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A generalized Hankel transformation

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Abstract. In this paper we introduce a new integral transformation depending on three parameters, which reduces to the Hankel, Hankel–Schwartz and other Hankel type transformations for suitable choice of the parameters. We study classical properties of this transformation and establish an inversion formula for it. Also, we solve several differential equations involving the operator $B_{\alpha_0,\alpha_1,\alpha_2} = x^{\alpha_0} Dx^{\alpha_1} Dx^{\alpha_2}$ by using the new integral transformation.

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1. Introduction. The integral transformation defined by

$$h_\nu\{f(x)\}(y) = \int_0^\infty f(x) x J_\nu(xy)\, dx$$

is called the Hankel transformation. Hankel [**6**] was the first to give an inversion formula for it.

Theorem 1. ([**16**]) *Let $F(x)$ be an arbitrary function of the real variable x subject to the condition that $\int_0^\infty F(x)\sqrt{x}\, dx$ exists and is absolutely convergent. Let the order ν of the Bessel functions be not less than $-\frac{1}{2}$. In these conditions,*

$$\int_0^\infty u\, du \int_0^\infty F(x) J_\nu(ux) J_\nu(ur) x\, dx = \frac{1}{2}(F(r+0) + F(r-0))$$

provided that the positive number r lies inside an interval where $F(x)$ is of bounded variation.

Essentially the same proof was given by Sheppard [**12**], who stressed the important fact that the value of the integral depends only on that part of the x–range of integration which is in the immediate neigbourhood of r. A different class of proof, based on the theory of discontinuous integrals, has been given by Sonine [**14**]. Basset [**2**] presented a proof of a more direct physical character but, according to Gray and Mathews [**5**], it is open to various objections. A proof depending on the theory of integral equations has been constructed by Weyl [**17**].

An important property of the Hankel transformation is the following Parseval's equation.

Theorem 2. *If the functions $f(x)$ and $g(x)$ satisfy the conditions of Theorem 1 and if $F(y)$ and $G(y)$ denote their Hankel transforms of order $\nu \geq -\frac{1}{2}$, then*

$$\int_0^\infty x f(x) g(x)\, dx = \int_0^\infty y F(y) G(y)\, dy \tag{1}$$

Other conditions under which (1) holds were given by Macaulay–Owen [9].

The Hankel transformation is relevant because besides other properties, the operational rule

$$h_\nu\{B_\nu f(x)\}(y) = -y^2 h_\nu\{f(x)\}(y)$$

holds for suitable functions $f(x)$. Here B_ν denotes the Bessel operator $x^{-\nu-1} Dx^{2\nu+1} Dx^{-\nu}$. This fact opens a wide field of applications for the Hankel transform (see Sneddon [13] and Gerardi [4]).

In the last years several variants of the Hankel transformation (see Zemanian [18], Schwartz [11], Hayek [7], Méndez [10] and others) have been studied. Each one generates a rich operational calculus for certain Bessel type operator.

In this paper we introduce a new integral transformation defined by

$$F_{\alpha_0,\alpha_1,\alpha_2}\{f(x)\}(y) = y^{\alpha_0-\alpha_2} \int_0^\infty J(xy, \alpha_2, \alpha_1, \alpha_0) f(x)\, dx$$

where $J(z, \alpha_0, \alpha_1, \alpha_2) = z^{(1-\alpha_1-2\alpha_2)/2} J_\nu(\frac{2}{2+k} z^{(2+k)/2})$, with $k = -\alpha_0 - \alpha_1 - \alpha_2, \nu = \frac{\alpha_1 - 1}{2+k} \geq -\frac{1}{2}$ and $2+k > 0$. This paper is organized as follows: first, we study classical properties of this transformation and establish an inversion formula for it. Note that $F_{\alpha_0,\alpha_1,\alpha_2}$–transformation reduces to the Hankel–type considered above (see also [1], [7], [10] and [18]) for suitable choises of the parameters α_0, α_1 and α_2. This new transformation satisfies the following operational rule

$$F_{\alpha_0,\alpha_1,\alpha_2}\{B_{\alpha_0,\alpha_1,\alpha_2} f(x)\}(y) = -y^{2+k} F_{\alpha_0,\alpha_1,\alpha_2}\{f(x)\}(y)$$

where $B_{\alpha_0,\alpha_1,\alpha_2} = x^{\alpha_0} Dx^{\alpha_1} Dx^{\alpha_2}$ and $f(t)$ is a suitable function. Hence, $F_{\alpha_0,\alpha_1,\alpha_2}$ is useful to solve ordinary and partial differential equations involving the operator $B_{\alpha_0,\alpha_1,\alpha_2}$. Finally, we analyze several applications of the $F_{\alpha_0,\alpha_1,\alpha_2}$–transformation.

2. The function $J(X; \alpha_0, \alpha_1, \alpha_2)$. Properties.

Let $\alpha_0, \alpha_1, \alpha_2$ be real numbers. We now consider the differential equation

$$(B_{\alpha_0,\alpha_1,\alpha_2} + 1) y(x) = 0 \tag{2}$$

with $2 - (\alpha_0 + \alpha_1 + \alpha_2) > 0$. In the sequel, to simplify notation, we will denote $k = -\alpha_0 - \alpha_1 - \alpha_2$.

The operator $B_{\alpha_0,\alpha_1,\alpha_2}$ reduces to the important Bessel operators related to certain Hankel type transformations. For example, the operators: $S_\mu = x^{-\mu-1/2} Dx^{2\mu+1} Dx^{-\mu-1/2}$ (Zemanian [18]), $DP_\mu = Dx^{\mu+1} Dx^{-\mu}$ (Mendez [10]), $B_\mu =$

$x^{-\mu-1}Dx^{2\mu+1}Dx^{-\mu}$ (Sneddon [**13**]) and $\Delta_\mu = x^{-2\mu-1}Dx^{2\mu+1}D$ (Altenburg [**1**]) can be obtained from $B_{\alpha_0,\alpha_1,\alpha_2}$ for suitable values of the parameters α_0, α_1 and α_2.

