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On collections of almost disjoint families

ON COLLECTIONS OF ALMOST DISJOINT FAMILIES

Bohuslav BALCAR and PETR SIMON

Dedicated to Professor Miroslav Katětov on his seventieth birthday

Abstract: We show, in ZFC only, that for every uncountable cardinal κ the quotient Boolean algebra $\mathscr{P}_\kappa(\kappa) = \mathscr{P}(\kappa)/[\kappa]^{<\kappa}$ is $(\omega, \cdot, \kappa^+)$, respectively $(\omega_1, \cdot, \kappa^{\aleph_0})$, nowhere distributive. This depends on the cofinality of κ. Moreover, we prove that for uncountable regular κ the forcing notion $\mathscr{P}_\kappa(\kappa)$ collapses a cardinal characteristic $b_\kappa > \kappa$ to ω. Nowhere distributivity of $\mathscr{P}_\kappa(\kappa)$ is formulated in terms of almost disjoint families on κ.

Key words: Almost disjoint family, non-distributivity of Boolean algebra, completion of Boolean algebra.

Classification: 03E05, 03E45, 06E05

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§ 1. Introduction. Soon after Cohen's discovery of forcing, it became apparent that the distributivity properties of a Boolean algebra decide the basic features of the generic extension. The first author investigated with his collaborators the distributivity of $\mathscr{P}(\kappa)/[\kappa]^{<\kappa}$ systematically for a long period of time. The present paper aims to give a survey of this topic. Though the partial results have been already published [BVop], [BPS], [BF], [BS], here we present the definite statements concerning the non-distributivity of $\mathscr{P}(\kappa)/[\kappa]^{<\kappa}$. The results concerning the collapsing of cardinals when forcing with $\mathscr{P}(\kappa)/[\kappa]^{<\kappa}$ are far from being complete except for regular κ . The paper extends [BS, § 4] from the Handbook of Boolean Algebras.

The notation used throughout the paper is the standard one. The Greek letter κ always means an infinite cardinal number, $[\kappa]^{<\kappa}$ is the ideal in the power set algebra $\mathscr{P}(\kappa)$ of all sets of size less than κ and the quotient Boolean algebra $\mathscr{P}(\kappa)/[\kappa]^{<\kappa}$ is denoted by $\mathscr{P}_\kappa(\kappa)$. We shall consider also its completion, for which we use the notation $\mathrm{Compl}(\mathscr{P}_\kappa(\kappa))$.

Few words on the organization of the paper. In the forthcoming § 2, the necessary notions are introduced and the main results formulated. The proofs will be done in §§ 3 - 5 for a regular case, for a singular with uncountable cofinality and for a singular with countable cofinality, respectively.

§ 2. The nondistributivity of $\mathcal{P}_\kappa (\kappa)$. For the reader's convenience, let us summarize here a few basic and well-known facts on the almost disjoint families on κ .

A family $\mathcal{A} \subseteq \mathcal{P}(\kappa)$ is called almost disjoint on κ , if all members of $\mathcal{A}$ have size κ and any two distinct $A, A' \in \mathcal{A}$ satisfy $|A \cap A'| < \kappa$. Thus an almost disjoint family on κ corresponds to a disjoint family in $\mathcal{P}_\kappa(\kappa)$. If a family $\mathcal{A} \subseteq [\kappa]^\kappa$ is almost disjoint and there is no almost disjoint family on κ properly containing $\mathcal{A}$, then $\mathcal{A}$ is called maximal almost disjoint, MAD. A MAD family on κ corresponds to a partition of unity in $\mathcal{P}_\kappa(\kappa)$.

There is no maximal almost disjoint family of size $\mathrm{cf}(\kappa)$ on κ . Next, on each κ , there is an almost disjoint family of size κ^+. Assuming, moreover, $2^{<\kappa} = \kappa$, then there is an AD family on κ of the maximal size possible, i.e. 2^κ . In particular, there is an almost disjoint family on ω of size 2^ω. On the other hand, J. Baumgartner [Ba] showed that it is consistent that all almost disjoint families on ω_1 have size strictly smaller than 2^{ω_1}. In the case of a singular, strange things may happen: P. Erdös and S.H. Hechler [EH] formed a MAD family of size κ on κ under the assumptions $\lambda = \mathrm{cf}(\kappa) < \kappa$ and $(\forall \tau < \kappa) \; \tau^\lambda < \kappa$.

In contrast to $\mathcal{P}(\kappa)$, the algebra $\mathcal{P}_\kappa (\kappa)$ has no atoms and therefore is not distributive. Moreover, every non zero element of $\mathcal{P}_\kappa (\kappa)$ admits a partition of size at least κ^+. An algebra $\mathcal{P}_\kappa (\kappa)$ is homogeneous. (Recall that $\mathcal{B}$ is homogeneous if for any non zero $x \in \mathcal{B}$, $\mathcal{B}$ is isomorphic to $\mathcal{B} \restriction x$.)

As commonly adopted, for $A, B \in [\kappa]^\kappa$, we shall write $A \subseteq^* B$ iff $|A \setminus B| < \kappa$, $A \subset^* B$ iff $A \subseteq^* B$ and $|B \setminus A| = \kappa$. This corresponds to the canonical order $\leq$ in the algebra $\mathcal{P}_\kappa(\kappa)$. For $M \subseteq \kappa$, its equivalence class modulo the ideal $[\kappa]^{<\kappa}$ (equivalently, the corresponding element in $\mathcal{P}_\kappa(\kappa)$) is denoted by $[M]$.

For two functions $f, g \in {}^\kappa\kappa$, $f \leq^* g$ means that $|\{\xi \in \kappa : f(\xi) > g(\xi)\}| < \kappa$.

The central notion discussed in the present paper is defined as follows.

2.1. Definition. Let $\mathcal{B}$ be a Boolean algebra, τ, λ, μ cardinal numbers, $\lambda \geq 2$.

· (i)   The algebra $\mathcal{B}$ is (τ,μ,λ)-distributive, if for every family $\{P_\alpha : \alpha < \tau\}$ of partitions of unity of $\mathcal{B}$ such that each $|P_\alpha| \leq \mu$ there is a partition of unity Q with the property that for every $q \in Q$ and for every $\alpha < \tau$, $|\{p \in P_\alpha : p \wedge q \neq 0\}| < \lambda$.

(ii)   The algebra $\mathcal{B}$ is (τ,μ,λ)-nowhere distributive, if there is a family $\{P_\alpha : \alpha < \tau\}$ of partitions of unity of $\mathcal{B}$ such that for each $\alpha < \tau$, $|P_\alpha| \leq \mu$ and for every $a \neq 0$, there is some $\alpha < \tau$ with $|\{p \in P_\alpha : p \wedge a \neq 0\}| \geq \lambda$.

