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A weakening of ♣, with applications to topology

István Juhász

Dedicated to Professor M. Katětov on his seventieth birthday

Abstract: Let (t) denote the following statement: there exists a sequence $\langle S_\lambda : \lambda \in L_1\rangle$ (L_1 is the set of all countable limit ordinals) and for each $\lambda \in L_1$ a disjoint partition $S_\lambda = \bigcup_{n\in\omega} S_\lambda^n$ such that S_λ is a cofinal ω-type subset of λ for each $X \in [\omega_1]^{\omega_1}$ there is some $\lambda \in L_1$ with $|X \cap S_\lambda^n| = \omega$ for all $n \in \omega$. Clearly, ♣ implies (t), moreover $V^c \models (t)$ if C adds a Cohen real to V. It is shown that (t) implies the existence of an "Ostaszewski type" S space and that of a compact T_2 space of countable tightness and π-weight in which every point has character ω_1.

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§1. The principle (t).

Let us start by recalling that ♣ is the following combinatorial principle: There is a sequence $\langle S_\lambda : \lambda \in L_1\rangle$ (here L_1 is the set of all countable limit ordinals) such that S_λ is a cofinal ω-type subset of λ and for every uncountable $X \subset \omega_1$ there is a $\lambda \in L_1$ with $S_\lambda \in X$. Thus the following principle that we call (t) is clearly a weakening of ♣.

Definition 1.1. Let (t) denote the next statement: There is a sequence $\langle s_\lambda : \lambda \in L_1\rangle$ and for each $\lambda \in L_1$ a disjoint partition $\bigcup_{n\in\omega} S_\lambda^n$ such that S_λ is a cofinal ω-type subset of λ and for every uncountable set $X \subset \omega_1$ there is a $\lambda \in L_1$ with $|X \cap S_\lambda^n| = \omega$ for all $n \in \omega$.

It follows from our next result that (t) is strictly weaker than ♣.

Theorem 1.2. *If one adds a Cohen real to a model of ZFC then* (t) *holds in the resulting extension.*

PROOF: Let $C = Fn(\omega, 2)$ be the standart partial order that adds a Cohen real to V. For each $\lambda \in L_1$ let P_λ be the natural partial order that adds a cofinal ω-sequence in λ, i.e. P_λ consits of all increasing functions $p : n \to \lambda$ where $n \in \omega$. Then, as is well known (see e.g. [**3**]), π_λ is forcing equivalent to C, hence in V^c there is a P_λ-generic (increasing) function $g_\lambda : \omega \to \lambda$ over V. ■

Now let us fix, in V, a partition $\omega = \bigcup_{n\in\omega} a_n$ of ω into infinitely many infinite subsets and set $S_\lambda^n = g_\lambda[a_n]$. We claim that,in V^C, the family $\{S_\lambda^n : \lambda \in L_1, n \in \omega\}$

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establishes (t). Indeed, it is clear that S_λ is a cofinal ω-type subset of λ and $S_\lambda = \bigcup_{n\in\omega} S^n_\lambda$ is a partition of S_λ.

Now, let $X \in [\omega_1]^{\omega_1}$ (in V^c) then (see [3]) there is an uncountable subset $X_0 \subset X$ with $X_0 \in V$. Then for any $\lambda \in X_0'$ (i.e. a limit point of X_0) it is straightforward to check that $|X_0 \cap S^n_\lambda| = \omega$ since both $X_0 \in V$ and $a_n \in V$ and g_n is P_λ-generic over V, hence $|X \cap S^n_\lambda| = \omega$ as well. Note that we have actually established more than is required by (t), namely a club set (X_0') of appropriate $\lambda's$.

For our later purposes we now give an equivalent reformulation of (t). For this we recall that a collection $\{S_n : n \in \omega\} \subset [S]^\omega$ is said to be independent in S if for any finite $0-1$ sequence $\varepsilon \in 2^{<\omega}$ we have $|S^\varepsilon| = \omega$, where $S^\varepsilon = \cap\{S_n^{\varepsilon(n)}; n \in \mathrm{dom}(\varepsilon)\}$ and $S^0_n = S_n, S^1_n = S \setminus S_n$.

