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Borel classes in AST. Measurability, cuts and equivalence

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Abstract. A kind of measures on the universe V and the Borel hierarchy are introduced. Using measure–theoretical methods Borel representable pairs of cuts, i.e. pairs of form $\langle \underline{X}, \overline{X} \rangle$ where X is a Borel class are characterized. Borel equivalence is studied and described in terms of cuts. Class–theoretical operations on Borel classes are related to arithmetics of Borel cardinals. A Ramsey type theorem relating Borel and real equivalence on sets is proved.

Keywords: Alternative set theory, measure, Borel class, real class, lower cut, upper cut, Borel equivalence, Borel cardinal

Classification: Primary 03H20, 04A15, 28A05, Secondary 03E70, 03H15, 28E05

This paper is a direct continuation of our preceding works [**K–Z a**], [**K–Z b**]. It starts with introducing of certain type of measures μ_d $(d \in N \setminus \{0\})$ on V and investigating the resulting notions of measurability which can be characterized cut–theoretically. However, they differ considerably from the cut–theoretical measure and measurability introduced by A.Tzouvaras [**Tz**]. In fact, our measures μ_d rather remind of the Loeb measure [**L**] which was reformulated into the framework of AST by M.Raškovič [**R**] (see also [**Č a**]). On the other hand, it differs in not being located to a single (hyperfinite) set but its elementary values $1/d$ are distributed over the whole universe V. The measures μ_d form a particular case of measures introduced by M.Kalina [**K**] using a sequential approach. However, all his measures behaving analogously to the classical ones are in a certain sense equivalent to some μ_d. From this point of view the presented construction of measures μ_d can be regarded as a partial alternative to the sequential approach from [**K**].

In the next section the Borel hierarchy over Sd–classes is introduced. Then the results on measurability are applied to the study of cuts of Borel classes. In particular, Borel representable pairs are fully described.

Section 4 is devoted to the study of equivalence and subvalence of Borel classes under Borel maps. A complete description of this equivalence in terms of the equivalence of near equality (introduced in [**K–Z a**]) of their cuts is obtained for Borel semisets. In order to extend this description smoothly also to nonsemiset Borel classes, the extension of the Borel hierarchy over Sd–classes to the Borel hierarchy over Sd*–classes (a fixed revealment of Sd–classes – see [**S–V**]) seems unevoidable. Then the notion of Borel cardinal can be introduced and represented in a natural way. Using the results of [**K–Z a**], for basic class–theoretical operations on Borel classes, the Borel cardinality of the result can be computed in terms of Borel cardinalities of the arguments.

Finally a compactness, or if you wish – a Ramsey type result relating the Borel and real equivalence of sets is established using the concept of compatible biequivalence from [G–Z a].

1. Preliminaries.

The reader is assumed to be acquainted with [V] and [K–Z a], [K–Z b]. Notions, notations, conventions and results from these sources will be used freely, sometimes without any reference.

Let us recall from [G–Z a] that a pair $\langle R, S\rangle$ is called a biequivalence if R is a π–equivalence, S is a σ–equivalence, $\mathrm{dom}(R) = \mathrm{dom}(S)$ and $R \subseteq S$. The Sd–class $\mathrm{dom}(R) = \mathrm{dom}(S)$ is called the domain of the biequivalence $\langle R, S\rangle$. A biequivalence $\langle R, S\rangle$ is called compatible if for each infinite set $u \subseteq \mathrm{dom}(R)$ it holds

$$(\forall x, y \in u)(xSy) \Rightarrow (\exists x, y \in u)(x \neq y \ \ \& \ \ xRy),$$

or equivalently

$$(\forall x, y \in u)(xRy \Rightarrow x = y) \Rightarrow (\exists x, y \in u)\neg(xSy).$$

From Theorem 7 in [G–Z a] it follows directly

1.1. Proposition. *If $\langle R, S\rangle$ is a compatible biequivalence, then for each infinite set u such that $u^2 \subseteq S$ there is an infinite subset $v \subseteq u$ such that $v^2 \subseteq R$.*

1.2. Lemma. *Let $\langle R, S\rangle$ be a (compatible) biequivalence with its domain Y and $F : X \to Y$ be an Sd-function. Then $\langle F^{-1} \circ R \circ F, F^{-1} \circ S \circ F\rangle$ is a (compatible) biequivalence with domain X.*

PROOF : becomes trivial as soon as one notices of the equality

$$F^{-1} \circ R \circ F = \{\langle x, y\rangle \in X^2; F(x)\, R\, F(y)\}$$

and analogously for S. ■

In particular, if $\langle R, S\rangle$ is a (compatible) biequivalence with domain Y and $X \subseteq Y$ is an Sd–class, then its restriction to X, i.e. $\langle R \cap X^2, S \cap X^2\rangle$, is a (compatible) biequivalence with domain X.

The following compatible biequivalence $\langle \simeq, \sim\rangle$ with domain Q (however in most cases restricted to N) will be frequently utilized in the sequel

$$p \simeq q \Leftrightarrow p = 0 = q \vee (p \neq 0 \neq q \ \ \& \ \ p/q \doteq 1),$$
$$p \sim q \Leftrightarrow p = 0 = q \vee (p \neq 0 \neq q \ \ \& \ \ 0 \neq p/q \in BQ)$$

where $\doteq$ denotes the usual equivalence of infinitesimal nearness on Q and $BQ = \{p \in Q; (\exists n)(|p| < n)\}$ is the class of bounded rationals. For how $\simeq$ can be extended from natural numbers to cuts on N, see [K–Z a].

A (not necessarily codable) system of classes $\mathcal{M}$ will be called a σ–ring if it is closed with respect to class–theoretical difference and countable unions (hence also countable intersections). If $\mathcal{M}$ is a σ–ring and μ is an operation defined on $\mathcal{M}$ such that $\mu(X)$ is a nonnegative real number or the symbol ∞ for each $X \in \mathcal{M}$, then μ is called a measure on $\mathcal{M}$ provided it is σ–additive, i.e.

$$\mu(\bigcup\{X_n; n \in FN\}) = \sum_n \mu(X_n)$$

for any sequence $\{X_n; n \in FN\} \subseteq \mathcal{M}$ such that $X_m \cap X_n = \emptyset$ for $m \neq n$. A measure μ on $\mathcal{M}$ is called complete if for any X, Y such that $X \in \mathcal{M}$, $\mu(X) = 0$ from $Y \subseteq X$ it follows $Y \in \mathcal{M}$, and consequently $\mu(Y) = 0$, too. Denoting the symmetric difference by

$$X \nabla Y = (X \cup Y) \setminus (X \cap Y) = (X \setminus Y) \cup (Y \setminus X),$$

it turns out that μ is complete iff for any X, Y such that $\mu(X \nabla Y) = 0$ it holds $X \in \mathcal{M} \Leftrightarrow Y \in \mathcal{M}$; then necessarily $\mu(X) = \mu(Y)$ or both values are undefined. Finally, μ is called σ–finite if each $X \in \mathcal{M}$ is a union of countably many classes X_n such that $\mu(X_n) < \infty$.