(iii)   We shall speak about $(\tau,\cdot,\lambda)$-distributivity ($(\tau,\cdot,\lambda)$-nowhere distributivity, resp.), if there is no demand on the size of P'_α s.

We shall omit an easy proof of the next proposition, which may be found e.g. in [BSV].

2.2. Proposition.   A Boolean algebra $\mathcal{B}$ is $(\tau,\cdot,\lambda)$-nowhere distributive iff no partial algebra $\mathcal{B} \restriction a$ $(a \neq 0)$ is $(\tau,\cdot,\lambda)$-distributive.

Let us apply the general definition of nowhere distributivity to the case of $\mathcal{P}_\kappa(\kappa)$. Clearly, every family witnessing to the $(\tau,\cdot,\lambda)$-nowhere distributivity is in fact a collection of almost disjoint families on κ. We shall try for the smallest possible τ and then for the greatest possible λ without any additional set-theoretical assumptions. The main emphasis is put on λ, because the τ is, in fact, known; see 2.5 below.

The first demand leads naturally to the notion of a height

2.3. Definition.   The height h_κ of an algebra $\mathcal{P}_\kappa(\kappa)$ is defined by $h_\kappa = \min\{\tau : \mathcal{P}_\kappa(\kappa)$ is not $(\tau,\cdot,2)$-distributive$\}$. The letter h without an index stands for h_ω .

2.4. Comments.   (a)   Since the algebra $\mathcal{P}_\kappa(\kappa)$ is homogeneous, h_κ equals to $\min\{\tau : \mathcal{P}_\kappa(\kappa)$ is $(\tau,\cdot,2)$-nowhere distributive$\}$.

(b)   Equivalently, h_κ is the least cardinal τ such that there is a collection $\{\mathcal{A}_\alpha : \alpha < \tau\}$ of MAD families on κ such that for every $X \in [\kappa]^\kappa$ there is some $\alpha < \tau$ and distinct $A, A' \in \mathcal{A}_\alpha$ with $|A \cap X| = \kappa = |A' \cap X|$.

(c)   Since every Boolean algebra is $(\tau,\cdot,2)$-distributive iff its completion is, h_κ is the smallest τ such that forcing with $\mathcal{P}_\kappa(\kappa)$ adds a new set of size τ .

(d)   Just in the spirit of the definition 2.3, one may also describe the splitting number as $s_\kappa = \min\{\tau : \mathcal{P}_\kappa(\kappa)$ is $(\tau,2,2)$-nowhere distributive$\}$.

For the interested reader, $s = s_\omega$ is extensively studied in [vD], and one can easily prove that $s_\kappa = \min\{\wp : 2^\wp > \kappa\}$ for an uncountable regular κ .

Immediately from the definition, $h_\kappa \leq s_\kappa$ for all cardinals κ. Concerning h_κ, the next should be stated.

2.5. Theorem.

(i) h_ω is a regular cardinal, $\omega_1 \leq h_\omega \leq cf(2^\omega)$ [BPS].

(ii) For an uncountable κ, if $cf(\kappa) > \omega$, then $h_\kappa = \omega$, [B Vop], if $cf(\kappa) = \omega$, then $h_\kappa = \omega_1$ [BS].

We shall omit the proof of 2.5 (i). An exhaustive information on h_ω can be found in [BPS] or [BS]. But it should be noted here that - contrary to the case of uncountable cardinals - the exact value of h_ω depends on additional principles of set theory.

We reprove 2.5 (ii) as it follows from the more detailed statements 2.7, 2.8. The full proof of them will be the contents of §§ 3 - 5.

2.6. Definition. Let κ be a regular cardinal. Define

$b_\kappa = \min \{|H| : H \leq^\kappa \kappa$ and H has no upper bound under $\leq^* \}$.

Notice that $b_\kappa > \kappa$ and b_κ is regular. Now we are ready to give the results.

2.7. Theorem. (i) Let κ be a regular uncountable cardinal. Then $\mathcal{P}_\kappa(\kappa)$ is $(\omega, \cdot, b_\kappa)$-nowhere distributive.

(ii) Let κ be a singular with uncountable cofinality. Then $\mathcal{P}_\kappa(\kappa)$ is $(\omega, \cdot, \kappa^+)$-nowhere distributive.

(iii) Let κ be a singular with countable cofinality. Then $\mathcal{P}_\kappa(\kappa)$ is $(\omega_1, \cdot, \kappa^\omega)$-nowhere distributive.

For a regular κ, we are able to prove a bit more.

2.8. Theorem. Let κ be a regular uncountable cardinal. Then there is a collection $\{\mathcal{A}_{n,\alpha} : n \in \omega, \alpha \in b_\kappa\}$ such that:

(i) For each $n < \omega$, $\bigcup \{\mathcal{A}_{n,\alpha} : \alpha \in b_\kappa\}$ is a MAD family on κ,

(ii) for each $n < \omega$, $\alpha < \beta < b_\kappa$, $\mathcal{A}_{n,\alpha} \cap \mathcal{A}_{n,\beta} = \emptyset$,

(iii) for every $M \in [\kappa]^\kappa$, there is some $n < \omega$ such that for each $\alpha < b_\kappa$, $|M \cap A| = \kappa$ for some $A \in \mathcal{A}_{n,\alpha}$.

Or, equivalently, there is a family $\{a_{n,\alpha} : n \in \omega, \alpha \in b_\kappa\}$ in Compl $(\mathcal{P}_\kappa(\kappa))$ such that

(i & ii)´ Every row $\{a_{n,\alpha} : \alpha \in b_\kappa\}$ is a partition of unity.

(iii)´ For every non zero $w \in$ Compl $(\mathcal{P}_\kappa(\kappa))$ there is some $n \in \omega$ such that $w \wedge a_{n,\alpha} \neq 0$ for all $\alpha \in b_\kappa$.

The family of partitions described just now can be also viewed as a name for a function from ω to b_κ, thus we have an immediate

2.9. Corollary. For a regular uncountable κ, forcing with $\mathcal{P}_\kappa (\kappa)$ collapses b_κ to ω.

2.10. Corollary. (i) Suppose $cf(\kappa) > \omega$ and $2^\kappa = \kappa^+$. Then Compl ($\mathcal{P}_\kappa (\kappa)$) is isomorphic to the Boolean algebra of all regular open subsets in the product of ω copies of a discrete space of size 2^κ, $\displaystyle\prod_{n \in \omega} 2^\kappa$ (i.e., in a generalized Baire space of weight 2^κ).

(ii) Suppose $\kappa > cf(\kappa) = \omega$ and $2^\kappa = \kappa^\omega$. Then Compl ($\mathcal{P}_\kappa (\kappa)$) is isomorphic to the Boolean algebra of all regular open subsets in G_δ-topology on a product of ω_1 copies of a discrete space of size 2^κ.