We now call (s) the following statement: there is a collection $\{S^n_\lambda : \lambda \in L_1, n \in \omega\}$ such that $S_\lambda = \bigcup_{n\in\omega} S^n_\lambda$ is a cofinal ω-type set in λ, $\{S^n_\lambda : n \in \omega\}$ is independent in S_λ, moreover for any uncountable set $X \subset \omega_1$ there is some $\lambda \in L_1$ with $|X \cap S^\varepsilon_\lambda| = \omega$ for every $\varepsilon \in 2^{<\omega}$.

Lemma 1.3. *(s) is equivalent to (t).*

PROOF: $(t) \to (s)$. Let $\langle S_\lambda : \lambda \in L_1\rangle$ with the partitions $S_\lambda = \bigcup_{n\in\omega} T^n_\lambda$ establish (t). Fix an independent collection $\{a_n : n \in \omega\}$ in ω and then set, for all $\lambda \in L_1$,

$$S^n_\lambda = \cup\{T^k_\lambda : k \in a_n\}.$$

Clearly, then $\{S^n_\lambda : \lambda \in L_1, n \in \omega\}$ establishes (s).

$(s) \to (t)$. For the converse, let $\{S^n_\lambda : \lambda \in L_1, n \in \omega\}$ establish (s) and set

$$T^n_\lambda = S^0_\lambda \cap \cdots \cap S^{n-1}_\lambda \setminus S^n_\lambda.$$

It is again straightforward to check that the S_λ's with the partitions $\{T^n_\lambda : n \in \omega\}$ establish (t). ∎

§2. Applications of (t) to topology.

♣ was invented by Ostaszewski in [5] for the purpose of constructing a locally compact T_2 topology τ on ω_1 such that every initial segment of ω_1 is open and every open set in τ is either countable or co-countable. Clearly, $\langle\omega_1, \tau\rangle$ is then an S space. For our present purposes we shall call such a space an Ostaszewski space. Actually, in [5] $♣ + CH = \diamondsuit$ was used and the resulting space was in addition also countably compact, hence also perfectly normal.

Those who know the method of [5] should easily realize that what one needs here is actually (t) and not the full force of ♣! Still we decided to give below some of the details because it will serve as a good warm-up for the second, much more complicated construction.

Theorem 2.1. *(t) implies the existence of an Ostaszewski space. Moreover, $(t) + CH$ implies the existence of a countably compact Ostaszewski space.*

PROOF: We only deal with the first statement because the modification using CH goes exactly as in [5].

We shall define topologies τ_λ on $\lambda \in L_1$ by induction as follows:

(1) τ_λ is locally compact T_2 and right separated in its natural order,

(2) if $\mu \in \lambda \cap L_1$ then $\tau_\mu = \tau_\lambda \cap P(\mu)$;

τ will be the topology on ω_1 generated by $\cup\{\tau_\lambda : \lambda \in L_1\}$.

We start the induction by putting $\tau_\omega = P(\omega)$. If λ is a limit of limits (i.e. $\lambda \in L_1'$) then τ_λ is defined as the topology generated by $\cup\{\tau_\mu : \mu \in \lambda \cap L_1\}$. If, on the other hand, $\lambda = \mu + \omega$ with $\mu \in L_1$ then first note that S_μ is closed discrete in $\langle \mu, \tau_\mu \rangle$, hence by (1) for each $\alpha \in S_\mu$ we can pick a compact open neighbourhood K_α such that the collection $\{K_\alpha : \alpha \in S_\mu\}$ is discrete. Now, a neighbourhood base for $\mu + n$ in τ_λ will consist of the sets

$$B_n(F) = \{\mu + n\} \cup \cup\{K_\alpha : \alpha \in S_\mu^n \setminus F\},$$

where F runs through $[S_\mu^n]^{<\omega}$. It is easy to check that (1) and (2) will be valid for τ_λ.

It is also straightforward to show that then $\langle \omega_1, \tau \rangle$ will be locally compact, T_2 and right separated. To see the rest first note that by our construction if $\mu \in L_1$ then S_μ^n converges to $\mu + n$, hence if $Z \subset S_\mu$ is such that $|Z \cap S_\mu^n| = \omega$ for all $n \in \omega$ then $\overline{Z} \supset [\mu, \mu + \omega)$, and by an easy induction $\overline{Z} \supset [\mu, \mu + \alpha \cdot \omega)$ for all $\alpha \in \omega_1$, i.e. $\overline{Z} \supset [\mu, \omega_1)$. Consequently, if $X \in [\omega_1]^{\omega_1}$ and $\lambda \in L_1$ is such that $|S_\lambda^n \cap X| = \omega$ for all $n \in \omega$, then $[\lambda, \omega_1) \subset \overline{X}$ as well. This shows that every uncountable closed set in $\langle \omega_1, \tau \rangle$ is co-countable. ∎

In [4] V.Malychin proved, that after adding a Cohen real, i.e. in V^C, there is a compact space of countable tightness in which every point has character ω_1. Below we give the stronger result that the existence of such a space already follows from (t). Note that A. Dow has recently shown in [1] that under PFA every compact space of countable tightness has points of countable character.