We chose among the solutions of (2) the function denoted by

$$J(x;\alpha_0,\alpha_1,\alpha_2) = x^{(1-\alpha_1-2\alpha_2)/2} J_\nu\Big(\frac{2}{2+k} x^{(2+k)/2}\Big) \tag{3}$$

when $\nu = \frac{\alpha_1-1}{2+k} \geq -\frac{1}{2}$. This function plays an underlying role in our work and reduces to known special functions for certain values of α_0, α_1 and α_2 (for example the Bessel function J_μ of the first kind and order μ is equal to $J(x; -\mu-1, 2\mu+1, -\mu)$).

The function $J(x;\alpha_0,\alpha_1,\alpha_2)$ possesses the following series expansion which converges for every $x \in (0,\infty)$

$$J(x;\alpha_0,\alpha_1,\alpha_2) = (2+k)^{(1-\alpha_1)/(2+k)} x^{-\alpha_2} \sum_{n=0}^{\infty} \frac{(-1)^n (2+k)^{-2n}}{n!\Gamma(n+\frac{\alpha_1-1}{2+k}+1)} x^{n(2+k)}.$$

We now list some properties of the function that will be useful in the sequel.

Proposition 1. *If* $\frac{\alpha_1-1}{2+k} \geq -\frac{1}{2}$ *and* $2+k>0$ *then*

$$B_{\alpha_0,\alpha_1,\alpha_2,x} J(xy;\alpha_0,\alpha_1,\alpha_2) = -y^{2+k} J(xy;\alpha_0,\alpha_1,\alpha_2)$$

To prove this assertion it is sufficient taking into account that $J(x;\alpha_0,\alpha_1,\alpha_2)$ is a solution of the differential equations (2).

Proposition 2. *In the same conditions that in Proposition 1, one has*

$$\frac{d}{dx}(x^{\alpha_2} J(x;\alpha_0,\alpha_1,\alpha_2)) = $$

$$= -x^{-\alpha_2+\frac{k}{2}} J(x;\alpha_0 - \frac{2+k}{2}, \alpha_1+2+k, \alpha_2 - \frac{2+k}{2}) \tag{4}$$

$$x^{(2+k)/2} \frac{d}{dx}(x^{\alpha_1+\alpha_2-1} J(x;\alpha_0,\alpha_1,\alpha_2)) = $$

$$= x^{-\alpha_0} J(x;\alpha_0 + \frac{2+k}{2}, \alpha_1-2-k, \alpha_2 + \frac{2+k}{2}) \tag{5}$$

(4) and (5) can be proved by using known properties of the Bessel function J_μ.

In the following proposition we present the asymptotic behaviours of the J–function.

Proposition 3. *In the conditions of Proposition 1, we have:*

$$x^{(\alpha_1+2\alpha_2-1)/2} J(x;\alpha_0,\alpha_1,\alpha_2) \cong \sqrt{\frac{2+k}{\pi}} x^{-(2+k)/4} \times$$

$$\times \cos\Big(\frac{2}{2+k} x^{(2+k)/2} - \frac{\pi}{2}\Big(\frac{\alpha_1-1}{2+k}+\frac{1}{2}\Big)\Big) + o(x^{-3(2+k)/4}), \text{ as } x \to \infty \tag{6}$$

$$J(x;\alpha_0,\alpha_1,\alpha_2) \cong \frac{(2+k)^{(1-\alpha_1)/(2+k)}}{\Gamma(1+\frac{\alpha_1-1}{2+k})} x^{-\alpha_2}, \text{ as } x \to 0. \tag{7}$$

3. The integral transformation $F_{\alpha_0,\alpha_1,\alpha_2}$.

Under certain conditions of convergence which will be conveniently specified, the following pair defines an integral transformation denoted by $F_{\alpha_0,\alpha_1,\alpha_2}$.

$$(8)\qquad \begin{aligned} F(y) &= F_{\alpha_0,\alpha_1,\alpha_2}\{f(x)\}(y) = y^{\alpha_0-\alpha_2}\int_0^\infty J(xy;\alpha_2,\alpha_1,\alpha_0)f(x)\,dx \\ f(x) &= F_{\alpha_0,\alpha_1,\alpha_2}\{F(y)\}(x) = x^{\alpha_0-\alpha_2}\int_0^\infty J(xy;\alpha_2,\alpha_1,\alpha_0)F(y)\,dy \end{aligned}$$

The main property of this new transformation is that it reduces to other ones studied earlier for suitable values of α_0, α_1 and α_2, namely: $H_\mu = F_{-1-\mu,1+2\mu,-\mu}$ (Hankel [6]), $h_\mu = F_{-\mu-\frac{1}{2},2\mu+1,-\mu-\frac{1}{2}}$ (Zemanian [18]), ${}_ch_{\mu,2} = F_{-\mu,\mu+1,0}$ (Hayek [7]), ${}_ch_{\mu,1} = F_{0,\mu+1,-\mu}$ (Mendez [10]) and $B_\mu = F_{-2\mu-1,2\mu+1,0}$ (Schwartz [11]).

3.1. Convergence.

In this section the conditions of convergence for the $F_{\alpha_0,\alpha_1,\alpha_2}$–transformation are stated.