We omit the standard proof. The set-theoretical assumptions and the non-distributivity from 2.7 enable us to routinely apply Mc Aloon's characteriza-tion of collapsing algebras. See e.g. [BSV, Theorem 1.15] or [CN, Theorem 12.13].

2.11. Remark. In [BS], a weaker form of 2.7 was given. Here, 2.7(ii), 2.7(iii) and 2.8 are new and solve several open questions from [BS].

The forthcoming three paragraphs are devoted to the proofs of both theo-rems. The authors apologize for a rather technical and complicated stuff. We shall start with 2.8.

§ 3. Regular uncountable cardinal. In this section, κ will stand for an uncountable regular cardinal. First, let us discuss in some detail the pro-perties of functions and closed unbounded sets on κ related to the cardinal b_κ.

3.1. Lemma. There is a family $\{f_\alpha : \alpha < b_\kappa\} \subseteq {}^\kappa \kappa$ such that:
(i) $f_0 \geq id$,
(ii) if $\alpha < \beta < b_\kappa$, then $f_\alpha \leq^* f_\beta$,
(iii) every f_α is continuous in the usual topology of ordinals,
(iv) there is no upper bound for $\{f_\alpha : \alpha < b_\kappa\}$ in $({}^\kappa\kappa, \leq^*)$.
(Hint: Given f_α, let $g_\alpha(\xi+1) = f_\alpha(\xi+1)$, $g_\alpha(\xi) = \sup_{\eta < \xi} f_\alpha(\eta)$ for ξ li-mit. If $\{f_\alpha : \alpha < b_\kappa\}$ is unbounded, then $\{g_\alpha : \alpha < b\}$ is.)

3.2. Lemma. The cardinal b_κ is the smallest one satisfying:
There is a family $\{C_\alpha : \alpha < b_\kappa\}$ consisting of closed unbounded subsets of κ such that

(i) if $\alpha < \beta$, then $C_\beta \subset^* C_\alpha$,

(ii) for every $M \in [\kappa]^\kappa$ there is some $\alpha < b_\kappa$ with $|M \setminus C_\alpha| = \kappa$.

Proof. Given $\{f_\alpha : \alpha < b_\kappa\}$ as in 3.1, let $C_\alpha = \{\xi \in \kappa : f_\alpha(\xi) = \xi\}$. The family $\{C_\alpha : \alpha < b_\kappa\}$ is as required.

Conversely, if $\tau < b_\kappa$ and $\{C_\alpha : \alpha < \tau\}$ are closed unbounded in κ and satisfy 3.2 (i), define $g_\alpha(\xi) = \min\{\eta \in C_\alpha : \eta \geq \xi\}$. By 3.1, there is a continuous g satisfying $g^* \geq g_\alpha$ for all $\alpha < \tau$. The set $M = \{\xi \in \kappa : g(\xi) = \xi\}$ contradicts 3.2 (ii). $\square$

Let us show two simple statements, both being immediate consequences of the lemma below.

3.3. Lemma. Let a family $\{D_\alpha : \alpha < b_\kappa\} \subseteq [\kappa]^\kappa$ satisfy the following :

($*$) For all $\alpha < \beta < b_\kappa$, $D_\beta \subset^* D_\alpha$.

Then there is a disjoint collection $\{\mathcal{A}_\xi : \xi < b_\kappa\}$ such that:

(i) $\bigcup\{\mathcal{A}_\xi : \xi < b_\kappa\}$ is a MAD family on κ ,

(ii) if $M \in [\kappa]^\kappa$ satisfies $(\forall \alpha < b_\kappa)(\exists \beta > \alpha)|M \cap (D_\alpha \setminus D_\beta)| = \kappa$, then for every $\xi < b_\kappa$ there is some $A \in \mathcal{A}_\xi$ such that $|M \cap A| = \kappa$.

Proof. We shall work in Compl ($\mathcal{P}_\kappa(\kappa)$) rather than in $\mathcal{P}(\kappa)$. That means, we shall look for a partition of unity $\{v_\xi : \xi < b_\kappa\}$ such that for every $M \in [\kappa]^\kappa$, if

($*$) for each $\alpha < b_\kappa$ there is some $\beta > \alpha$ with $[M] \wedge ([D_\alpha] \setminus [D_\beta]) \neq 0$,

then $[M] \wedge v_\xi \neq 0$ for all $\xi < b_\kappa$.

Let $w_\alpha = \bigwedge_{\beta < \alpha}[D_\beta] \setminus [D_\alpha]$ for $0 < \alpha < b_\kappa$, $w_0 = 1 \setminus \bigvee\{w_\alpha : 0 < \alpha < b_\kappa\}$. Clearly $\{w_\alpha : \alpha < b_\kappa\}$ is a partition of unity in Compl ($\mathcal{P}_\kappa(\kappa)$), some w_α's may equal to 0.

Suppose $M \subseteq \kappa$ satisfies ($*$) and consider the set $\Theta = \{\alpha < b_\kappa : [M] \wedge w_\alpha \neq 0\}$. We claim that Θ is κ-closed unbounded in b_κ .

Indeed, if $\alpha_0 < b_\kappa$, then by ($*$) $d = [M] \wedge ([D_{\alpha_0}] \setminus [D_\beta]) \neq 0$ for some $\beta > \alpha_0$. Since $\{w_\alpha : \alpha < b_\kappa\}$ is a partition of unity and since obviously $d \wedge w_0 = 0$, there must be some γ, $\alpha_0 < \gamma \leq \beta$ with $d \wedge w_\gamma \neq 0$. Thus Θ is unbounded.

Further, if $\{\alpha_\xi : \xi < \kappa\}$ is a strictly increasing sequence converging to $\alpha < b_\kappa$ and contained in Θ , then $[M] \wedge w_{\alpha_\xi} \neq 0$ for all $\xi < \kappa$. In particular, the same holds for all successors $\xi + 1 < \kappa$, so $|M \cap (D_{\alpha_\xi} \setminus D_{\alpha_{\xi+1}})| = \kappa$ for all $\xi < \kappa$. Choose inductively a set $X \subseteq M \setminus D_\alpha$, $|X| = \kappa$ satisfying $|X \setminus D_{\alpha_\xi}| < \kappa$ for all $\xi < \kappa$. Then $[X] \leq w_\alpha$, consequently,

$[M] \wedge w_{\alpha} \neq 0$. This shows the κ-closedness of θ .