Theorem 2.2. *(t) implies that there exists a compact T_2 space X such that $t(X) = \pi(X) = \omega$ but $\chi(p, X) = \omega_1$ for each $p \in X$.*

The construction of this space X will be an inverse limit, similarly as in [4], moreover, in the course of this construction we shall use (s) rather than (t). In what follows we call a space good if it is compact, T_2, 0–dimensional and first countable. The spaces in our eventual inverse system will all be good.

We start with a simple lemma given in [4] concerning good spaces.

Lemma 2.3. *If X is good, $p \in x$ and $\{K_n : n \in \omega\}$ is a disjoint collection of compact sets in $X \setminus \{p\}$ with K_n converging to p (i.e. for any open $V \ni p$ we have $K_n \subset V$ for all but finitely many $n \in \omega$), then there is a decreasing clopen neighbourhood base $\{V_n : n \in \omega\}$ of p in X such that $K_n \subset V_n \setminus V_{n+1}$ for every $n \in \omega$,*

Now, our inverse system will be of the from $\mathcal{S} = \{X_\alpha, \pi_\beta^\alpha; \alpha, \beta \in \omega_1\}$, where, as mentioned, X_α is good, the underlying set of X_α is 2^α and the bonding map π_β^α is the natural projection of 2^α onto 2^β, i.e. for $x \in 2^\alpha$ we have

$$\pi_\beta^\alpha(X) = x \restriction \beta.$$

Consequently, our resulting space

$$X = \varprojlim \mathcal{S}$$

can be considered as a space on 2^{ω_1} and the maps $\pi_\alpha : X \to X_\alpha$ will be the natural projections (restriction to α) again. Now, this means that $X_{\alpha+1}$ is always of the form $X_\alpha \times 2$, i.e. $X_{\alpha+1}$ is obtained by some "doubling" of X_α. Below we first give a very general way of doubling a space X that will be crucial to our construction. Moreover, this general doubling procedure may be of some interest in itself.

Definition 2.4. Let X be an arbitrary topological space with the topology τ and $u : X \to \tau^2$ be a map such that for $p \in X$ we have $u(p) = \langle U_p^0, U_p^1 \rangle, p \notin U_p^i, U_p^0 \cap U_p^1 = \emptyset$ and $U_p = \{p\} \cup U_p^0 \cup U_p^1$ is a neighbourhood of p in X. We define $D(X, u)$, the double of X with respect to u, as follows: the underlying set of $D(X, u)$ is $X \times 2$ and a neighbourhood base of $\langle p, i \rangle \in X \times 2$ is formed by the sets

$$B(V, p, i) = \{\langle p, i \rangle\} \cup [(V \cap U_p^i) \times 2]$$

where V runs through all neighbourhoods of p in X.

Let us illustrate this definition with two examples. First, for any T_1 space X let $u : X \to \tau^2$ be defined by $u(p) = \langle X \setminus \{p\}, \emptyset \rangle$. It is obious that $D(X, u)$ then is identical with what is called in [2] the Alexandrov double of X.

Next, let X be any LOTS, with $\prec$ as the generating ordering. Let us put for $p \in X$ $u(p) = \langle (\leftarrow, p), (p, \rightarrow) \rangle$, then $D(X, u)$ will be $X \times 2$ with the lexicographic order topology. In particular, if $X = [0, 1]$ then we obtain the double arrow space.

Below we formulate several properties of this double construction. The simple proofs are left to the reader.

Lemma 2.5. *If X is T_2 or compact or 0–dimensional or first countable then so is $D(X, u)$. Hence if X is good then so is $D(X, u)$. The natural projection of $D(X, u)$ onto X (that sends $\langle x, i \rangle$ to x) is continuous.*

Let us now return to our proposed inverse system $\mathcal{S}$ of length ω_1. We shall put $X_n = 2^n$ for all $n \in \omega$, of course here we have to take the discrete topologies on these finite spaces. Then we do the construction in "chunks" of length ω, from one limit ordinal to the next. Our next step is to give a general treatment of this ω–type "sub" inverse limit.