We consider the integral

$$(9)\qquad \int_0^\infty J(x;\alpha_2,\alpha_1,\alpha_0)f(x)\,dx$$

where $f(x)$ is a locally integrable function on $0 < x < \infty$ such that

$$\begin{aligned} f(x) &= o(x^\alpha),\ \text{as } x \to 0, \qquad \text{and} \\ f(x) &= o(x^\beta),\ \text{as } x \to \infty \end{aligned}$$

According to (6) and (7) it can be deduced that the integral in (9) is absolutely convergent when

$$\alpha > \alpha_0 - 1 \qquad \text{and} \qquad \beta < \frac{\alpha_1 - 1 + 2\alpha_0}{2} + \frac{2+k}{4} - 1$$

Hence we get

Proposition 4. *The integral defining $F_{\alpha_0,\alpha_1,\alpha_2}\{f(x)\}(y)$ is absolutely convergent provided that $f(x)$ is a locally integrable function on $0 < x < \infty$ such that $\alpha > \alpha_0 - 1$ and $\beta < \frac{\alpha_1-1+2\alpha_0}{2} + \frac{2+k}{4} - 1$.*

3.2. Operational Calculus.

We now prove several operational rules for the $F_{\alpha_0,\alpha_1,\alpha_2}$-transformation, some of them are useful in certain applications.

In view of the previous definitions we can write

$$F_{\alpha_0,\alpha_1,\alpha_2}\{B_{\alpha_0,\alpha_1,\alpha_2}f(x)\}(y) = \int_0^\infty x^{\alpha_2}J(xy;\alpha_0,\alpha_1,\alpha_2)Dx^{\alpha_1}Dx^{\alpha_2}f(x)\,dx$$

and by using (4) and integrating by parts, one has

$$F_{\alpha_0,\alpha_1,\alpha_2}\{B_{\alpha_0,\alpha_1,\alpha_2}f(x)\}(y) = [x^{\alpha_2}J(xy;\alpha_0,\alpha_1,\alpha_2)x^{\alpha_1}Dx^{\alpha_2}f(x)- x^{\alpha_1+\alpha_2}f(x)D(x^{\alpha_2}J(xy;\alpha_0,\alpha_1,\alpha_2)]_0^\infty - y^{2+k}F_{\alpha_0,\alpha_1,\alpha_2}\{f(x)\}(y)$$

Therefore

(10) $$F_{\alpha_0,\alpha_1,\alpha_2}\{B_{\alpha_0,\alpha_1,\alpha_2}f(x)\}(y) = -y^{2+k}F_{\alpha_0,\alpha_1,\alpha_2}\{f(x)\}(y)$$

provided that:

(11) $$x^{\alpha_2+\alpha_1}J(x;\alpha_0,\alpha_1,\alpha_2)D(x^{\alpha_2}f(x))]_0^\infty = 0$$

(12) $$x^{\alpha_1+\alpha_2}f(x)D(x^{\alpha_2}J(x;\alpha_0,\alpha_1,\alpha_2))]_0^\infty = 0$$

Note that there exists a wide class of functions satisfying (11) and (12). For example, all enough smooth functions $f(x)$ such that

$$f(x) = o(x^\rho), \qquad f'(x) = o(x^\gamma) \qquad \text{as } x \to 0$$
$$f(x) = o(x^\beta), \qquad f'(x) = o(x^\alpha) \qquad \text{as } x \to \infty$$

with $\rho > 1-\alpha_1-\alpha_2, \alpha < \frac{2}{2+k}-\alpha_1-2\alpha_2, \gamma > -\alpha_2-\alpha_1$ and $\beta < \min(\frac{1}{2}-\alpha_1-2\alpha_2-\frac{k}{4}, \frac{k}{4}+\frac{3}{2}-2\alpha_2-\alpha_1)$, satisfy (11) and (12).

With respect to the integral operation the following operational rule is true

(13) $$F_{\alpha_0-\frac{2+k}{2},\alpha_1+2+k,\alpha_2-\frac{2+k}{2}}\{x^{\alpha_0+k/2}I(x^{-\alpha_0}f(x))\}(y) = \\ = y^{-(2+k)/2}F_{\alpha_0,\alpha_1,\alpha_2}\{f(x)\}(y) - y^{-\alpha_2-(2+k)/2}\times \\ \times(xy)^{\alpha_0}J(x;\alpha_2,\alpha_1,\alpha_0)I\{x^{-\alpha_0}f(x)\}]_0^\infty$$

which can be seen by making use of (4) and integration by parts.

On the other hand, it can be easily proved that

(14) $$F_{\alpha_0,\alpha_1,\alpha_2}\{f(ax)\}(y) = a^{\alpha_0-\alpha_2-1}F_{\alpha_0,\alpha_1,\alpha_2}\{f(x)\}(\frac{y}{a})$$

a being a positive real constant.

3.3. Parseval's relations.

Let $f(x)$ and $G(y)$ be functions such that $f(x)x^{-\alpha_0}$ is in $L_1(0,\infty)$ and $G(y)y^{-\alpha_0}$ is also in $L_1(0,\infty)$. If

$$\frac{\alpha_1-1}{2+k} \geq -\frac{1}{2}, F(y) = F_{\alpha_0,\alpha_1,\alpha_2}\{f(x)\}(y) \text{ and } g(x) = F_{\alpha_0,\alpha_1,\alpha_2}\{G(y)\}(x),$$

one has

$$\int_0^\infty f(x)g(x)x^{\alpha_2-\alpha_0}\,dx = \int_0^\infty f(x)\int_0^\infty J(xy;\alpha_2,\alpha_1,\alpha_0)G(y)\,dydx =$$
$$= \int_0^\infty G(y)y^{\alpha_2-\alpha_0}\{y^{\alpha_0-\alpha_2}\int_0^\infty J(xy;\alpha_2,\alpha_1,\alpha_0)f(x)\,dx\}\,dy =$$
$$= \int_0^\infty G(y)F(y)y^{\alpha_2-\alpha_0}\,dy$$