It remains to select a partition $\{S_{\xi} : \xi < b_{\kappa}\}$ of b_{κ} consisting of κ-stationary subsets of b_{κ} and define $v_{\xi} = \bigvee_{\alpha \in S_{\xi}} w_{\alpha}$. The partition $\{S_{\xi} : \xi < b_{\kappa}\}$ exists by a Fodor-Solovay theorem [F], [So]. The statement follows. $\square$

3.4. Proposition. Let $\{C_{\alpha} : \alpha < b_{\kappa}\}$ be a family of closed unbounded sets on κ satisfying 3.2 (i), (ii). Then there is a family $\{\mathcal{A}_{\xi} : \xi < b_{\kappa}\}$ such that

(1) if $\xi < \eta < b_{\kappa}$, then $\mathcal{A}_{\xi} \cap \mathcal{A}_{\eta} = \emptyset$,

(ii) $\bigcup \{\mathcal{A}_{\xi} : \xi < b_{\kappa}\}$ is a MAD family on κ ,

(iii) if $M \in [\kappa]^{\kappa}$ satisfies $|M \cap C_{\alpha}| = \kappa$ for all $\alpha < b_{\kappa}$, then for each $\xi < b_{\kappa}$, $|M \cap A| = \kappa$ for some $A \in \mathcal{A}_{\xi}$.

Proof. Apply Lemma 3.3. $\square$

3.5. Proposition. Let $\{Q_{\gamma} : \gamma < \kappa\} \subseteq [\kappa]^{\kappa}$ be a disjoint family. Then there is a family $\{\mathcal{A}_{\xi} : \xi < b_{\kappa}\}$ such that

(i) if $\xi < \eta < b_{\kappa}$, then $\mathcal{A}_{\xi} \cap \mathcal{A}_{\eta} = \emptyset$,

(ii) $\bigcup \{\mathcal{A}_{\xi} : \xi < b_{\kappa}\}$ is a MAD family on κ ,

(iii) if $M \in [\kappa]^{\kappa}$ satisfies $|M \cap Q_{\gamma}| = \kappa$ for cofinally many γ's, then for all $\xi < b_{\kappa}$, $|M \cap A| = \kappa$ for some $A \in \mathcal{A}_{\xi}$.

Equivalently, there is a partition of unity in Compl ($\mathcal{P}_{\kappa} (\kappa)$), $\{c_{\xi} : \xi < b_{\kappa}\}$, such that for each $M \in [\kappa]^{\kappa}$, if $[M] \wedge [Q_{\gamma}] \neq 0$ for cofinally many γ's, then $[M] \wedge c_{\xi} \neq 0$ for all $\xi < b_{\kappa}$.

Proof. Clearly, we may assume without any loss of generality that $\{Q_{\gamma} : \gamma < \kappa\}$ is a partition of κ . Identify κ with $\kappa \times \kappa$ and imagine Q_{γ} as $\{\gamma\} \times \kappa$. Choose a family $\{f_{\alpha} : \alpha < b_{\kappa}\} \subseteq {}^{\kappa}\kappa$ without an upper bound, with each f_{α} strictly increasing and such that for $\alpha < \beta < b_{\kappa}$, $f_{\alpha} \not\leq^{*} f_{\beta}$. Denote $D_{\alpha} = \{(\xi, \eta) \in \kappa \times \kappa : f_{\alpha}(\xi) \leq \eta\}$.

We want to apply Lemma 3.3. Let $M \in [\kappa]^{\kappa}$ satisfy $|M \cap Q_{\gamma}| = \kappa$ for cofinally many γ's. We show that then M fulfils the assumptions of 3.3 (ii). Indeed, let $\alpha < b_{\kappa}$ be arbitrary. Define g: $\kappa \longrightarrow \kappa$ as follows: $g(\xi) = $ $= \min \{\eta : (\xi, \eta) \in M, \eta \geq f_{\alpha}(\xi)\} + 1$ for all $\xi < \kappa$ with $|M \cap (\{\xi\} \times \kappa)| = \kappa$, $g(\xi) = \min \{g(\bar{\xi}) : \bar{\xi} > \xi$ and $|M \cap (\{\bar{\xi}\} \times \kappa)| = \kappa\}$ otherwise.

Since $\{f_{\alpha} : \alpha < b_{\kappa}\}$ is unbounded in b_{κ}, there is some $\beta > \alpha$ such that $|\{\xi < \kappa : g(\xi) \not\leq f_{\beta}(\xi)\}| = \kappa$. If $\xi < \kappa$ and $g(\xi) \not\leq f_{\beta}(\xi)$, consider the least $\bar{\xi}$ satisfying $|M \cap (\{\bar{\xi}\} \times \kappa)| = \kappa$, $\xi \leq \bar{\xi}$. We have an $\eta < \kappa$ such that $(\bar{\xi}, \eta) \in M$, $g(\bar{\xi}) > f_{\alpha}(\bar{\xi})$ and $g(\bar{\xi}) = \eta + 1 = g(\xi) \not\leq f_{\beta}(\xi) < f_{\beta}(\bar{\xi})$. Therefore $(\bar{\xi}, \eta) \in M \cap (D_{\alpha} \setminus D_{\beta})$ and the regularity of κ gives the rest:

$|M \cap (D_\alpha \setminus D_\beta)| = \kappa$.

It remains to use 3.3. $\square$

3.6. Proof of 2.8. We shall define a system of subsets of κ , D_φ , for all finite increasing sequences φ of ordinals less than b_κ . To do this, fix some family $\{C_\alpha : \alpha < b_\kappa\}$ of closed unbounded subsets of κ satisfying 3.2 (i), (ii) and such that $0 \in C_\alpha$ for all $\alpha < b_\kappa$.

We proceed by an induction on the length of φ . Let $D_\emptyset = \kappa$, $f_\emptyset = id_\kappa$. On the first level, set $D_\alpha = \kappa \setminus C_\alpha$ and let $f_\alpha : D_\alpha \longrightarrow C_\alpha$ be defined by $f_\alpha(\xi) = \sup \xi \cap C_\alpha$. Since $0 \in C_\alpha$, f_α is well-defined and for all $\xi \in D_\alpha$, $f_\alpha(\xi) < \xi$. Moreover, for each $\eta \in f_\alpha[D_\alpha]$, $|f_\alpha^{-1}{}''\eta| < \kappa$, because C_α is unbounded.