Definition 2.6. Let X be a 0–dimensional first countable space and $\mathcal{V} = \{V_p^n : p \in X, n \in \omega\}$ be a system of clopen sets in X such that $\{V_p^n : n \in \omega\}$ forms a decreasing neighbourhood base at p for each $p \in X$. We shall denote by L_p^n the clopen set $V_p^n \setminus V_p^{n+1}$. We also fix an independent family $\mathcal{A} = \{a_n : n \in \omega\} \subset [\omega]^\omega$.

Put $X_0 = X$ and let $u_0 : X_0 \to \tau_0^2$ be defined by

$$u_0(p) = \langle \cup\{L_p^k : k \in a_0 = a_0^0\}, \cup\{L_p^k : k \in \omega \setminus a_0 = a_0^1\} \rangle.$$

We then set $X_1 = D(X_0, u_0)$ and let π_1 be the natural projection from $X_1 = X_0 \times 2$ to X_0. Continuing this induction we let $X_{n+1} = D(X_n, u_n)$, where u_n is defined on $X_n = X \times 2^n$ as follows:

$$u_n(\langle p, \varepsilon \rangle) = \langle \cup\{L_p^k \times 2^n : k \in a^{\varepsilon \frown 0} = a^\varepsilon \cap a_n\}, \cup\{L_p^k \times 2^n : k \in a^{\varepsilon \frown 1} = a^\varepsilon \setminus a_n\} \rangle.$$

Of course, π_{n+1} is the natural projection of $X_{n+1} = X_n \times 2$ onto X_n.

Thus we obtain an inverse system $\mathcal{S}(X, \mathcal{V}, \mathcal{A}) = \{X_n, \pi_n^{n+1} = \pi_{n+1}; n \in \omega\}$ of length ω, whose inverse limit we denote by $\mathcal{L}(X, \mathcal{V}, \mathcal{A})$. Clearly, we may assume that underlying set of $\mathcal{L}(X, \mathcal{V}, \mathcal{A})$ is $X \times 2^\omega$. The bounding map $\pi_\omega : \mathcal{L}(X, \mathcal{V}, \mathcal{A}) \to X$ is then equal to the natural projection of $X \times 2^\omega$ onto X.

Note that if X is a good space then, by 2.5, so is every X_n, hence so is their inverse limit $\mathcal{L}(X, \mathcal{V}, \mathcal{A})$.

Next we formulate a property of these inverse limits $\mathcal{L}(X, \mathcal{V}, \mathcal{A})$ that will be crucial in our construction.

Lemma 2.7. *Let $X, \mathcal{V}, \mathcal{A}$ be as in 2.6. and $p \in X$. Assume that $b \subset \omega$ is such that $|b \cap a^\varepsilon| = \omega$ for each $\varepsilon \in 2^{<\omega}$ (in this case we say that b is $\mathcal{A}$-big), moreover $H \subset X$ and $H \cap L_p^k \neq \emptyset$ for every $k \in b$. Then in $\mathcal{L}(X, \mathcal{V}, \mathcal{A})$ we have*

$$\pi_\omega^{-1}\{p\} \subset \overline{\{\langle x, h_x\rangle : x \in H\}},$$

no matter how $h_x \in 2^\omega$ is chosen for each $x \in H$.

PROOF: Indeed, let $h \in 2^\omega$ be arbitrary, then a basic neighbourhood of the point $\langle p, h\rangle$ in $\mathcal{L}(X, \mathcal{V}, \mathcal{A})$ has the form $U = \pi_\omega^{-1}(\{\langle p, h \restriction n\rangle\} \cup \bigcup\{L_p^k \times 2^n : k \in a^{h \restriction n} \setminus l\})$. Now, let $k \in b \cap a^{h \restriction n} \setminus l$, then $L_p^k \cap H \neq \emptyset$, hence if $x \in L_p^k \cap H$ we have $\langle x, h_x \restriction n\rangle \in L_p^k \times 2^n$, i.e. $U \cap \{\langle x, h_x\rangle : x \in H\} \neq \emptyset$. But this shows that $\langle p, h\rangle \in \overline{\{\langle x, h_x\rangle : x \in H\}}$ indeed. ∎

We need one more preparatory lemma.