It is possible to interchange the integration order because $z^{\alpha_0}J(z;\alpha_2,\alpha_1,\alpha_0)$ is bounded on $0 < z < \infty$ (see (6) and (7)) and

$$\int_0^\infty |f(x)x^{-\alpha_0}|dx \int_0^\infty |G(y)y^{-\alpha_0}|\,dy < \infty$$

Hence we can state

Proposition 5. *If $f(x)x^{-\alpha_0}$ and $G(y)y^{-\alpha_0}$ are in $L_1(0,\infty)$ and, in addition, $\frac{\alpha_1-1}{2+k} \geq -\frac{1}{2}, F(y) = F_{\alpha_0,\alpha_1,\alpha_2}\{f(x)\}(y)$ and $g(x) = F_{\alpha_0,\alpha_1,\alpha_2}\{G(y)\}(x)$, then*

$$\int_0^\infty x^{\alpha_2-\alpha_0}f(x)g(x)\,dx = \int_0^\infty y^{\alpha_2-\alpha_0}F(y)G(y)\,dy$$

Other integral products are formally the following ones:

a) *If $f(x) = g(x)$ then $F(y) = G(y)$ and*

$$\int_0^\infty x^{\alpha_2-\alpha_0}[f(x)]^2\,dx = \int_0^\infty y^{\alpha_2-\alpha_0}[F(y)]^2\,dy$$

b)

$$\int_0^\infty x^{\alpha_2-\alpha_0}f(x)G(x)\,dx = \int_0^\infty y^{\alpha_2-\alpha_0}F(y)g(y)\,dy$$

4. The inversion theorem.

In this section we give a direct proof of the inversion formula for the $F_{\alpha_0,\alpha_1,\alpha_2}$–transformation. This proof is supported in the properties of the function $J(x;\alpha_0,\alpha_1,\alpha_2)$ although the employed procedure is similar to the way followed by Watson [**16**], Sneddon [**13**], and Titchmarsh [**15**] to establish the inversion theorem for the ordinary Hankel transform.

Our inversion formula is obtained as a consequence of the results that we will present in the following three propositions.

Proposition 6. *If $\frac{\alpha_1-1}{2+k} \geq -\frac{1}{2}$ and the integral $\int_0^\infty |f(x)|x^{(1-\alpha_1-2\alpha_0)/2-(2+k)/4}\,dx$ exists, then*

$$\int_0^\infty u^{\alpha_0-\alpha_2}\int_0^\infty f(x)J(ux;\alpha_2,\alpha_1,\alpha_0)J(ur;\alpha_2,\alpha_1,\alpha_0)\,dxdu =$$
$$= \lim_{\lambda\to\infty}\int_0^\infty f(x)\int_0^\lambda u^{\alpha_0-\alpha_2}J(ux;\alpha_2,\alpha_1,\alpha_0)J(ur;\alpha_2,\alpha_1,\alpha_0)\,dudx$$

provided that the last limit has sense.

PROOF: We define the function

$$\Phi(x,u) = f(x)J(ru;\alpha_2,\alpha_1,\alpha_0)J(ux;\alpha_2,\alpha_1,\alpha_0)u^{\alpha_0-\alpha_2}$$

It can be written

$$\Phi(x,u) = f(x)x^{\frac{1-\alpha_1-2\alpha_0}{2}-\frac{2+k}{4}}u^{\frac{k}{2}}r^{\frac{1-\alpha_1-2\alpha_0}{2}-\frac{2+k}{4}}\times \\ \times (ux)^{\frac{\alpha_1+2\alpha_0-1}{2}+\frac{2+k}{4}}J(ux;\alpha_2,\alpha_1,\alpha_0)(ur)^{\frac{\alpha_1+2\alpha_0-1}{2}+\frac{2+k}{4}}J(ur;\alpha_2,\alpha_1,\alpha_0)$$

and, since in virtue of (6) and (7) $u^{\frac{\alpha_1-1+2\alpha_0}{2}+\frac{2+k}{4}}J(u;\alpha_2,\alpha_1,\alpha_0)$ is bounded on $0<u<\infty$, there exists a constant $A=A(\alpha_1,k)$ such that

$$|\Phi(x,u)| < A^2 r^{\frac{1-\alpha_1-2\alpha_0}{2}-\frac{2+k}{4}}x^{\frac{1-\alpha_1-2\alpha_0}{2}-\frac{2+k}{4}}u^{\frac{k}{2}}|f(x)|$$

Therefore, $\int_0^\infty\int_0^\lambda|\Phi(x,u)|\,dudx$ exists provided that the integral $\int_0^\infty|f(x)|x^{\frac{1-\alpha_1-2\alpha_0}{2}-\frac{2+k}{4}}\,dx$ exists and by using Fubini's Theorem one has

$$\int_0^\infty\int_0^\lambda\Phi(x,u)\,dudx = \int_0^\lambda\int_0^\infty\Phi(x,u)\,dxdu$$

Therefore if $\int_0^\infty|f(x)|x^{\frac{1-\alpha_1-2\alpha_0}{2}-\frac{2+k}{4}}\,dx$ exists then

$$\int_0^\infty u^{\alpha_0-\alpha_2}\int_0^\infty f(x)J(ux;\alpha_2,\alpha_1,\alpha_0)J(ur;\alpha_2,\alpha_1,\alpha_0)\,dxdu = \\ = \lim_{\lambda\to\infty}\int_0^\infty f(x)\int_0^\lambda u^{\alpha_0-\alpha_2}J(ux;\alpha_2,\alpha_1,\alpha_0)J(ur;\alpha_2,\alpha_1,\alpha_0)\,dudx \quad \blacksquare$$

By using a procedure similar to the one followed by Watson **[16]** pp. 457, the following statement can be proved.

