Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 30 (1989), No. 1, 71--73

Persistent URL: http://dml.cz/dmlcz/106704

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Planarity thresholds for two types of random subgraphs of the n–cube

KARIN MAHRHOLD, KARL WEBER

Abstract. Solving a problem posed by the second author (cf. [2]) we determine the threshold probability $p_f = 2^{-n/11}n^{-4/11}$ ($p_g = 2^{-n/14}n^{-4/14}$) for planarity of random induced (spanning) subgraphs of the n–cube.

Keywords: Random graph, n–cube, planarity, random subgraph

Classification: 05C80

The n–cube Q_n is the graph consisting of the 2^n vertices $(a_1, \ldots, a_n), a_i \in \{0,1\}$, and the $n2^{n-1}$ edges between vertices differing in exactly one coordinate. A spanning subgraph g of Q_n has the same vertex set as Q_n. An induced subgraph f of Q_n with the vertex set $A \subseteq Q_n$ contains exactly those edges of Q_n that join two vertices in A. (Note that by Q_n or f are not only denoted the graphs but also their vertex sets, g stands for the edge set of g too.) Choosing the edges of g (the vertices of f) at random, independently of each other with the same probability p, we arrive at a random spanning (induced) subgraph whose probabilities are defined as Prob $(g) = p^{|g|}q^{n2^{n-1}-|g|}$ and Prob $(f) = p^{|f|}q^{2^n-|f|}$, respectively, where $q = 1-p$. We say g (or f) has a given property almost surely (a.s.) if the probability that g (or f) has this property tends to 1 as $n \to \infty$. A probability $\widetilde{p}$ is called a threshold for the property E if $\widetilde{p} = o(p)$ implies E is almost sure whereas $p = o(\widetilde{p})$ implies $\overline{E}$ is almost sure.

In the sequel we write $\alpha << \beta$ instead of $\alpha = o(\beta)$. We write $\alpha \asymp \beta$ if α and β have the same order of magnitude, i.e.$\alpha = O(\beta)$ and $\beta = O(\alpha)$. All limits, asymptotics, etc., are understood as $n \to \infty$.

Our main result is the following

Theorem 1. *The probability $p_f = 2^{-n/11}n^{-4/11}$ is the threshold probability for planarity of random induced subgraphs of Q_n, and the probability $p_g = 2^{-n/14}n^{-4/14}$ is the threshold probability for planarity of random spanning subgraphs of Q_n.*

A graph is called cubical if it can be embedded into some Q_n, i.e. if it is isomorphic to a subgraph of Q_n ([1]). A cubical subdivision of K_5 ($K_{3,3}$) with minimum number of vertices is called a minimum subdivision of K_5 ($K_{3,3}$). The key result for the proof of Theorem 1 is

Theorem 2 ([3]). *Every minimum subdivision of $K_{3,3}$ or K_5 is isomorphic to S_1 or S_2, respectively (cf. Fig. 1). Both S_1 and S_2 are (up to isometric transformations of Q_n) uniquely embeddable into Q_n for $n \geq 4$.*

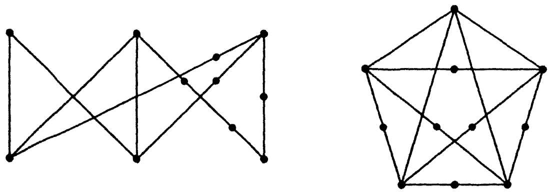

Fig. 1

Note that S_1 contains 11 vertices and 14 edges and S_2 contains 11 vertices and 16 edges.

PROOF: of Theorem 1. First we deal with random induced subgraphs f. Obviously $2^n n^{11} p^{12}$ is the order of magnitude for the expected number of connected subgraphs of f with 12 vertices. Hence for $p << 2^{-n/12} n^{-11/12} = p_1$ this expectation tends to zero and so f contains a.s. no connected subgraph of order 12 (or greater). Now denote by $X(f)$ the number of copies of S_1 or S_2 in f. Then by Theorem 2

$$EX \asymp 2\binom{n}{4} 2^{n-4} p^{11} \asymp n^4 2^n p^{11}.$$

(Note that the number of 4–cubes in Q_n is $\binom{n}{4}2^{n-4}$, and each 4–cube contains a bounded number of copies of S_1 and S_2. Moreover p^{11} is the probability that such a fixed copy is a subgraph of f.)

For $p << p_f = 2^{-n/11} n^{-4/11}$ we have $EX \to 0$ and f contains a.s no copy of S_1 or S_2 and, since $p_f < p_1$, no nonplanar subgraph in general.

In order to show that for $p >> p_f$, f contains a copy of S_1 or S_2 (actually of both of them) a.s. we use the second moment method. Because of $\text{Prob}(X = 0) \leq D^2X/(EX)^2$, our assertion follows if $D^2X = o((EX)^2)$ or $EX^2 = (EX)^2(1+o(1))$, respectively, can be shown.

Denote the copies of S_1 and S_2 in Q_n by $K_1, K_2, \ldots, K_T$ ($T \asymp n^4 2^n$ by Theorem 2) and define $X_i(f) = 1$ if f contains K_i and $X_i(f) = 0$ otherwise. Then we have $EX^2 = \sum E(X_i X_j)$, where the sum is taken over all (ordered) pairs $(i, j), 1 \leq i, j \leq T$. Now, by Theorem 2, K_i is contained in exactly one 4–dimensional subcube W_i of Q_n. Conversely, every W_i contains the same constant number of $K_j's$. Hence we get

$$\sum_{(i,j):|W_i \cap W_j| = 2^k} E(X_i X_j) = 0(n^{8-k} 2^n p^{22-2^k}) = o((EX)^2)$$

for $p >> p_f, k = 0, 1, 2, 3$. Moreover,

$$\sum_{(i,j):W_i = W_j} E(X_i X_j) = 0(EX) = o((EX)^2) \text{ for } p >> p_f$$

and since $K_i \cap K_j = \emptyset$ implies $E(X_i.X_j) = (EX_i)(EX_j)$ we have

$$\sum_{(i,j):W_i \cap W_j = \emptyset} E(X_i X_j) \leq (EX)^2.$$

Now the proof for random spanning subgraphs g goes along the same line. The expectation for the number of connected subgraphs of g with 15 edges is of the order $2^n n^{15} p^{15}$. Thus for $p << 2^{-n/15} n^{-1} = p_2$ the graph g contains no connected subgraph with 15 (or more) edges. Denote by $Y(g)$ the number of copies of S_1 in g. Then (arguing as above)

$$EY \asymp n^4 2^n p^{15},$$

and we may proceed as above. (In this case we have $E(X_i X_j) = (EX_i)(EX_j)$ even for $|W_i \cap W_j| \leq 1$.) ∎

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(*Received* August 11,1988, *revised* October 20,1988)