assume (H2). Let $u(t, u_0)$ be a global solution with $\omega(u_0) \neq \emptyset$. If $w \in \omega(u_0)$, w is an equilibrium, then $\|w\|_{H^1(D)} \leq J$ for some positive constant J depending on u_0.*

PROOF : Since w is an equilibrium, we have $\int_D |\nabla w|^2 = \int_{\partial D} wf(w)$, therefore for $0 < \varepsilon < q - 1$

$$(2+\varepsilon)V(w) = \frac{\varepsilon}{2}\int_D |\nabla w|^2 + \int_{\partial D}(wf(w) - (q+1)F(w)) + (q-1-\varepsilon)\int_{\partial D} F(w) \geq$$
$$\geq \frac{\varepsilon}{2}\int_D |\nabla w|^2 + k_1 \int_{\partial D} |w|^{q+1} - k_2 \geq k_3\|w\|^2_{H^1(D)} - k_4$$

k_i are positive constants. The assertion follows from (6). ■

Lemmas 1-3 yield now the main result.

Theorem. *Let (H1), (H2) hold. If $u(t, u_0)$ is a global classical solution of (1)-(3), then*

$$\sup_{t\geq 0}\|u(t, u_0)\|_{H^1(D)} < \infty.$$

It is shown in [Fo] that for solutions of a problem which includes (1)-(3) with f satisfying (H1), (H2) it holds

$$\|u(t, u_0)\|_{C(\overline{D})} \leq K(\|u_0\|_{C(\overline{D})}, \sup_{0\leq s\leq t}\|u(s, u_0)\|_{L^r(\partial D)})$$
$$\text{for} \quad 0 < t < t_{\max}(u_0) \quad \text{if} \quad r > (p-1)(N-1),\ N > 1.$$

From this estimate with $r = p + 1$ (together with (9)) we obtain

Corollary. *Let the assumptions of the theorem be satisfied. Then*

$$\sup_{t\geq 0}\|u(t, u_0)\|_{C(\overline{D})} < \infty.$$

Remark. The method of proof of the theorem works also for systems of the form

$$u^i_t = \Delta u^i + g^i(u^1, \dots, u^m),$$
$$\frac{\partial u^i}{\partial \nu} = f^i(u^1, \dots, u^m),$$

where $(g^1, \dots, g^m) = \text{grad}\, G$, $(f^1, \dots, f^m) = \text{grad}\, F$ for some G, F and g^i, f^i satisfy

(i) Lipschitz and growth conditions (like (H1)) under which the problem generates a local semiflow in $(H^1(D))^m$, an imbedding like (8) holds,

(ii) structure conditions (like (H2)) which ensure the applicability of the concavity method.

We finish with an application of the theorem.

Example. Consider the problem (1)–(3) with $f(u) = |u|^{p-1}u$, $p > 1$, $p < \frac{N}{N-2}$ if $N > 1$. If $u_0 \geq 0$, $u_0 \not\equiv 0$, $\partial u_0/\partial\nu = u_0^p$ then $t_{\max}(u_0) < \infty$.

PROOF : Suppose $t_{\max}(u_0) = \infty$. Choose $t_0 > 0$. According to the maximum principle there is a number $\varepsilon > 0$ such that $u(t, u_0) \geq \varepsilon$ for $t \geq t_0$. By the theorem $\|u(\cdot, u_0)\|_{H^1(D)}$ is bounded, hence $\{u(t, u_0) : t \geq t_0\}$ is relatively compact in $H^1(D)$ (cf. [**A2**]) and the ω–limit set consists of equilibria. But it is easily seen that there are no positive equilibria – a contradiction. ■

Acknowledgement. The author is indebted to Pavol Quittner whose critical comments led to a significant improvement of the original version of this paper.

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(*Received* May 18,1989)