Lemma 1. *If $f(x)x^{-\frac{2+k}{4}}$ is absolutely integrable in $(a,b)(0<a<b)$ then*

$$\lambda^{\frac{\alpha_1-1+2\alpha_2}{2}}\int_a^b f(x)x^{\frac{\alpha_1-1+2\alpha_2}{2}}J(x\lambda,\alpha_0,\alpha_1,\alpha_2)\,dx = o(\lambda^{-\frac{2+k}{4}}),$$

as $\lambda\to\infty$ for $\frac{\alpha_1-1}{2+k}\geq-\frac{1}{2}$.

An integral involving J–functions arises in the proof of the following propositions. This is solved in Lemma 2.

Lemma 2. *If* $\frac{\alpha_1-1}{2+k} \geq -\frac{1}{2}$ *then*

$$\int_0^\lambda u^{\alpha_0-\alpha_2} J(xu;\alpha_2,\alpha_1,\alpha_0)J(ur;\alpha_2,\alpha_1,\alpha_0)\,du =$$

$$= \frac{\lambda^{\alpha_1+2\alpha_0-1+\frac{2+k}{2}}}{x^{2+k}-r^{2+k}}\Big(x^{\frac{2+k}{2}} J(x\lambda;\alpha_2-\frac{2+k}{2},\alpha_1+2+k,\alpha_0-\frac{2+k}{2})\times$$
$$\times J(r\lambda;\alpha_2,\alpha_1,\alpha_0) - r^{\frac{2+k}{2}} J(r\lambda;\alpha_2-\frac{2+k}{2},\alpha_1+2+k,\alpha_0-\frac{2+k}{2})\times$$
$$\times J(x\lambda;\alpha_2,\alpha_1,\alpha_0)\Big).$$

PROOF: In view of Proposition 2 and integrating by parts, one has

$$I = \int_0^\lambda u^{\alpha_0-\alpha_2} J(xu;\alpha_2,\alpha_1,\alpha_0)J(ur;\alpha_2,\alpha_1,\alpha_0)\,du =$$
$$= r^{-\frac{2+k}{2}}\Big[u^{\alpha_1+2\alpha_0+\frac{k}{2}} J(ru;\alpha_2-\frac{2+k}{2},\alpha_1+2+k,\alpha_0-\frac{2+k}{2})\times$$
$$\times J(xu;\alpha_2,\alpha_1,\alpha_0)\Big]_{u=0}^{u=\lambda} + (\frac{x}{r})^{\frac{2+k}{2}}\times$$
$$\times\int_0^\lambda u^{\alpha_0-\alpha_2} J(xu;\alpha_2-\frac{2+k}{2},\alpha_1+2+k,\alpha_0-\frac{2+k}{2})\times$$
$$\times J(ur;\alpha_2-\frac{2+k}{2},\alpha_1+2+k,\alpha_0-\frac{2+k}{2})\,du$$

Hence, since $\frac{\alpha_1-1}{2+k} \geq -\frac{1}{2}$ we get

$$I = r^{-2-k}\Big(r^{\frac{2+k}{2}}\lambda^{\alpha_1+2\alpha_0+\frac{k}{2}} J(r\lambda;\alpha_2-\frac{2+k}{2},\alpha_1+2+k,\alpha_0-\frac{2+k}{2})\times$$
$$\times J(x\lambda;\alpha_2,\alpha_1,\alpha_0) + (rx)^{\frac{2+k}{2}}\times$$
$$\times\int_0^\lambda u^{\alpha_0-\alpha_2} J(xu;\alpha_2-\frac{2+k}{2},\alpha_1+2+k,\alpha_0-\frac{2+k}{2})\times$$
$$\times J(ur;\alpha_2-\frac{2+k}{2},\alpha_1+2+k,\alpha_0-\frac{2+k}{2})\,du\Big)$$

By repeating the procedure, we can write

$$I = \frac{\lambda^{\alpha_1+2\alpha_0-1+\frac{2+k}{2}}}{x^{2+k}-r^{2+k}}\Big(x^{\frac{2+k}{2}} J(x\lambda;\alpha_2-\frac{2+k}{2},\alpha_1+2+k,\alpha_0-\frac{2+k}{2})\times$$
$$\times\, J(r\lambda;\alpha_2,\alpha_1,\alpha_0) - r^{\frac{2+k}{2}} J(r\lambda;;\alpha_2-\frac{2+k}{2},\alpha_1+2+k,\alpha_0-\frac{2+k}{2})\times$$
$$\times\, J(x\lambda;\alpha_2,\alpha_1,\alpha_0)\Big)$$

■

The next proposition shows that the only part of the x-range of integration which contributes to the value of the repeated integral under consideration is the immediate vicinity of the point $x = r$.

Proposition 7. *Under in the same conditions of Proposition 6, if $r \notin [a,b]$ then*

$$\int_0^\infty u^{\alpha_0-\alpha_2} \int_a^b f(x) J(ux;\alpha_2,\alpha_1,\alpha_0) J(ur;\alpha_2,\alpha_1,\alpha_0)\, dx du = 0$$

PROOF: To prove this result we use Proposition 6 to obtain

$$\int_0^\infty u^{\alpha_0-\alpha_2} \int_a^b f(x) J(ux;\alpha_2,\alpha_1,\alpha_0) J(ur;\alpha_2,\alpha_1,\alpha_0)\, dx du =$$

$$= \lim_{\lambda\to\infty} \int_a^b f(x) \int_0^\lambda u^{\alpha_0-\alpha_2} J(ux;\alpha_2,\alpha_1,\alpha_0) J(ur;\alpha_2,\alpha_1,\alpha_0)\, du dx$$