The following result was in fact established during the proof of Theorems 2.1, 2.2 in [**K–Z b**], but was not stated explicitly there.

1.3. Lemma. *Let X be a real class and a be a natural number such that $\underline{X} \leq a$. Then there is a sequence of sets $\{u_n; n \in FN\}$ such that $X \subseteq \bigcup\{u_n; n \in FN\}$ and $u_n \hat{\precsim} a$ for each n.*

We will also utilize a result on real functions proved in [**Č–V**].

1.4.Lemma. *Let F be a real function. Then there is a sequence $\{F_n; n \in FN\}$ of Sd-functions such that $F \subseteq \bigcup\{F_n; n \in FN\}$.*

In fact both 1.3, 1.4 are special cases of a fairly general theorem from [**Č–V**].

Two real classes X, Y are said to be really equivalent, notation $X \hat{\approx} Y$ if there is a real function $F : X \approx Y$ (i.e. a bijective real map of X onto Y). X is really subvalent to Y, notation $X \hat{\precsim} Y$, if there is a one–one real function $F : X \to Y$; X is strictly really subvalent to Y, notation $X \hat{\prec} Y$, if $X \hat{\precsim} Y$ but not $X \hat{\approx} Y$. Basic properties of the notions just introduced can be found in [**Č–V**]. In particular, the Cantor – Bernstein theorem $X \hat{\precsim} Y \;\&\; Y \hat{\precsim} X \Leftrightarrow X \hat{\approx} Y$ holds.

Combining the last theorem from [**Č–V**] and Lemma 3.2 from [**K–Z b**] one obtains.

1.5. Proposition. *For each infinite set u there are real classes $Y, Z \subseteq u$ such that $Y \cap Z = \emptyset$, $\underline{Y} = \underline{Z} = |u|/FN$, $\overline{Y} = \overline{Z} = \mathrm{int}(|u|)$, but neither $Y \hat{\precsim} Z$ nor $Z \hat{\precsim} Y$ holds.*

Let us recall from [**Č–V a**] also the following

1.6. Lemma.

(a) *For any u, v it holds $u \hat{\approx} v$ if and only if $|u| = |v|$, or $|u|, |v| \notin FN$ and $|u| \sim |v|$.*

(b) *For each infinite set u it holds $|u|/FN \hat{\prec} u \hat{\approx} |u| \cdot FN$.*

2. Measurability of classes.

Throughout the whole paragraph c and d denote fixed nonzero natural numbers. If to each element $x \in V$ the same "weight" $1/d$ is assigned, then the rational number $|u|/d \geq 0$ becomes the "weight" of the set u. But, because of their vagueness, to general classes from the extended universe the described "weight" function cannot, at least not directly, be applied. Nevertheless, for some of them one can expect that approximating them by sub– and supersets their "weight" can be measured with the precision up to the equivalence $\doteq$ of infinitesimal nearness.

A class X is said to be of bounded d–measure, abbreviation $M(d,X)$, if there is a $q \in BQ$ such that

$$(\forall n > 0)(\exists u, v)(u \subseteq X \subseteq v \ \ \& \ \ \frac{|v|}{d} - \frac{1}{n} < q < \frac{|u|}{d} + \frac{1}{n}).$$

Note that the number $q \in BQ$ is determined uniquely up to the equivalence $\doteq$, and it can always be taken nonnegative. Hence taking its monad in $\doteq$ a single real number $\mu_d(X) \geq 0$, called the d–measure of X, is assigned to each class X of bounded d–measure.

A class X (such that $M(d,X)$ holds) is said to be of zero d–measure if $\mu_d(X) = 0$.

2.1. Proposition.

(a) *For each X it holds $\mu_d(X) = 0$ if and only if*

$$(\forall n > 0)(\exists v)(X \subseteq v \ \ \& \ \ |v|/d < 1/n).$$

(b) *If $Y \subseteq X$ and $\mu_d(X) = 0$ then also $\mu_d(Y) = 0$.*

(c) *Let $\{X_n; n \in FN\}$ be a sequence of classes such that $\mu_d(X_n) = 0$ for each n and $X = \bigcup\{X_n; n \in FN\}$. Then $\mu_d(X) = 0$.*

Proof : (a) and (b) are trivial.

(c) Let $m > 0$. Then there is a sequence $\{v_n; n \in FN\}$ such that $X_n \subseteq v_n$ and $|v_n|/d < 1/2^{n+1}m$ holds for each n. By the prolongation axiom there is a $v \supseteq \bigcup\{v_n; n \in FN\}$ such that $|v|/d < 1/m$. ■

2.2. Theorem. *For each class X the following four conditions are equivalent:*

(a) $M(d,X)$;

(b) $(\forall n > 0)(\exists u, v)(u \subseteq X \subseteq v \ \ \& \ \ |v \setminus u|/d < 1/n) \ \ \&$
$\ \ \& \ \ (\exists m)(\exists w)(X \subseteq w \precsim m \cdot d)$;

(c) $\overline{X} \div \underline{X} \leq d/FN \ \ \& \ \ \overline{X} < d \cdot FN$;

(d) $(\exists s)(\mu_d(X \nabla s) = 0 \ \ \& \ \ |s| < d.FN)$.

Proof : (a)⇒(b) is trivial.

(b)⇔ (c) Obviously, the second members of the conjunctions (b),(c) are equivalent, and in view of 3.1.4 from [K–Z a] so are the first ones, as well.

(b)⇒(d) Let $\{u_n; 0 < n \in FN\}, \{v_n; 0 < n \in FN\}$ be sequences of sets such that for each $n > 0$ it holds $u_n \subseteq X \subseteq v_n$ and $|v_n \setminus u_n|/d < 1/n$. In view of the second condition in (b), we can assume that $|v_n|/d \in BQ$ for each n. Then there is a set S such that $u_n \subseteq s \subseteq v_n$ for each n. It is routine to check that s satisfies both the conditions required.