Lemma 2.8. *Let $\mathcal{S} = \{X_n, \pi_m^n : n, m \in \omega\}$ be an ω-length inverse system of good spaces, $X = \varprojlim \mathcal{S}$ and pick $p \in X$ and $p_n \in X_n$ for $n \in \omega$ such that for each $n \in \omega$ we have $p_n \neq \pi_n(p)$, but $\pi_m^n(p_n) = \pi_m(p)$ holds for all $m < n$. Now if we put $K_n = \pi_n^{-1}\{p_n\}$ then $\{K_n : n \in \omega\}$ is a disjoint collection of compact sets in $X \setminus \{p\}$ that converges to p. Thus Lemma 2.3. applies.*

We leave the easy proof to the reader.

Now we are ready to construct the required inverse system. First we put $X_n = 2^n$ for $n \in \omega$. Now assume $\lambda \in L_1$ and the inverse system $\mathcal{S}_\lambda = \{X_\alpha, \pi_\beta^\alpha; \alpha, \beta \in \lambda\}$ has been already obtained with the required properties, i.e. each X_α is good with underlying set 2^α and π_β^α is the natural projection of 2^α onto 2^β.

Then we first put $X_\lambda = \varprojlim \mathcal{S}_\lambda$. Next consider the cofinal ω-type subset S_λ of λ and note that $X_\lambda = \varprojlim\{X_{\sigma_\lambda^n+1}, \pi_{\sigma_\lambda^m+1}^{\sigma_\lambda^n+1}; n, m \in \omega\}$ as well, where σ_λ^n denotes the n^{th} element of S_λ in its natural order. Let us pick $p \in X_\lambda$ and define the point $p_n \in X_{\sigma_\lambda^n+1}$ by $p_n \restriction \sigma_\lambda^n = p \restriction \sigma_\lambda^n$ and $p_n(\sigma_\lambda^n) \neq p(\sigma_\lambda^n)$ for every $n \in \omega$.

Let us put

$$K_p^n = \pi_n^{-1}\{p_n\} = \{q \in 2^\lambda; q \restriction \sigma_\lambda^n = p \restriction \sigma_\lambda^n \text{ and } q(\sigma_\lambda^n) \neq p(\sigma_\lambda^n)\},$$

then by 2.8 the disjoint compact sets $K_p^n \subset X \setminus \{p\}$ converge to p, hence from 2.3 we obtain a decreasing clopen neighbourhood base $\{V_p^n : n \in \omega\}$ for p such that

$$K_p^n \subset L_p^n = V_p^n \setminus V_p^{n+1}$$

for each $n \in \omega$. Put $\mathcal{V}_\lambda = \{V_p^n : p \in X_\lambda, n \in \omega\}$, moreover

$$a_\lambda^n = \{k \in \omega : \sigma_\lambda^k \in S_\lambda^n\},$$

hence $\mathcal{A} = \{a_\lambda^n : n \in \omega\} \subset [\omega]^\omega$ is an independent family. Thus we may build the ω-type inverse system $\mathcal{S}(X_\lambda, \mathcal{V}_\lambda, \mathcal{A}_\lambda)$ which will form the following block of length ω of $\mathcal{S}$. It is clear that our inductive assumptions will remain valid this way by passing from $\mathcal{S}_\lambda$ to $\mathcal{S}_{\lambda+\omega}$ after making the necessary identifications, e.g. $2^\lambda \times 2^n \approx 2^{\lambda+n}$.

Finally, having completed the induction we put $\mathcal{S} = \{X_\alpha, \pi^\alpha_\beta; \alpha, \beta \in \omega_1\}$ and $X = \varprojlim X_\alpha$, where the underlying set of X is 2^{ω_1} and $\pi_\alpha : X \to X_\alpha$ is the natural projection.

Let us start checking the properties of X. First, X is trivially compact T_2 and $w(X) \le \omega_1$. It is also clear that if $p \in X$ and G is any G_δ set containing p then there is an $\alpha \in \omega_1$ such that $q \in G$ whenever $q \restriction \alpha = p \restriction \alpha$, hence $\chi(p, X) = \omega_1$ holds for all $p \in X$.