By applying Lemma 2, we have

$$\int_0^\infty u^{\alpha_0-\alpha_2} \int_a^b f(x) J(ux;\alpha_2,\alpha_1,\alpha_0) J(ur;\alpha_2,\alpha_1,\alpha_0)\, dx du =$$

$$= \lim_{\lambda\to\infty} \lambda^{\alpha_1+2\alpha_0-1+\frac{2+k}{2}} J(r\lambda;\alpha_2,\alpha_1,\alpha_0) \int_a^b J(x\lambda;\alpha_2-\frac{2+k}{2},\alpha_1+2+k,\alpha_0-\frac{2+k}{2})\times$$

$$\times \frac{f(x)x^{\frac{2+k}{2}}}{x^{2+k}-r^{2+k}}\, dx - \lim_{\lambda\to\infty} \lambda^{\alpha_1+2\alpha_0-1+\frac{2+k}{2}} J(r\lambda;\alpha_2-\frac{2+k}{2},\alpha_1+2+k,\alpha_0-\frac{2+k}{2})\times$$

$$\times r^{\frac{2+k}{2}} \int_a^b \frac{f(x)}{x^{2+k}-r^{2+k}} J(x\lambda;\alpha_2,\alpha_1,\alpha_0)\, dx$$

We denote

$$I_1(\lambda) = \lambda^{\frac{\alpha_1+2\alpha_0-1}{2}} \int_a^b J(x\lambda;\alpha_2-\frac{2+k}{2},\alpha_1+2+k,\alpha_0-\frac{2+k}{2}) \frac{f(x)x^{\frac{2+k}{2}}}{x^{2+k}-r^{2+k}}\, dx$$

$$I_2(\lambda) = \lambda^{\frac{\alpha_1+2\alpha_0-1}{2}} \int_a^b \frac{f(x)}{x^{2+k}-r^{2+k}} J(x\lambda;\alpha_2,\alpha_1,\alpha_0)\, dx$$

Hence, since $f(x)x^{\frac{1-\alpha_1-2\alpha_0}{2}-\frac{2+k}{4}}$ is integrable (if it is an improper integral it is absolutely integrable) on (a,b) and the functions

$$\frac{x^{\frac{2+k}{2}}}{x^{2+k}-r^{2+k}} \text{ and } \frac{1}{x^{2+k}-r^{2+k}}$$

are bounded on $x \in [a,b]$ (note that $r \notin [a,b]$), we can infer by using Lemma 1 that

$$I_i = o(\lambda^{-\frac{2+k}{4}}) \text{ as } \lambda \to \infty, \text{ for } i = 1,2.$$

Therefore

$$\int_0^\infty u^{\alpha_0-\alpha_2} \int_a^b f(x) J(ux;\alpha_2,\alpha_1,\alpha_0) J(ur;\alpha_2,\alpha_1,\alpha_0)\, dx du = 0$$

when $r \notin [a,b]$, since the function $u^{\frac{\alpha_1-1+2\alpha_0}{2}+\frac{2+k}{4}} J(u;\alpha_2,\alpha_1,\alpha_0)$ is bounded on $0<u<\infty$ provided that $\frac{\alpha_1-1}{2+k} \geq -\frac{1}{2}$.

We now establish the contribution from the immediate vicinity of $x = r$. ∎

Proposition 8. *If f is of bounded variation in a neigborhood of the point $r > 0$ and δ is a positive number, then*

$$\lim_{\lambda\to\infty} r^{\alpha_0-\alpha_2}\int_{r-\delta}^{r+\delta}\int_0^{\lambda} f(x)u^{\alpha_0-\alpha_2}J(ux;\alpha_2,\alpha_1,\alpha_0)\,dudx =$$
$$= \frac{1}{2}\{f(x+0)+f(x-0)\},\ \textit{provided that}\ \frac{\alpha_1-1}{2+k}\geq=-\frac{1}{2}.$$

PROOF: We consider the following integral

$$\int_{r-\delta}^{r+\delta}\int_0^{\lambda} f(x)u^{\alpha_0-\alpha_2}J(ux;\alpha_2,\alpha_1,\alpha_0)J(ur;\alpha_2,\alpha_1,\alpha_0)\,dudx$$

If f is of bounded variation in a neigborhood of $r > 0$ then

$$f(x)x^{(3\alpha_2+\alpha_1-\alpha_0)/4} = X_1(x) - X_2(x)$$

on the said neigborhood, where X_1 and X_2 are monotonic positive increasing functions.

By applying the second mean–value theorem to $X_1(x)$ we can write

$$\int_{r}^{r+\delta}\int_0^{\lambda} X_1(x)x^{-(3\alpha_2+\alpha_1-\alpha_0)/4}u^{\alpha_0-\alpha_2}J(ux;\alpha_2,\alpha_1,\alpha_0)\times$$
$$\times J(ur;\alpha_2,\alpha_1,\alpha_0)\,dudx = X_1(r+0)\int_{r}^{r+\delta}\int_0^{\lambda} x^{-(3\alpha_2+\alpha_1-\alpha_0)/4}\times$$
$$\times u^{\alpha_0-\alpha_2}J(ux;\alpha_2,\alpha_1,\alpha_0)J(ur;\alpha_2,\alpha_1,\alpha_0)\,dudx+$$
$$+\{X_1(r+\delta)-X_1(r+0)\}\int_{r+\xi}^{r+\delta}\int_0^{\lambda} x^{-(3\alpha_2+\alpha_1-\alpha_0)/4}u^{\alpha_0-\alpha_2}\times$$
$$\times J(ux;\alpha_2,\alpha_1,\alpha_0)J(ur;\alpha_2,\alpha_1,\alpha_0)\,dudx$$

ξ being in $(0,\delta)$.