(d)⇒(a) It suffices to put $q = |s|/d$. ■

2.3. Corollary.

(a) *Let X, Y be classes having the same cuts, or more generally, let $\underline{X} \simeq \underline{Y}$ and $\overline{X} \simeq \overline{Y}$ hold. Then $M(d, X) \Leftrightarrow M(d, Y)$.*
(b) *If $c \sim d$, then $M(c, X) \Leftrightarrow M(d, X)$ for each X.*
(c) *If $c \leq d$, then $M(c, X) \Rightarrow M(d, X)$ for each X.*
(d) *$M(d, u) \Leftrightarrow |u| < d \cdot FN$ for each set u.*
(e) *If $d \in FN$, then $M(d, X) \Leftrightarrow \mathrm{Fin}(X)$ for each X.*

2.4. Proposition. *Let X, Y be classes such that both $M(d, X), M(d, Y)$ hold, and F be an Sd-function. Then also $M(d, X \cup Y), M(d, X \cap Y), M(d, X \setminus Y)$ and $M(d, F''X)$ hold.*

PROOF : The assertion concerning the union is completely trivial. Since $X \cap Y = X \setminus (X \setminus Y)$ and $X \setminus Y = (X \cup Y) \setminus Y$, it suffices to prove $M(d, X \setminus Y)$ under the assumption $Y \subseteq X$. In that case for each $n > 0$ there are sets u, v, w, z such that $u \subseteq X \subseteq v, w \subseteq Y \subseteq z$ and $|v \setminus u|/d < 1/2n, |z \setminus w|/d < 1/2n$. Then $u \setminus z \subseteq X \setminus Y \subseteq v \setminus w$ and

$$\frac{(v \setminus w) \setminus (u \setminus z)|}{d} = \frac{|v| - |w| - |u \cup z| + |z|}{d} \leq$$
$$\frac{|v| - |u|}{d} + \frac{|z| - |w|}{d} < \frac{1}{n}.$$

If v is taken in a way that $|v| < d \cdot FN$, then also $|v \setminus w| < d \cdot FN$.

In view of 2.2 (c), the assertion $M(d, F''X)$ is a direct consequence of 3.1.5 and 3.1.2 from [**K–Z a**]. ∎

A class X will be called d–measurable if there is a sequence $\{X_n; n \in FN\}$ such that $M(d, X_n)$ holds for each n and $X = \bigcup\{X_n; n \in FN\}$. The system of all d–measurable classes will be denoted by $\mathcal{M}_d$.

Obviously, each class of bounded d–measure is d–measurable and each d–measurable class is a semiset. From 2.3 (b) it follows $c \sim d \Rightarrow \mathcal{M}_c = \mathcal{M}_d$ and so does $c \leq d \Rightarrow \mathcal{M}_c \subseteq \mathcal{M}_d$ from 2.3 (c). Finally, 2.3 (e) implies that $\mathcal{M}_d$ is the codable class of all at most countable classes for $d \in FN$.

The d–measure μ_d can be extended to the whole system $\mathcal{M}_d$ putting $\mu_d(X) = \infty$ whenever $X \in \mathcal{M}_d$ and $\neg M(d, X)$.

Obviously, if $c \simeq d$ then not only $\mathcal{M}_c = \mathcal{M}_d$, but also $\mu_c(X) = \mu_d(X)$ for each $X \in \mathcal{M}_c$, i.e. $\mu_c = \mu_d$. Similarly, if $X, Y \in \mathcal{M}_d$ have the same cuts, then $\mu_d(X) = \mu_d(Y)$.

The following theorem on cuts of d–measurable classes is a direct consequence of 2.2 and of the results form [**K–Z a**].

2.5. Theorem. *For each class $X \in \mathcal{M}_d$ exactly one of the following three conditions is satisfied:*

(a) $\overline{X} \leq d/FN$;
(b) $d/FN < \underline{X} \simeq \overline{X} < d \cdot FN$;
(c) $|X| = d \cdot FN$.

Moreover, for any class X (a) is equivalent to $\mu_d(X) = 0$ and so is (b) to $0 < \mu_d(X) < \infty$. However, (c) is equivalent to $\mu_d(X) = \infty$ only for $X \in \mathcal{M}_d$.

2.6. Theorem. *Let $\{X_n; n \in FN\}$ be a sequence of pairwise disjoint d–measurable classes and $X = \bigcup\{X_n; n \in FN\}$. Then*

$$\mu_d(X) = \sum_n \mu_d(X_n).$$

PROOF : If the series $\sum_n \mu_d(X_n)$ diverges (which is always the case if $\mu_d(X_n) = \infty$ for some n), then obviously $\mu_d(X) = \infty$. Assume that it converges. Let $q_n \in \mu_d(X_n)$ for each n. Now taking any $k > 0$, there are two sequences $\{u_n; n \in FN\}, \{v_n; n \in FN\}$ such that $u_n \subseteq X_n \subseteq v_n$ and

$$\frac{v_n}{d} - \frac{1}{2^{n+2}k} < q_n < \frac{|u_n|}{d} + \frac{1}{2^{n+2}k}$$

holds for each n. From the convergence of the series it follows that there are set prolongations $\{u_i; i \leq a\}, \{v_i; i \leq a\}, \{q_i; i \leq a\}$ of the mentioned sequences such that for all $i, j \leq a, i \neq j$, it holds $u_i \cap u_j = \emptyset, u_i \subseteq v_i$,

$$\frac{|v_i|}{d} - \frac{1}{2^{i+2}k} < q_i < \frac{|u_i|}{d} + \frac{1}{2^{i+2}k},$$

and an $m \in FN$ such that $\sum_{i=m}^{a} q_i < 1/2k$. We put

$$u = \bigcup\{u_i; 0 \leq i < m\}, \quad v = \bigcup\{v_i; 0 \leq i \leq a\}$$

and

$$q = \sum_{i=0}^{a} q_i.$$

Then $u \subseteq X \subseteq v$ and the real number $\sum_n \mu_d(X_n)$ is the monad of q. A simple computation gives

$$q - |u|/d = \sum_{i=0}^{m-1}(q_i - |u_i|/d) + \sum_{i=m}^{a} q_i < \sum_{i=0}^{m-1} \frac{1}{2^{i+2}k} + 1/2k < 1/k,$$

$$|v|/d - q = \sum_{i=0}^{a}(|v_i|/d - q_i) < \sum_{i=0}^{a} \frac{1}{2^{i+2}k} < 1/k.$$