If D_φ and f_φ have been defined for all $\varphi = \langle \alpha_0, \alpha_1, \ldots, \alpha_{n-1} \rangle$ increasing, for an arbitrary $\alpha_n > \alpha_{n-1}$ put $D_{\varphi \cap \alpha_n} = D_\varphi \setminus f_\varphi^{-1}[C_{\alpha_n}]$, $f_{\varphi \cap \alpha_n} = f_{\alpha_n} \circ f_\varphi$. One can quickly check that for each increasing $\varphi = \langle \alpha_0, \ldots \ldots, \alpha_{n-1} \rangle$, $\{D_{\varphi \cap \alpha} : \alpha > \alpha_{n-1}\}$ is an $\subseteq^*$-increasing family of subsets of D_φ such that for each $M \in [D_\varphi]^\kappa$ there is some $\alpha < b_\kappa$ with $|M \cap D_{\varphi \cap \alpha}| = \kappa$. Also, for all $\eta < \kappa$, $|f_\varphi^{-1}{}''\eta| < \kappa$ and for every $\xi \in \text{dom } f_{\varphi \cap \alpha}$, $f_{\varphi \cap \alpha}(\xi) < < f_\varphi(\xi)$.

Using D_φ's just defined, we shall find countably many partitions P_n ($n < \omega$) of 1 in Compl ($\mathscr{P}_\kappa(\kappa)$) as follows. Let $v_\emptyset = 1$, $P_0 = \{v_\emptyset\}$, $v_\alpha = [D_\alpha] \setminus \setminus \bigvee_{\beta < \alpha} [D_\beta]$ for $\alpha < b_\kappa$, $P_1 = \{v_\alpha : \alpha < b_\kappa\}$.

Suppose $n < \omega$ and $P_n = \{v_\varphi : |\varphi| = n$ and φ strictly increasing $\}$ is known, then put

$$v_{\varphi \cap \alpha} = v_\varphi \wedge ([D_{\varphi \cap \alpha}] \setminus \bigvee_{\varphi(n-1) < \beta < \alpha} [D_{\varphi \cap \beta}]),$$

$P_{n+1} = \{v_\varphi : |\varphi| = n+1$ and $\varphi : n+1 \longrightarrow b_\kappa$ is strictly increasing$\}$.

We have got an auxiliary family of partitions of 1 in Compl ($\mathscr{P}_\kappa(\kappa)$). Let us show one important feature of it.

3.7. Claim. Whenever $M \in [\kappa]^\kappa$, then there is some $\varphi = \langle \alpha_0, \ldots, \alpha_{n-1} \rangle$ such that

$$|\{\alpha < b_\kappa : [M] \wedge v_{\varphi \cap \alpha} \neq 0\}| \geq \kappa .$$

Proof of the claim. Suppose not, let M be the counterexample. Denote $\Phi = \{\varphi \in {}^{<\omega}b_\kappa : [M] \wedge v_\varphi \neq 0\}$.

By an induction on $|\varphi|$, we shall define a tree of subsets of M.

$|\varphi|=1$: Enumerate $\{\alpha_\xi : \xi < \rho\}$ all $\alpha < b_\kappa$ with $[M] \wedge v_\alpha \neq 0$. By the assumption, $\rho < \kappa$. Define $M_{\alpha_0} = M \cap D_{\alpha_0}$, $M_{\alpha_\xi} = M \cap D_{\alpha_\xi} \smallsetminus \bigcup_{\eta < \xi} M_{\alpha_\eta}$ for $\xi < \rho$.

Then $|M_{\alpha_\xi}| = \kappa$ for all $\xi < \rho$, because the ideal $[\kappa]^{<\kappa}$ is κ -additive, and $[M_{\alpha_\xi}] \leq [M] \wedge v_{\alpha_\xi}$ for all $\xi < \rho$. Moreover, $|M \smallsetminus \bigcup_{\xi < \rho} M_{\alpha_\xi}| < \kappa$,

since in the opposite, the non zero element $[M \smallsetminus \bigcup_{\xi < \rho} M_{\alpha_\xi}]$ must meet some v_{α_ξ},

which contradicts the definition of M_{α_ξ} .

If $|\varphi| \geq 1$, $M_\varphi \subseteq M$ is known, proceed in the same manner with all $\alpha < b_\kappa$ such that $[M_\varphi] \wedge v_{\varphi \cap \alpha} \neq 0$, to reach $M_{\varphi \cap \alpha}$.

For each $n \in \omega$, $|M \smallsetminus \bigcup_{\substack{\varphi \in \phi \\ |\varphi|=n}} M_\varphi| < \kappa$, therefore there is some $\xi \in M \smallsetminus$

$\smallsetminus \bigcup_{\varphi \in \phi} M_\varphi$. Due to our construction, there is a unique sequence $\langle \alpha_n : n \in \omega \rangle$ such that $M_{\alpha_0} \supseteq M_{\alpha_0, \alpha_1} \supseteq \ldots \supseteq M_{\alpha_0, \alpha_1, \ldots, \alpha_n} \supseteq \ldots$ having ξ in its intersecti-

on. For this ξ , $f_{\alpha_0}(\xi) > f_{\alpha_0, \alpha_1}(\xi) > \ldots > f_{\alpha_0, \alpha_1, \ldots, \alpha_n}(\xi) > \ldots$, a cont-

radiction, which proves the claim.

3.8. Now we shall refine all P_n's in order to get the family $\{a_{n,\alpha} : \alpha < b_\kappa\}$.

Whenever $M \in [\kappa]^\kappa$, according to Claim 3.7, there is some φ and some σ with $cf(\sigma) = \kappa$ such that $[M] \wedge v_{\varphi \cap \xi} \neq 0$ for cofinally many $\xi < \sigma$. This situation clearly resembles the one described in Proposition 3.5, however, $v_{\varphi \cap \xi}$ are not subsets of κ , but non zero members of Compl $(\mathcal{P}_\kappa(\kappa))$. Nevertheless, $v_{\varphi \cap \xi}$ were created with the intention of a possible applicati-on of 3.5; let us do it.

If $|\varphi| = n-1$ and $\sigma < b_\kappa$, $\sigma > \varphi$ (n-2) is an ordinal with $cf(\sigma) = \kappa$, choose an increasing sequence $\langle \alpha_\gamma : \gamma < \kappa \rangle$ converging to σ . Use Proposi-tion 3.5 as follows. Take $D_{\varphi \cap \alpha_\gamma} \smallsetminus \bigcup_{\xi < \gamma} D_{\varphi \cap \xi}$ in the role of Q_γ, let

$\{c_\xi (\varphi \cap \sigma) : \xi < b_\kappa\}$ be the result. If 3.5 is not applicable - which may happen e.g. if a lot of Q_γ's are of size $< \kappa$ - do nothing, i.e. $c_\xi (\varphi \cap \sigma) = 0$ by definition.

Now, define $a_{n,\xi}$ to be

$$\bigvee \{v_{\varphi \cap \sigma} \wedge c_\xi (\varphi \cap \sigma) : |\varphi| = n-1, cf(\sigma) = \kappa \}$$

for $0 < \xi < b_\kappa$, $a_{n,0} = 1 \smallsetminus \bigvee_{0 < \xi < b_\kappa} a_{n,\xi}$.