Hence

$$\lim_{\lambda\to\infty}\int_{r}^{r+\delta}\int_0^{\lambda} X_1(x)x^{-(3\alpha_2+\alpha_1-\alpha_0)/4}u^{\alpha_0-\alpha_2}J(ux;\alpha_2,\alpha_1,\alpha_0)\times$$
$$\times\ J(ur;\alpha_2,\alpha_1,\alpha_0)\,dudx = \frac{1}{2}X_1(r+0)r^{(-3\alpha_2+\alpha_1-\alpha_0)/4-(2+k)/4}$$

The procedure can be repeated for $X_2(x)$ and the x–integral is extended on $x\in(r-\delta,r)$.

Finally we obtain

$$\lim_{\lambda\to\infty} r^{\alpha_0-\alpha_2}\int_{r-\delta}^{r+\delta}\int_0^{\lambda} f(x)u^{\alpha_0-\alpha_2}J(ux;\alpha_2,\alpha_1,\alpha_0)J(ur;\alpha_2,\alpha_1,\alpha_0)\,dudx =$$
$$= \frac{1}{2}\{f(x+0)+f(x-0)\}.$$

■

In virtue of Proposition 6,7 and 8 we can state

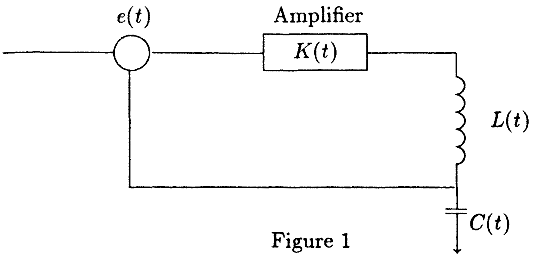
Theorem 3. *If f is of bounded variation in a vicinity of the point $r > 0$, $\frac{\alpha_1 - 1}{2+k} \geq -\frac{1}{2}$ and $\int_0^\infty |f(x)|^{\frac{1-\alpha_1-2\alpha_0}{2} - \frac{2+k}{4}}\, dx$ exists, then*

$$\lim_{\lambda\to\infty} r^{\alpha_0-\alpha_2} \int_0^\infty f(x) \int_0^\lambda u^{\alpha_0-\alpha_2} J(ux;\alpha_2,\alpha_1,\alpha_0) J(ur;\alpha_2,\alpha_1,\alpha_0)\, du dx =$$
$$= \frac{1}{2}\{f(x+0)+f(x-0)\}.$$

5. Applications.

We now show several applications of the new integral transformation $F_{\alpha_0,\alpha_1,\alpha_2}$. We solve certain differential equations involving the Bessel type operator $B_{\alpha_0,\alpha_1,\alpha_2}$ which describe time varying networks.

Example 1. Given the positive feedback circuit of Figure 1 with a variable gain amplifier and variable networks elements



$$L(t) = at^{\alpha_1}, \quad a > 0$$
$$C(t) = bt^{\alpha_0}, \quad b > 0$$
$$K(t) = Kt^{\alpha_2}, \quad K > 0$$

Figure 1

with $k = -2, (\alpha_1 - 1)^2 - \frac{4K}{ab} > 0$ and $\alpha_2 < 4$. We wish to determinate the charge $q(t)$ flowing the inductor and capacitor during the time $0 < t < \infty$. $q(t)$ satisfies the following differential equation

$$\text{(15)} \qquad ab(t^{\alpha_0} D t^{\alpha_1} D - \frac{K}{ab} t^{\alpha_2}) q(t) + q(t) = bKt^{\alpha_2+\alpha_0} e(t).$$

In virtue of the previous conditions there exist three real numbers k_0, k_1 and k_2 such that $2 - (k_0 + k_1 + k_2) > 0$ and $t^{k_0} D t^{k_1} D t^{k_2} = t^{\alpha_0} D t^{\alpha_1} D - \frac{K}{ab} t^{\alpha_2}$. Hence (15) can be rewritten

$$(abt^{k_0} D t^{k_1} D t^{k_2} + 1) q(t) = bKt^{\alpha_2+\alpha_0} e(t)$$

By applying the F_{k_0,k_1,k_2}–transformation and according to the operational rule (10) one has

$$Q(x) = \frac{1}{1 + abx^{4-\alpha_2}} E(x)$$

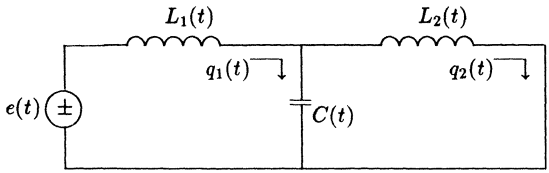
where

$$Q(x) = F_{k_0,k_1,k_2}\{q(t)\}(x) \text{ and}$$
$$E(x) = F_{k_0,k_1,k_2}\{bKt^{\alpha_2+\alpha_0} e(t)\}(x).$$

Therefore, by invoking the inversion formula we get

$$q(t) = F_{k_0,k_1,k_2}\{\frac{1}{1+abx^{4-\alpha_2}}E(x)\}(t).$$

Example 2. Consider the electrical network shown in Figure 2 for the time $0 < t < \infty$. The network consists of a voltage source $e(t)$, two inductors $L_1(t)$ and $L_2(t)$ and a capacitor $C(t)$. Let q_1 and q_2 be the mesh charges as shown.



$$L_1(t) = at^{\alpha_1}, \quad a > 0$$
$$L_2(t) = bt^{\alpha_1}, \quad b > 0$$
$$C(t) = ct^{\alpha_0}, \quad c > 0$$