Hence $q \in \mu_d(X)$. This shows the desired equality. ∎

2.7. Corollary. *Let $M(d, X)$ and $Y \subseteq X$ hold. Then $Y \in \mathcal{M}_d$ iff $M(d, Y)$.*

Summarizing, one obtains

2.8. Theorem. *The system $\mathcal{M}_d$ of all d–measurable classes is a σ–ring and μ_d is a complete, σ–finite measure on $\mathcal{M}_d$.*

PROOF : Let $X = \bigcup\{X_n; n \in FN\}, Y = \bigcup\{Y_n; n \in FN\}$, where $M(d, X_n)$, $M(d, Y_n)$ hold for each n, belong to $\mathcal{M}_d$. It remains to show that $X \setminus Y \in \mathcal{M}_d$. For all m, n we put $Z_{mn} = X_m \cap Y_n$ and $Z_m = \bigcup\{Z_{mn}; n \in FN\}$. Then $M(d, Z_{mn})$ holds by 2.4; hence $Z_m \in \mathcal{M}_d$ and $M(d, Z_m)$ follows from the inclusion $Z_m \subseteq X_m$ and 2.7. The equality $X \setminus Y = \bigcup\{X_m \setminus Z_m; m \in FN\}$ concludes the proof. The rest has already been proved. ∎

According to the previous results, each class $X \in \mathcal{M}_d$ can be represented both in the forms

$$X = \bigcup\{X_n; n \in FN\} = \bigcup\{Y_n; n \in FN\}$$

where for each n it holds $M(d, X_n), X_n \subseteq X_{n+1}$, and $M(d, Y_n), Y_m \cap Y_n = \emptyset$ for $m \neq n$. Then

$$\mu_d(X) = \sup_n \mu_d(X_n) = \sum_n \mu_d(Y_n).$$

The following theorem is perhaps rather surprising, but otherwise a trivial consequence of 2.4.

2.9. Theorem. *Let $X \in \mathcal{M}_d$ and F be an Sd-function. Then $F''X \in \mathcal{M}_d$ and $\mu_d(F''X) \leq \mu_d(X)$.*

2.10. Corollary. *If F is a one–one Sd-function and $X \subseteq \mathrm{dom}(F), X \in \mathcal{M}_d$, then*

$$\mu_d(F''X) = \mu_d(X).$$

As a consequence, the measure μ_d is invariant with respect to the group of all bijective Sd–functions $F : V \approx V$.

2.11. Theorem. *Let $X \in \mathcal{M}_c, Y \in \mathcal{M}_d$. Then $X \times Y \in \mathcal{M}_{cd}$ and $\mu_{cd}(X \times Y) = \mu_c(X) \cdot \mu_d(Y)$.*

PROOF : If $M(c, X), M(d, Y)$ hold, then it is obvious. The transfer to the general case is also trivial. ∎

3. Cuts of Borel classes.

The system of all Borel classes is the least σ–ring containing all the Sd–classes. More precisely, one can introduce the hierarchy of Borel classes indexed by countable ordinals $\alpha \in \Omega$ as follows:

X is a σ_0–class iff X is a π_0–class iff X is an Sd–class.

If $0 < \alpha \in \Omega$, then X is a σ_α–class (π_α–class) iff there is a sequence $\{X_n; n \in FN\}$ such that each X_n is a π_β–class (σ_β–class) for some $\beta < \alpha$ and $X = \bigcup\{X_n; n \in FN\}$ ($X = \bigcap\{X_n; n \in FN\}$, respectively).

X is a δ_α–class if X is both a σ_α–class and a π_α–class.

Finally, X is a Borel class if X is a σ_α–class, or equivalently a π_α–class, for some $\alpha \in \Omega$.

From the hierarchy described it follows that the Borel classes form a codable class.

Obviously, each Borel class is a real one.

Instead of the terms σ_1–class and π_1–class we will say simply σ–class and π–class, respectively. Note that δ_1–classes and Sd–classes coincide.

As it is not our aim to undertake a deeper study of the Borel hierarchy just introduced, whose properties moreover are quite analogous to those of the Borel hierarchies over classical metric spaces, we are going to state here only one result of general nature which will be necessary for our purpose.

Since inverse images under functions preserve all the operations generating the Borel family, the following lemma can be easily proved by transfinite induction over Ω.

3.1. Lemma. *Let F be an Sd–function, $\alpha \in \Omega$ and X be a σ_α–class (π_α–class, δ_α–class). Then $(F^{-1})''X$ also is a σ_α–class (π_α–class, δ_α–class, respectively).*

As a consequence, inverse images of Borel classes under Sd–functions are Borel classes. In particular, if X, Y are Borel classes, then X^{-1} and $X \times Y$ are Borel classes, as well.

The following result concerning the cuts of Borel classes is an analogue of Theorem 2.4 from [K–Z b] on cuts of real classes and it plays a similarly important role.

3.2. Theorem. *Let X be a Borel class and $a \in N$. If $\underline{X} \leq a \leq \overline{X}$, then*

$$\mathrm{int}(a) \leq \underline{X} \leq a \leq \overline{X} \leq \mathrm{cl}(a).$$

Proof : Let $\underline{X} \leq a \leq \overline{X}$. We will prove the inequality $\mathrm{int}(a) \leq \underline{X}$, or equivalently, that $\lfloor a(1-1/k) \rfloor < \underline{X}$ holds for each $k > 0$. By 1.3 there is a sequence $\{w_n; n \in FN\}$ of sets such that $X \subseteq \bigcup\{w_n; n \in FN\}$ and $|w_n| \leq a$ for each n. We put $Y_0 = X \cap w_0, Y_{n+1} = (X \cap w_{n+1}) \setminus (Y_0 \cup \cdots \cup Y_n)$. Then $\{Y_n; n \in FN\}$ is a sequence of pairwise disjoint Borel classes and $X = \{Y_n; n \in FN\}$. As $Y_n \leq w_n \widehat{\precsim} a \leq \overline{X}$, there has to be an m such that $a \leq \overline{Y_0 \cup \cdots \cup Y_m}$. Let for each $n \leq m$ u_n, v_n denote sets such that $u_n \leq Y_n \subseteq v_n$ and $|v_n \setminus u_n|/a < 1/2^{n+1}k$. Then $u = u_0 \cup \cdots \cup u_m \subseteq X$, $|u| = |u_0| + \cdots + |u_m|$ and $a \leq |v_0 \cup \cdots \cup v_m| \leq |v_0| + \cdots + |v_m|$. Let us compute

$$|u| = \sum_{i=0}^{m} |u_i| > \sum_{i=0}^{m} (|v_i| - \frac{a}{2^{i+1}k}) > a - a/k \geq \lfloor a(1-1/k) \rfloor.$$