If a set M belongs to $[\kappa]^\kappa$, then by Claim, there is some φ with $|\{\xi < b_\kappa : |M \cap (D_{\varphi \frown \xi} \setminus \bigcup_{\eta < \xi} D_{\varphi \frown \eta})| = \kappa\}| \geq \kappa$. Moreover, the assumption $[M] \leq v_\varphi$ does not induce any loss of generality. Therefore there is some $\delta < b_\kappa$ when 3.5 could be applied. Then $[M] \wedge c_\xi (\varphi \frown \delta) \neq 0$ for all $\xi < b_\kappa$, consequently, $[M] \wedge a_{n,\xi} \neq 0$, too.

To complete the proof, find for $n < \omega$ and $\xi < b_\kappa$ a family $\mathcal{A}_{n,\xi} \subseteq [\kappa]^\kappa$ maximal with respect to:

(a) $\mathcal{A}_{n,\xi}$ is almost disjoint and

(b)   for each $A \in \mathcal{A}_{n,\xi}$, $[A] \leq a_{n,\xi}$. $\square$

§ 4. Singular cardinal with uncountable cofinality.

Here we prove that $\mathcal{P}_\kappa(\kappa)$ is $(\omega, \cdot, \kappa^+)$-nowhere distributive provided $\omega < \lambda = \mathrm{cf}(\kappa) < \kappa$. The letters λ, κ will have this meaning till the end of the present section.

We have to show that there is a collection $\{\mathcal{A}_n : n \in \omega\}$ of MAD families on κ such that for every $M \in [\kappa]^\kappa$ there is some $n \in \omega$ satisfying $|\{A \in \mathcal{A}_n : |A \cap M| = \kappa\}| > \kappa$. This will be done in four steps.

4.1. Let us fix an increasing sequence of regular cardinals $\langle \kappa_\xi : \xi < \lambda \rangle$ converging to κ with $\kappa_0 > \lambda$. Define $r_\xi = \kappa_\xi \setminus \bigcup_{\eta < \xi} \kappa_\eta$ for $\xi < \lambda$; notice that $|r_\xi| = \kappa_\xi$ and $\{r_\xi : \xi < \lambda\}$ is a partition of κ. For $X \subseteq \lambda$, let $\varphi(X) = \bigcup_{\xi \in X} r_\xi$. Clearly $\varphi : \mathcal{P}(\lambda) \longrightarrow \mathcal{P}(\kappa)$ uniquely determines a regular embedding $\mathcal{P}_\lambda(\lambda) \longrightarrow \mathcal{P}_\kappa(\kappa)$. The reason is that $|\varphi(X)| = \kappa$ iff $|X| = \lambda$ and $\varphi'' \mathcal{Q}$ is MAD on κ iff $\mathcal{Q}$ is MAD on λ.

4.2. According to 2.8, there is a collection $\{\mathcal{Q}_{n,\alpha} : n \in \omega, \alpha \in b_\lambda\}$ such that

(i)   for each $n \in \omega$, $\bigcup \{\mathcal{Q}_{n,\alpha} : \alpha \in b_\lambda\}$ is a MAD family on λ,

(ii)   if $\alpha < \beta$, then $\mathcal{Q}_{n,\alpha} \cap \mathcal{Q}_{n,\beta} = \emptyset$,

(iii)   for every $X \in [\lambda]^\lambda$ there is some $n \in \omega$ such that for each $\alpha \in b_\lambda$, $|X \cap Q| = \lambda$ for some $Q \in \mathcal{Q}_{n,\alpha}$.

For $n \in \omega$ and $\alpha \in b_\lambda$, let $\mathcal{B}_{n,\alpha} = \varphi'' \mathcal{Q}_{n,\alpha}$. Then by 4.1, $\{\bigcup_{\alpha \in b_\lambda} \mathcal{B}_{n,\alpha} : n \in \omega\}$ is a collection of MAD families on κ guaranteeing $(\omega, \cdot, b_\lambda)$-nowhere distributivity of $\mathcal{P}_\kappa(\kappa)$.

If $b_\lambda > \kappa$, we need not do anything more.

4.3. Let us go on and choose a family $\{f_\alpha : \alpha < b_\lambda\}$ of strictly in-

creasing functions without an upper bound in $^\lambda\lambda$. Define an "inverse" function g_α by the rule $g_\alpha(\xi)=\sup f_\alpha^{-1}{}''\xi$. For every $\alpha<b_\lambda$, the ideal of f_α-small sets on κ will be $\mathcal{m}_\alpha=\{M\subseteq\kappa:(\exists\xi_0<\lambda)(\forall\xi>\xi_0,\xi<\lambda)$ $|M\cap r_\xi|\leq\kappa_{g_\alpha(\xi)}\}$.

It is easy to see that each $X\in[\kappa]^\kappa$ contains an f_α-small subset still of size κ, for any $\alpha<b_\lambda$. Therefore there are MAD families on κ consisting of f_α-small subsets.

   Claim. If $\mathcal{A}$ is a MAD family on κ and if each member of $\mathcal{A}$ is f_α-small, then for any $M\in[\kappa]^\kappa$ which is not f_α-small, we have $|\{A\in\mathcal{A}:$ $:|M\cap A|=\kappa\}|\geq\kappa^+$.

   Proof. We may assume that for every $\xi<\lambda$, $|M\cap r_\xi|>\kappa_{g_\alpha}(\xi)$, since in general, the set of these ξ's is cofinal in (regular) λ. Therefore the standard diagonal argument, choosing $M'_\xi\subseteq M\cap r_\xi\setminus\bigcup_{\delta<\kappa_{g_\alpha(\xi)}}A_\delta$,

where $\{A_\delta:\delta<\kappa\}\subseteq\mathcal{A}$, enables us to find a subset $M'\subseteq M$, $|M'|=\kappa$, which is almost disjoint with at most κ-many members of $\mathcal{A}$. The claim follows.

   4.4. Now, fix $n<\omega$ and $\alpha<b_\lambda$. For $B\in\mathcal{B}_{n,\alpha}$ select a MAD family $\mathcal{A}(B)$ on B consisting of f_α-small sets and set $\mathcal{A}_{n,\alpha}=\bigcup\{\mathcal{A}(B):B\in\mathcal{B}_{n,\alpha}\}$. It remains to show that $\{\bigcup_{\alpha<b_\lambda}\mathcal{B}_{n,\alpha}:n\in\omega\}$ is the desired collection.