Figure 2

Upon applying a mesh analysis we obtain simultaneous differential equations

$$\begin{aligned} ct^{\alpha_0}e(t) &= acB_{\alpha_0,\alpha_1,0}\, q_1(t) + q_1(t) - q_2(t) \\ 0 &= q_2(t) - q_1(t) + cbB_{\alpha_0,\alpha_1,0}\, q_2(t) \end{aligned}$$

These equations have the right form for an analysis via the $F_{\alpha_0,\alpha_1,\alpha_2}$–transformation. By applying $F_{\alpha_0,\alpha_1,\alpha_2}$ one has

$$\begin{aligned} E(x) &= (1 - ac\,x^{2-\alpha_0-\alpha_1})Q_1(x) - Q_2(x) \\ 0 &= (1 - cb\,x^{2-\alpha_0-\alpha_1})Q_2(x) - Q_1(x) \end{aligned}$$

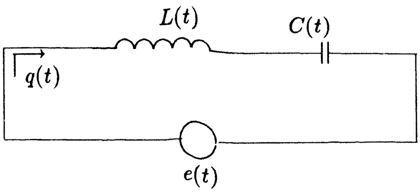
where E, Q_1 and Q_2 denote the $F_{\alpha_0,\alpha_1,0}$–transforms of $ct^{\alpha_0}e(t), q_1$ and q_2, respectively. Upon solving the last system equations for Q_1 and Q_2, we get

$$Q_1(x) = \frac{1 - cb\,x^{2-(\alpha_0+\alpha_1)}}{ac^2b\,x^{4-2(\alpha_0+\alpha_1)} - (a+b)c\,x^{2-(\alpha_0+\alpha_1)}}E(x)$$

$$Q_2(x) = \frac{1}{ac^2b\,x^{4-2(\alpha_0+\alpha_1)} - (a+b)c\,x^{2-(\alpha_0+\alpha_1)}}E(x).$$

By invoking the inversion formula we obtain the solution of our problem.

Example 3. Consider the series circuit of Figure 3 consisting of a voltage source $e(t)$, a capacitor $C(t)$ and an inductor $L(t)$ which vary with the time as indicated.



$$L(t) = at^{\alpha_1}, \quad a > 0$$
$$C(t) = bt^{\alpha_0}, \quad b > 0$$

Figure 3

The mesh charges $q(t)$ satisfy the differential equation

$$ab\,B_{\alpha_0,\alpha_1,0}q(t) + q(t) = b\,t^{\alpha_0}e(t)$$

and by applying the $F_{\alpha_0,\alpha_1,0}$–transformation and in virtue of the inversion formula we get

$$q(t) = F_{\alpha_0,\alpha_1,0}\{\frac{1}{1 - ab\,y^{2-(\alpha_0-\alpha_2)}}F_{\alpha_0,\alpha_1,0}\{b\,t^{\alpha_0}e(t)\}(y)\}(t)$$

Example 4. Given the circuit of Figure 4 we wish to find the three charges q_1, q_2 and q_3 by writing loop equations and transforming.

Let $L_1(t) = at^{\alpha_1}$, $a > 0$, $C_1(t) = ct^{\alpha_0}$, $c > 0$, $L_2(t) = bt^{\alpha_1}$, $b > 0$, $C_2(t) = dt^{\alpha_0}$, $d > 0$.

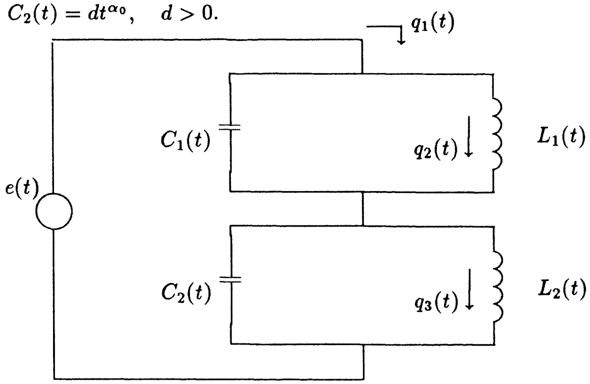


Figure 4

Figure 4 yields the following system of differential equations

$$\begin{aligned} cd\,t^{\alpha_0}e(t) &= (d+c)q_1(t) - d\,q_2(t) - c\,q_3(t) \\ 0 &= q_2(t) - q_1(t) + ac\,B_{\alpha_0,\alpha_1,0}q_2(t) \\ 0 &= q_3(t) - q_1(t) + bd\,B_{\alpha_0,\alpha_1,0}q_3(t) \end{aligned}$$

Transforming all three equations with the $F_{\alpha_0,\alpha_1,0}$-transformation it follows

$$\begin{aligned} E(x) &= (d+c)Q_1(x) - d\,Q_2(x) - c\,Q_3(x) \\ 0 &= Q_2(x) - Q_1(x) - ac\,x^{2-(\alpha_0+\alpha_1)}Q_2(x) \\ 0 &= Q_3(x) - Q_1(x) - bd\,x^{2-(\alpha_0+\alpha_1)}Q_3(x) \end{aligned}$$

where $Q_i(x) = F_{\alpha_0,\alpha_1,0}\{q_i\}(x)$ and $E(x) = F_{\alpha_0,\alpha_1,0}\{cdt^{\alpha_0}e(t)\}(x)$.

Solving the above algebraic system and invoking the inversion formula for $F_{\alpha_0,\alpha_1,0}$ we can obtain the charges q_1, q_2 and q_3.

This technique can be applied to any finite number of loops as long as there is no resistence and every inductance and capacitance in the networks varies with time as they do in this problem.

Remark. The problems solved in this section generalize the ones studied by Gerardi [**4**], Zemanian [**18**] and Koh [**8**] by using the Hankel and others transformations.

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