Consequently, $\lfloor a(1-1/k) \rfloor \leq \underline{X}$. Now, assume that $\overline{X} \leq \mathrm{cl}(a)$ does not hold. Then there is a $b > a$, $b \not\doteq a$ such that $\underline{X} \leq b \leq \overline{X}$. Then, as just proved $a < \mathrm{int}(b) \leq \underline{X}$, contradicting $\underline{X} \leq a$. ∎

3.3. Corollary. *Let X be a Borel class.*

(a) *If either $\underline{X}$ or $\overline{X}$ is an additive cut, then $\underline{X} = \overline{X}$.*
(b) *If $\underline{X}$, $\overline{X}$ are nonadditive, then $\mathrm{int}(a) \leq \underline{X} \leq \overline{X} \leq \mathrm{cl}(a)$ for some a. If additionally $\underline{X} \neq \overline{X}$, then each $a \in \overline{X} \setminus \underline{X}$ satisfies the above inequality.*
(c) $\underline{X} \simeq \overline{X}$.
(d) $\underline{X} = \overline{X} \vee (\forall a \in \overline{X} \setminus \underline{X}) M(a, X)$.

A pair of cuts $\langle A, B\rangle$ will be called Borel representable if there is a Borel class X such that $A = \underline{X}, B = \overline{X}$.

Obviously, each Borel representable pair of cuts is really representable, it consists of two nearly equal cuts and each of them is a σ– or a π–class.

3.4. Theorem. *A pair of cuts is Borel representable if and only if it is exactly of one of the following two types:*

$$\langle A, A\rangle \ (A \text{ - } \textit{real, i.e. } \sigma\text{- or } \pi\textit{-cut}); \tag{1}$$

$$\langle a - A, a + A\rangle \ (A \text{ - } \textit{additive, real, } 0 < A < a). \tag{2}$$

PROOF : In the proof of Theorem 3.3 in [**K–Z b**], characterizing really representable pairs of cuts, it was in fact shown that both the types (1), (2) are Borel representable. Obviously, they cannot overlap. It suffices to prove that none of the remaining eight types of really representable pairs of cuts is Borel representable. This will proceed in a uniform way. For any $i = 3, 4, \ldots, 10$ we will assume that there is a Borel class X_i with cuts of the i–th type. Starting from X_i a Borel class Y_i with cuts contradicting to 3.2 will be constructed.

$$\underline{X_3} = a + b/FN, \quad \overline{X_3} = a + \mathrm{int}(b) \qquad (FN < b). \tag{3}$$

There is a $u \subseteq X_3$, such that $|u| = a$. We put $Y_3 = X_3 \setminus u$. Then $\underline{Y_3} = b/FN$, $\overline{Y_3} = \mathrm{int}(b)$.

$$\underline{X_4} = a + b/FN, \quad \overline{X_4} = a + \mathrm{cl}(b) \qquad (FN < b). \tag{4}$$

Again there is a $u \subseteq X_4$, $|u| = a$. For $Y_4 = X_4 \setminus u$ it holds $\underline{Y_4} = b/FN, \overline{Y_4} = \mathrm{cl}(b)$.

$$\underline{X_5} = a - b/FN, \quad \overline{X_5} = a + \mathrm{int}(b) \qquad (FN < b < a \cdot FN). \tag{5}$$

There is a $v \supseteq X_5$ such that $|v| = a + b$. We put $Y_5 = v \setminus X_5$. Then $Y_5 = b/FN$, $\overline{Y_5} = \mathrm{cl}(b)$.

$$\underline{X_6} = a - b/FN, \quad \overline{X_6} = a + \mathrm{cl}(b) \qquad (FN < b < a \cdot FN). \tag{6}$$

There is a $v \supseteq X_6, |v| = a + 2b$. For $Y_6 = v \setminus X_6$ it holds $\underline{Y_6} = \mathrm{int}(b)$, $\overline{Y_6} = \mathrm{cl}(2b)$.

$$\underline{X_7} = a + b/FN, \quad \overline{X_7} = a + b \cdot FN \qquad (FN < b). \tag{7}$$

There is a $u \subseteq X_7$ such that $|u| = a$. We put $Y_7 = X_7 \setminus u$. Then $\underline{Y_7} = b/FN$, $\overline{Y_7} = b \cdot FN$.

$$\underline{X_8} = a - b/FN, \quad \overline{X_8} = a + b \cdot FN \qquad (FN < b < a \cdot FN). \tag{8}$$

If $a/FN < b < a \cdot FN$, then even the class $Y_8 = X_8$ contradicts 3.2, since $\underline{X_8} = \mathrm{int}(a)$, $\overline{X_8} = a \cdot FN$. If $b \cdot FN < a$, then there is a set $v \supseteq X_8$ such that

$|v| = 2a$. For $Y_8 = v \setminus X_8$ it holds $\underline{Y_8} = a - b \cdot FN, \overline{Y_8} = a + b/FN$. The fact that the existence of such a Borel class Y_8 is in contradiction with 3.2 will be shown in (10) below.

(9) $$\underline{X_9} = a - b \cdot FN, \quad \overline{X_9} = a - b/FN \qquad (FN < b < a/FN).$$

There is a set $v \supseteq X_9$, $|v| = a$. For $Y_9 = v \setminus X_9$ it holds $\underline{Y_9} = b/FN, \quad \overline{Y_9} = b \cdot FN$.

(10) $$\underline{X_{10}} = a - b \cdot FN, \quad \overline{X_{10}} = a + b/FN \qquad (FN < b < a/FN).$$

There is a $w \supseteq X_{10}$ such that $|w| = a + b$. We put $Y_{10} = w \setminus X_{10}$. Then
$\underline{Y_{10}} = \mathrm{int}(b), \quad \overline{Y_{10}} = b \cdot FN$. ∎

Let $\langle C, D\rangle$ be a Borel representable pair of cuts. We put $Y = (C \times \{0\}) \cup ((D \div C) \times \{1\})$. It can be easily verified that $\underline{Y} = C, \quad \overline{Y} = D$, whenever $\langle C, D\rangle$ is of type (1) or (2). This not only proves the Borel representability of pairs of cuts of types (1), (2), for which we have referred to [**K–Z b**], but also the following theorem.