   To this end, let $M\in[\kappa]^\kappa$. By an induction, define an increasing function $g:\lambda\longrightarrow\lambda$ as $g(\xi)=\min\{\eta:\eta>\sup g''\xi$ and $|r_\eta\cap M|>\kappa_\xi\}$. Since $\{f_\alpha:\alpha<b_\lambda\}$ have no upper bound, there is some $\beta<b_\lambda$ with the set $X=$ $=\{\xi<\lambda:g(\xi)<f_\beta(\xi)\}$ of size λ. As $g''X$ is of size λ, too, we can apply (iii) from 4.2. There exists some $n\in\omega$ such that for every $\alpha<b_\lambda$ there is some $Q\in\mathcal{Q}_{n,\alpha}$ with $|Q\cap g''X|=\lambda$. In particular, for the above β, we have $|T\cap g''X|=\lambda$ for some $T\in\mathcal{Q}_{n,\beta}$. Let $B=\varphi''T$. As $B\in\mathcal{B}_{n,\beta}$ and as all members of $\mathcal{A}(B)$ are f_β-small, the definition of X together with the claim gives that $|\{A\in\mathcal{A}(B):|M\cap A|=\kappa\}|\geq\kappa^+$, which was to be proved.

$\square$

   § 5. Singular cardinal with countable cofinality. The aim of this section is to prove 2.7 (iii), that means, we have to find a collection $\{\mathcal{A}_\alpha:\alpha<\omega_1\}$ of almost disjoint families on κ with $\kappa>\mathrm{cf}(\kappa)=\omega$

such that for every $M \in [\kappa]^\kappa$ there is some $\alpha < \omega_1$ satisfying $|\{A \in \mathcal{A}_\alpha :$ $: |M \cap A| = \kappa\}| \geq \kappa^\omega$.

Using an easy diagonal argument one can show that for every centered countable family $\mathcal{F}$ in $\mathcal{P}_\kappa(\kappa)$ there is a non zero member u with $u \leq v$ for all $v \in \mathcal{F}$. Therefore $\mathcal{P}_\kappa(\kappa)$ is ω-distributive, so it is worth noticing that h_κ is the least one from the possible candidates.

The forthcoming lemma will turn up to be the crucial point in the proof. It holds for an arbitrary cardinal.

5.1. Lemma. Let $M \in [\kappa]^\kappa$, let $f : M \longrightarrow \kappa$ be a 1-1 function. Then there is some $L \in [M]^\kappa$ and $g : L \longrightarrow \kappa$ such that g is 1-1 and for all $\xi \in L$, $g(\xi) < f(\xi)$.

Proof. Let $h : f[M] \longrightarrow \kappa$ be a $\searrow$ increasing mapping onto κ. Let $L = \{\xi \in M : h \circ f(\xi)$ is a successor ordinal$\}$ and let $g(\xi)$ be defined by the equality $h(g(\xi)) + 1 = h(f(\xi))$.

As h as well as f are 1-1, g is 1-1, too. Since h is increasing, $g(\xi) <$ $< f(\xi)$ for all $\xi \in L$. As h is an onto mapping, g is well-defined. $\square$

5.2. The construction of the desired collection. Using a transfinite induction, we shall construct a collection $\{\mathcal{A}_\alpha : \alpha < \omega_1\}$ together with a family of functions $\{f_A : A \in \mathcal{A}_\alpha, \alpha < \omega_1\}$ such that:

(i) $\mathcal{A}_0 = \{\kappa\}$, f_κ is the identity on κ,

(ii) every $\mathcal{A}_\alpha$ is a MAD family on κ and for each $A \in \mathcal{A}_\alpha$, f_A is a 1-1 mapping from A to κ,

(iii) whenever $\alpha < \beta < \omega_1$ and $B \in \mathcal{A}_\beta$, then for some $A \in \mathcal{A}_\alpha$, $B \subset^* A$,

(iv) if $\alpha < \beta < \omega_1$, $A \in \mathcal{A}_\alpha$, $B \in \mathcal{A}_\beta$ and $B \subset^* A$, then for all $\xi \in A \cap$ $\cap B$, $f_B(\xi) < f_A(\xi)$,

(v) if $\alpha < \beta < \omega_1$, then there is some $\tau < \kappa$ such that for all $A \in$ $\in \mathcal{A}_\alpha$, $B \in \mathcal{A}_\beta$, if $B \subseteq^* A$, then $|B \setminus A| < \tau$. $\mathcal{A}_0$ is fully described by (i), so suppose $\alpha < \omega_1$ and $\mathcal{A}_\alpha$ is known together with $\{f_A : A \in \mathcal{A}_\alpha\}$.

Let us use Lemma 5.1: Whenever $C \in [A]^\kappa$ there is some $B \subseteq C$ with $f_B :$ $: B \longrightarrow \kappa$ such that $|B| = \kappa$, f_B is 1-1 and $f_B(\xi) < f_A(\xi)$ for all $\xi < \kappa$. So choose some infinite MAD family $\mathcal{B}(A)$ consisting of such B's and let $\mathcal{A}_{\alpha+1} = \bigcup \{\mathcal{B}(A) : A \in \mathcal{A}_\alpha\}$.

If $\beta < \omega_1$ is a limit ordinal and $\{\mathcal{A}_\alpha : \alpha < \beta\}$ have been found, fix some sequence of ordinals $\alpha_n \nearrow \beta$ and some sequence of cardinals $\kappa_n \nearrow \kappa$.

For each $\supseteq^*$-decreasing sequence $\mathcal{S} = \{A_n : n \in \omega\}$, where $A_n \in \mathcal{A}_{\alpha_n}$, let $\mathcal{B}(\mathcal{S})$ be a maximal almost disjoint family consisting of B's satisfying $|B \setminus A_n| < \kappa_n$ for all n, $|B| = \kappa$. Denote by $\mathcal{C} = \bigcup \{\mathcal{B}(\mathcal{S}), \mathcal{S}$ is a $\supseteq^*$-decreasing chain contained in $\bigcup_{n \in \omega} \mathcal{A}_{\alpha_n}\}$. Clearly, $\mathcal{C}$ is a MAD family on κ and each $C \in \mathcal{C}$ has the property that for every $\alpha < \beta$ there is a unique $A \in \mathcal{A}_\alpha$ with $C \subseteq^* A$. Notice that (v) holds for $\mathcal{C}$ in place of $\mathcal{A}_\beta$.

Now, consider for $C \in \mathcal{C}$ the mapping h_C defined by $h_C(\xi) =$
$= \min \{f_A(\xi) : A \in \bigcup_{\alpha < \beta} \mathcal{A}_\alpha \& A^* \supseteq C \& \xi \in A \cap C\}$. Being a minimum of a countable family of 1-1 mappings, $|h_C^{-1}(\{\alpha\})| \leq \omega$ for all $\alpha \in \kappa$. Therefore there is some MAD family $\mathcal{D} \subseteq \mathcal{P}(\kappa)$ such that for each $D \in \mathcal{D}$ there is some $C \in \mathcal{C}$ with $D \subseteq C$ and $f_D = f_C \restriction D$ is 1-1.