Next we show that $t(X) = \omega$, so let $p \in \overline{A} \subset X$. We have to show that there is a countable set $B \subset A$ with $p \in \overline{B}$. For each $\alpha \in \omega_1$, there is a point $p_\alpha \in X$ such that $p \restriction \alpha = p_\alpha \restriction \alpha$ and $p_\alpha \in \overline{B}_\alpha$ for some countable subset B_α of A. Indeed, since X_α is good we can choose $B_\alpha \in [A]^{\le\omega}$, so that $\pi_\alpha(p) = p \restriction \alpha \in \overline{\pi_\alpha(B_\alpha)}^\alpha$ and $\overline{\pi_\alpha(B_\alpha)}^\alpha \subset \pi_\alpha(\overline{B_\alpha})$ because π_α is a closed map. Hence there is a $p_\alpha \in \overline{B_\alpha}$ with $\pi_\alpha(p_\alpha) = p_\alpha \restriction \alpha = p \restriction \alpha$.

If there is an $\alpha \in \omega_1$ with $p = p_\alpha$ then we are done. Otherwise the set

$$H = \{\sigma(p, p_\alpha) : \alpha \in \omega_1\}$$

is uncountable, where $\sigma(p, q)$ for $p, q \in 2^\varphi$ denotes the smallest ordinal σ with $p(\sigma) \ne q(\sigma)$. Then we may choose $\lambda \in L_1$ to H as given by (s) and claim that

$$p \in \overline{\{p_{\alpha_\sigma} : \sigma \in H \cap S_\lambda\}}$$

where, for $\sigma \in H, \alpha_\sigma$ is closen with $\sigma(p, p_{\alpha_\sigma}) = \sigma$. This will suffice because then $p \in \overline{\bigcup\{B_{\alpha_\sigma} : \sigma \in H \cap S_\lambda\}}$ holds as well.

Our claim will follow if we can show that for all $\nu \in \omega_1 \setminus \{0\}$ we have

$$I(\nu) : \qquad (\pi^{\lambda+\omega\cdot\nu}_\lambda)^{-1}\{p \restriction \lambda\} \subset \overline{\{p_{\alpha_\sigma} \restriction \lambda + \omega\cdot\nu : \sigma \in H \cap S_\lambda\}},$$

and we show this by induction on ν.

For $\nu = 1$ this holds because $\{k : \sigma^k_\lambda \in H \cap S_\lambda\}$ is $\mathcal{A}_\lambda$-big, i.e. $|H \cap S^\varepsilon_\lambda| = \omega$ for each $\varepsilon \in 2^\omega$, and thus 2.7 may be applied to $p \restriction \lambda \in X_\lambda$ and the set $\{p_{\alpha_\sigma} \restriction \lambda : \sigma \in H \cap S_\lambda\}$. If ν is a limit ordinal then $I(\mu)$ for all $\mu < \nu$ implies $I(\nu)$ trivially, by the properties of inverse limits. Finally, if $\nu = \mu + 1$ then we can apply 2.7 again, this time for any point of $(\pi^{\lambda+\omega\cdot\mu}_\lambda)^{-1}\{p \restriction \lambda\}$ and the set

$$\{p_{\alpha_\sigma} \restriction \lambda + \omega\cdot\mu : \sigma \in H \cap S_\lambda\},$$

noting that $\{k : \sigma^k_{\lambda+\omega\cdot\mu} > \lambda\}$ is co-finite in ω, hence trivially $\mathcal{A}_{\lambda+\omega\cdot\nu}$-big. This completes the proof of $t(X) = \omega$.

Finally, to show $\pi(X) = \omega$ we first prove that the ordinary "product" open subsets of 2^{ω_1} form a π-base in X. This will follow if we show, by induction on $\alpha \in \omega_1$. that the ordinary open sets of 2^α form a π-base in X_α. For $\alpha = \omega$ this is trivial and so it is for limit α if already known for all $\beta < \alpha$. Finally, the successor step follows from the fact that if $u : X \to \tau^2$ is such that $U^0_p \ne \emptyset \ne U^1_p$ for all $p \in X$ (in addition to the other requirements of 2.4) then the "ordinary" open sets in $X \times 2 = X \oplus X$ will clearly form a π-base in $D(X, u)$.

Since 2^{ω_1} with its usual product topology is separable we obtain that so is our space X. Finally, we know by [6] that X also has countable π-character, hence in fact countable π-weight.

We have thus checked that X satisfies all the properties claimed in 2.2.

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