3.5. Theorem. *For every Borel class X there is a δ_2–class Y (in fact Y can be chosen to be a union of a σ–class and a π–class) such that $\underline{X} = \underline{Y}$ and $\overline{X} = \overline{Y}$.*

Consequently, for the study of the behaviour of Borel classes with respect to the measures μ_d it suffices, in some sense, to deal with δ_2–classes, only.

4. Borel equivalence.

Preliminarily we have to list some basic facts concerning Borel functions, i.e. functions which at the same time are Borel classes.

4.1. Lemma. *Let F be a Borel function. Then there is a sequence $\{X_n; n \in FN\}$ of Borel classes and a sequence $\{F_n; n \in FN\}$ of Sd–functions such that*

$$\mathrm{dom}(F) = \bigcup\{X_n; n \in FN\}$$

and

$$F = \bigcup\{F_n \restriction X_n; n \in FN\}.$$

PROOF : Let $\{F_n; n \in FN\}$ be the sequence of Sd–functions guaranteed to F by Lemma 1.4. For each n let $\widetilde{F}_n$ denote the Sd–function defined by $\widetilde{F}_n(x) = \langle F_n(x), x\rangle$ for $x \in \mathrm{dom}(F_n)$. We put

$$X_n = \{x; F_n(x) = F(x)\} = (\widetilde{F}_n^{-1})''F.$$

After 3.1, X_n are Borel classes, and both the remaining conditions are trivially satisfied. ∎

4.2. Theorem.

(a) *If F is a Borel function and X is a Borel class then $(F^{-1})''X$ is a Borel class.*

(b) *If F, G are Borel functions then $F \circ G$ is a Borel function, as well.*

PROOF :

(a) Let $\{X_n; n \in FN\}, \{F_n; n \in FN\}$ be the sequences guaranteed to F by the previous Lemma. Then
$$(F^{-1})''X = \bigcup\{(F_n^{-1})''(X \cap X_n); n \in FN\}$$
obviously is a Borel class.

(b) Let additionally $\{Y_n; n \in FN\}, \{G_n; n \in FN\}$ be such sequences for G. Then
$$\begin{aligned} F \circ G &= \bigcup\{(F_m \restriction X_m) \circ (G_n \restriction Y_n); m, n \in FN\} \\ &= \bigcup\{(F_m \circ G_n) \restriction (Y_n \cap (G_n^{-1})''X_m); m, n \in FN\} \end{aligned}$$
is a Borel function, again. ∎

4.3. Remark. There are also functions F which are not Borel, but $(F^{-1})''X$ is a Borel class for each Borel X. Namely, in [V] for every pair of infinite sets u, v a bijection $F : u \approx v$ is constructed such that $(\forall X \subseteq u)(\mathrm{Set}(X) \Leftrightarrow \mathrm{Set}(F''X))$ holds. Nevertheless, if $|u| \not\sim |v|$, then owing to 1.6 (a) F cannot even be a real class.

4.4. Remark. Note that a composition $R \circ S$ of Borel relations R, S need not be a Borel relation.

Borel classes X, Y will be called Borel equivalent, notation $X \overset{b}{\approx} Y$, if there is a one–one Borel function F such that $\mathrm{dom}(F) = X, \mathrm{rng}(F) = Y$. X is Borel subvalent to Y, notation $X \overset{b}{\precsim} Y$, if there is a one–one Borel function $F : X \to Y$. Finally, X is strictly Borel subvalent to Y, notation $X \overset{b}{\prec} Y$, if $X \overset{b}{\precsim} Y$ and not $X \overset{b}{\approx} Y$.

Since the relation of Borel equivalence is trivially reflexive and symmetric and its transitivity follows from 4.2 (b), the term "Borel equivalence" is fully justified. Also, both the relations of Borel subvalence and strict subvalence are tranzitive. The last statement, however, is based on the Cantor – Bernstein theorem on Borel equivalence.

4.5. Theorem. *Let X, Y be Borel classes. Then $X \overset{b}{\approx} Y$ if and only if $X \overset{b}{\precsim} Y$ and $Y \overset{b}{\precsim} X$.*

PROOF : Let us concentrate on the nontrivial implication, only. Let $F : X \to Y$, $G : Y \to X$ be one–one Borel functions. By induction over FN two sequences $\{Z_n; n \in FN\}$, $\{W_n; n \in FN\}$ can be constructed putting $Z_0 = Y \setminus F''X, W_n = G''Z_n, \quad Z_{n+1} = F''W_n$. Owing to 4.4 (a), all the W_n, Z_n are Borel classes, hence $W = \bigcup\{W_n; n \in FN\}$ is a Borel class, as well. The fact that the Borel function $F \restriction (X \setminus W) \cup G^{-1} \restriction W$ establishes the equivalence between X and Y is already well known. ∎

The following theorem is trivial.

4.6. Theorem. *Let $\{X_n; n \in FN\}, \{Y_n; n \in FN\}$ be two sequences of Borel classes, such that $X_n \overset{b}{\approx} Y_n$ holds for each n. Then*

(a) $X_0 \times X_1 \overset{b}{\approx} Y_0 \times Y_1$;
(b) *if $X_m \cap X_n = Y_m \cap Y_n = \emptyset$ for any $m \neq n$, then* $\bigcup\{X_n; n \in FN\} \overset{b}{\approx} \bigcup\{Y_n; n \in FN\}$.

4.7. Proposition. *Let A, B be real (i.e. Borel) cuts. Then $A \overset{b}{\approx} B$ if and only if $A \simeq B$, and $A \overset{b}{\precsim} B$ if and only if $A \lesssim B$.*

PROOF : First let us assume that there is a Borel bijection $F : A \approx B$. Then by 3.1.3 from [**K–Z** a] it holds $\underline{F} \leq A, B \leq \overline{F}$. By 3.3 (c) we have $\underline{F} \simeq \overline{F}$, hence also $A \simeq B$. Now assume that $A \simeq B$. Then either $A = B$ or there is an a such that $A \simeq a \simeq B$. According to 4.5 it suffices to prove $\mathrm{int}(a) \overset{b}{\approx} \mathrm{cl}(a)$ for each a. This is a special case of the following lemma. ∎

4.8. Lemma. *Let $a \in N$ and $C \leq a$ be a real additive cut. Then $a - C \overset{b}{\approx} a + C$.*

PROOF : It suffices to deal with the case $a \notin FN$, $0 < C < a$. The Borel bijection $F : a - C \approx a + C$ can be constructed directly putting $F(3c) = c, F(3c+1) = a - c$, $F(3c+2) = a + c$ for $c \in C$ and $F(b) = b$ for $b \in (a - C) \setminus C$.

The rest of the proof of 4.7 is also trivial, now. ∎

4.9.Lemma.

(a) *Let X be a σ-class. Then $X \overset{b}{\approx} |X|$.*

(b) *Let X be a π-class and a semiset. Then $X \overset{b}{\approx} |X|$.*

PROOF : (a) X can be written in the form $X = \bigcup\{X_n; n \in FN\}$, where X_n are Sd–classes and $X_n \subseteq X_{n+1}$ for each n. If all the X_n are even sets, then obviously $|X| = \bigcup\{|X_n|; n \in FN\}$ and a Borel bijection $F : X \approx |X|$ can be constructed easily by induction. If at least one of the classes X_n is proper, then obviously $|X| = |X_n| = N$ and there is an Sd–function $F : X_n \approx N$ for such an n. $X \overset{b}{\approx} N$ then follows from the Cantor – Bernstein theorem 4.5.

(b) Since X is a semiset, it can be written in the form $X = \bigcap\{u_n; n \in FN\}$, where $u_{n+1} \subseteq u_n$ for each n. Let $\{g_n; n \in FN\}$ be a sequence of functions such that $g_n : u_n \approx |u_n|$. Then there are set prolongations $\{u_i; i \leq a\}, \{g_i; i \leq a\}$ such that $u_{i+1} \subseteq u_i$ for each $i < a$ and $g_i : u_i \approx |u_i|$ for each $i \leq a$. A set sequence $\{f_i; i \leq a\}$ such that $f_i : u_i \approx |u_i|$ for each $i \leq a$ and $f_{i+1} \subseteq f_i$ for each $i < a$ will be constructed by induction downwards, putting $f_a = g_a$ and

$$f_{j-1} = f_j \cup g_{j-1} \restriction (u_{j-1} \setminus u_j)$$

for $j \leq a, j \neq 0$. Then the Borel function $F = \bigcup\{f_j; FN < j \leq a\} = \bigcap\{f_n; n \in FN\}$ establishes the equivalence $X \overset{b}{\approx} |X|$. ∎

4.10. Theorem. *Let X be a Borel semiset. Then $\underline{X} \overset{b}{\approx} X \overset{b}{\approx} \overline{X}$.*

PROOF : According to 1.10 from [**K–Z b**] there are semisets Y, Z such that $Y \subseteq X \subseteq Z$, each of them is a σ– or π–class and $|Y| = \underline{X}, |Z| = \overline{X}$. By 4.9 it holds $|Y| \overset{b}{\approx} Y, Z \overset{b}{\approx} |Z|$. The Borel subvalences $Y \overset{b}{\precsim} X \overset{b}{\precsim} Z$ are trivial. By 3.3 (c) it holds $|Y| = \underline{X} \simeq \overline{X} = |Z|$, hence by 4.7 also $\underline{X} = |Y| \overset{b}{\approx} |Z| = \overline{X}$. The Cantor – Bernstein theorem 4.5 completes the proof. ∎

4.11. Remark. Neither 4.9 (b) nor 4.10 can be generalized omitting the semiset presumption. In fact, as pointed out by K.Čuda [**Č b**] there is a π–class X which is not a semiset, i.e. $|X| = N$, but there is even no real bijection $F : X \approx N$. For X a class of indiscernibles, whose existence was established in [**S–Ve**], can be taken. We leave out the proof of this fact requiring several notions and results going beyond the scope of the present article.

In view of the Remark we have to decide which one of the following two possible ways we will follow. On the one hand, we can try to classify the structure of Borel classes with cut N with respect to the Borel equivalence. However, this equivalence is no more describable in terms of their cuts, and probably would turn out rather dependent of the chosen extension of the basic axiomatic system of AST. On the other hand, we can regard the remarked phenomenon as a kind of pathology and look for tools enabling to surmount it. The notion of Sd*–class offers the possibility to proceed in the latter way which seems more appropriate to us.

The codable system $\{X; \mathrm{Sd}^*(X)\}$ of all Sd*–classes is a fixed revealment of the codable system $\{X; \mathrm{Sd}(X)\}$ of all Sd–classes (see [**S–V**]). Roughly speaking, the Sd*–classes form a system of revealed and even sharp classes containing the Sd–classes as a subsystem and behaving in exactly the same way with respect to normal formulas of the language FL_V. Additionally, for each sequence $\{X_n; n \in FN\}$ of Sd*–classes there is an Sd*–relation R such that $X_n = R''\{n\}$ for each n; i.e. enabling to apply the prolongation technics, Sd*–classes behave much more like sets than Sd–classes. (All essential properties of Sd*–classes are briefly listed also in [**G–Z b**], p. 685).

The codable system of all *Borel classes can be constructed from Sd*–classes in exactly the same way as the system of all Borel classes from Sd–classes. Terms like σ^*–class, π^*–class, etc. are self–explanatory. Everything holding for Borel classes remains true under appropriate "starification". In particular this concerns the notions of *Borel equivalence, subvalence and strict subvalence, in symbols $X \overset{*}{\approx} Y, X \overset{*}{\precsim} Y$ and $X \overset{*}{\prec} Y$, respectively.

Since each *Borel semiset is a Borel one, $X \overset{b}{\approx} Y \Leftrightarrow X \overset{*}{\approx} Y$ for Borel semisets. Similarly, as each *Borel cut is a Borel one $A \overset{b}{\approx} B \Leftrightarrow A \overset{*}{\approx} B$ for Borel cuts. Analogous results relate both type of subvalences and strict subvalences.

On the other hand, the pathology from Remark 4.11, and also that from 1.8 in [**K–Z b**], disappear.

4.12. Theorem. *For arbitrary *Borel classes X, Y the following statements hold:*

(a) $\underline{X} \overset{*}{\approx} X \overset{*}{\approx} \overline{X}$;

(b) $X \overset{*}{\approx} Y \Leftrightarrow \underline{X} \simeq \underline{Y} \Leftrightarrow \overline{X} \simeq \overline{Y}$;
(c) $X \overset{*}{\precsim} Y \vee Y \overset{*}{\precsim} X$.

PROOF : (a) It suffices to show $X \overset{*}{\approx} N$ for each *Borel class which is not a semiset. By 1.9 from [**K–Z b**] there is a π^*–class $Y \subseteq N$ such that $|Y| = N$. Using the prolongation property of Sd*–classes, there is a proper Sd*–class $Z \subseteq Y$. Obviously $Z \overset{*}{\approx} N$, hence $X \overset{*}{\approx} N$ by the Cantor–Bernstein theorem.

(b) follows directly from (a) and 3.3 (c).

(c) is a consequence of (b) and of the fact that $A \precsim B$ is a dichotomic relation on cuts, stated in [**K–Z a**]. ∎

Thus the system of all Borel classes behaves much better with respect to the equivalence $\overset{*}{\approx}$ than with respect to $\overset{b}{\approx}$. Let us recall that $\mathcal{C}_0$ denotes the codable system of all Borel cuts and that the linearly ordered algebra $\langle \mathcal{C}_0/\simeq; +, ., \sum, \precsim\rangle$ with isotone operations, called the algebra of Borel cardinals, was introduced in [**K–Z a**]. Now its name can be justified. Each Borel class X determines uniquely its Borel cardinality $\beta(X)$ as the equivalence class of its (no matter which one) lower or upper cut with respect to $\simeq$. If somebody would prefer to represent Borel cardinals directly as classes from the extended universe, it suffices to fix some selector from the equivalence $\simeq$ on $\mathcal{C}_0$. Let us mention the following three possibilities of doing this.

(1) Similarly as in the Cantor set theory, $\beta(X)$ can be represented as the least cut A such that $A \overset{*}{\approx} X$, i.e. $\beta(X) = \mathrm{int}(\underline{X}) = \mathrm{int}(\overline{X})$.

(2) Dually, we can put $\beta(X)$ equal to the largest cut B such that $B \overset{*}{\approx} X$; i.e. $\beta(X) = \mathrm{cl}(\underline{X}) = \overline{cl}(\overline{X})$.

(3) Finally, we can fix a selector S from the equivalence $\simeq$ on N and put $\beta(X)$ to be equal to the unique element $a \in S$ such that $a \simeq \underline{X} \simeq \overline{X}$ provided X has nonadditive cuts, and $\beta(X) = |X|$ if X has an additive cut (like in the previous two cases).

There is no need to specify such a choice, that's why the equality of Borel cardinals of X, Y will be denoted rather by $\beta(X) \simeq \beta(Y)$ than $\beta(X) = \beta(Y)$.

Theorem 4.6 and the results on cut arithmetic from [**K–Z a**] yield immediately

4.13. Theorem. *Let $\{X_n; n \in FN\}$ be a sequence of Borel classes.*

(a) *If $X_0 \cap X_1 = \emptyset$, then $\beta(X_0 \cup X_1) \simeq \beta(X_0) + \beta(X_1)$.*
(b) $\beta(X_0 \times X_1) \simeq \beta(X_0) \cdot \beta(X_1)$.
(c) *If $X_m \cap X_n = \emptyset$ for $m \neq n$, then*

$$\beta(\bigcup\{X_n; n \in FN\}) \simeq \sum\{\beta(X_n); n \in FN\}.$$

The most substantial difference between the Borel and real equivalence seems to be the following strengthening of 1.5.

4.14. Theorem. *For each infinite set u there are real classes $Y, Z \subseteq u$ such that $\underline{Y} \prec Y \prec \overline{Y}, \underline{Z} \prec Z \prec \overline{Z}$ and $\neg(Y \precsim Z \vee Z \precsim Y)$.*

PROOF : Owing to 1.10 from [**K–Z b**] and 4.9 for each real semiset X it holds $\underline{X} \precsim X \precsim \overline{X}$. Let $Y, Z \subseteq u$ be the classes guaranteed by 1.5. Since $\underline{Y} = \underline{Z} = |u|/FN \prec u \approx \mathrm{int}(|u|) = \overline{Y} = \overline{Z}$ by 1.6, none of the four real subvalences $\underline{Y} \precsim Y \precsim \overline{Y}, \underline{Z} \precsim Z \precsim \overline{Z}$ can turn out to be real equivalence. ∎

If x, y are sets, then $x \overset{b}{\approx} y \Leftrightarrow |x| \simeq |y|$. (Essentially the same conjecture for hyperfinite sets within the scope of nonstandard analysis was raised by C.Henson [**H**] during a meeting in Oberwolfach; an affirmative solution was then announced as a result of a discussion between C.Henson and D.Ross [**H–R**].) Hence $\{\langle x, y\rangle; x \overset{b}{\approx} y\}$ is a π–equivalence on V. Similarly, $x \approx y \Leftrightarrow |x| = |y| \vee (|x|, |y| \notin FN \;\&\; |x| \sim |y|)$, hence $\{\langle x, y\rangle; x \approx y\}$ is a δ_2–equivalence). Therefore the pair $\langle \overset{b}{\approx}, \approx\rangle$ of equivalences on V coincides on infinite sets with the pair $\langle\{\langle x, y\rangle; |x| \simeq |y|\}, \{\langle x, y\rangle; |x| \sim |y|\}\rangle$ which, by 1.2, is a compatible biequivalence on V, since $\langle \simeq, \sim\rangle$ is a compatible biequivalence on N.

Now, let u be an infinite set such that $x \approx y$ for all $x, y \in u$. Then either there is an infinite set $v \subseteq u$ such that elements of v are finite, or an infinite set $w \subseteq u$ such that all elements of w are infinite. In the first case there is an $n \in FN$ and an infinite set $v_0 \subseteq v$ such that $|x| = n$ for each $x \in v_0$. In the second case, by 1.1, there is an infinite set $w_0 \subseteq w$ such that $|x| \simeq |y|$ for all $x, y \in w_0$.

We have proved the following Ramsey type theorem

4.15. Theorem. *Let u be an infinite set such that $x \approx y$ for all $x, y \in u$. Then there is an infinite subset $u_0 \subseteq u$ such that $x \overset{b}{\approx} y$ for all $x, y \in u_0$.*

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