In order to pass from $\mathcal{D}$ to the desired $\mathcal{A}_\beta$, proceed as in the successor step.

This completes the inductive definition.

It remains to show that $\{\mathcal{A}_\alpha : \alpha < \omega_1\}$ is really what we need.

5.3. Here we shall try for the $(\omega_1, \cdot, \kappa^\omega)$-nowhere distributivity showing first that for each $M \in [\kappa]^\kappa$ there is some $\alpha < \omega_1$ with $|\{A \in \mathcal{A}_\alpha : |M \cap A| = \kappa\}| \geq 2$.

Suppose not, let $M \in [\kappa]^\kappa$ be a counterexample: For every $\alpha < \omega_1$ there is only one $A_\alpha \in \mathcal{A}_\alpha$ with $|M \cap A_\alpha| = \kappa$ (here must be some, because $\mathcal{A}_\alpha$ is MAD !). As $\mathcal{A}_\alpha$ is a MAD family, then $A_\alpha \supseteq^* M$. For brevity, let $f_\alpha = f_{A_\alpha}$ and let $I_n = \{\alpha < \omega_1 : |M \setminus A_\alpha| \leq \kappa_n\}$. Since $\bigcup_{n \in \omega} I_n = \omega_1$, there is some $n < \omega$ with $|I_n| = \omega_1$. For $\alpha \in I_n$, we have $|M \setminus A_\alpha| \leq \kappa_n$, hence $|\bigcup_{\alpha \in I_n} (M \setminus A_\alpha)|$
$\leq \omega_1 \cdot \kappa_n < \kappa$, so there is some $\xi \in M$, $\xi \in \bigcap_{\alpha \in I_n} A_\alpha$. But then $\{f_\alpha(\xi) : \alpha \in I_n\}$ is a strictly decreasing ω_1-sequence of ordinals, a contradiction.

Then, at least two members from some $\mathcal{A}_\alpha$ must meet the set M.

5.4. Now, let us prove that for each $M \in [\kappa]^\kappa$ there is some $\alpha < \omega_1$ with $|\{A \in \mathcal{A}_\alpha : |A \cap M| = \kappa\}| \geq \kappa^+$.

Pick an arbitrary $M \in [\kappa]^\kappa$. Using 5.3 repeatedly, find $\alpha_n \in \omega_1$ and $A_n \in \mathcal{A}_{\alpha_n}$ such that for each $n \in \omega$,

$$|M \cap A_0 \cap A_1 \cap \ldots \cap A_{n-1} \cap A_n| = \kappa ,$$

as well as

$|M \cap A_0 \cap \ldots \cap A_{n-1} \setminus A_n| = \kappa$.

Obviously, $\{\alpha_n : n \in \omega\}$ is a strictly increasing sequence of ordinals and $\{A_n : n \in \omega\}$ is $\supseteq^*$-decreasing.

Consider $\beta = \sup \{\alpha_n : n \in \omega\}$. Since our collection satisfies 5.2 (v), an analogous reasoning as in Claim in 4.3 yields $|\{A \in \mathcal{A}_\beta : |M \cap A| = \kappa\}| \geq \kappa^+$.

5.5. Finally, we prove that for each $M \in [\kappa]^\kappa$ there is some $\alpha < \omega_1$ such that $|\{A \in \mathcal{A}_\alpha : |M \cap A| = \kappa\}| \geq \kappa^\omega$.

Fix an arbitrary $M \in [\kappa]^\kappa$ and consider the partially ordered set $(T, \leq) = = (\{A \in \bigcup_{\alpha < \omega_1} \mathcal{A}_\alpha : |A \cap M| = \kappa\}, \supseteq^*)$. By the construction of the collection $\{\mathcal{A}_\alpha : \alpha < \omega_1\}$ and by 5.4 one can immediately check the properties of $(T, \leq)$ that are essential for our proof:

(i) $(T, \leq)$ is a tree of height ω_1,

(ii) all branches in T have length ω_1,

(iii) for every $t \in T$ there is some $\alpha < \omega_1$ such that $|\{s \in T_\alpha : s \geq t\}| \geq \kappa^+$.

We have to show that $|T_\beta| \geq \kappa^\omega$ for some $\beta < \omega_1$. According to (ii) above it is enough to show that the initial subtree $\bigcup_{\alpha < \beta} T_n$ has κ^ω many branches for some $\beta < \omega_1$.

Define $\tau = \min\{\nu \leq \kappa : \nu$ is a cardinal and $\nu^\omega > \kappa\}$. Since $\varrho^\omega = = \sum_{\xi < \varrho} \xi^\omega$ for every ϱ with uncountable cofinality, we have either $\tau = 2$ or $\tau > \omega = \mathrm{cf}(\tau)$. In each case, $\tau^\omega = \kappa^\omega$.

If $\tau = 2$, one can easily find a full dyadic tree of height ω embedded into $(T, \leq)$, using (iii).

If $\tau > \omega = \mathrm{cf}(\tau)$, fix a sequence of regular cardinals $\tau_n \nearrow \tau$. For every $t \in T$ let $\alpha(t)$ be the first ordinal such that the set $G(t) = \{s \in T_{\alpha(t)} : : s \geq t\}$ is of size at least κ^+. Select an arbitrary subset $H(t) \subseteq G(t)$ of size τ and enumerate it as $\{s_\xi(t) : \xi < \tau\}$.

For $f \in \prod_{n \in \omega} \tau_n$, we shall inductively define a pair $(C(f), \varphi(f))$, where $C(f)$ is a chain in T and $\varphi(f)$ is an increasing mapping from ω to ω_1. Let $C(f)(0) = t_\emptyset$, the root of T and $\varphi(f)(0) = \alpha_0 = 0$. If $C(f)(n) = t_n$ and $\varphi(f)(n) = \alpha_n$ is known, let $\varphi(f)(n+1) = \alpha_{n+1} = \alpha(t_n)$, $C(f)(n+1) = t_{n+1} = = s_{f(n)}(t_n)$.

As $2^\omega < \tau$, $\omega_1^\omega < \tau$, too, and as $|\prod_{n \in \omega} \tau_n| = \tau^\omega > \tau$, there is

some $g \in {}^{\omega}\omega_1$, $g=\langle \alpha_0, \alpha_1, \ldots \rangle$ such that $|\{f \in \prod_{n \in \omega} \tau_n : \varphi(f)=g\}| \geq \tau^+$.

Let $\beta = \sup \{\alpha_n : n \in \omega\}$ and consider the subtree $S=\{C(f)(n) : n \in \omega$, $\varphi(f)=g\}$. The height of S is ω, the n'th level of S is of size $\leq \tau_n$ (an obvious induction on n gives that) and still S has at least τ^+ branches. It remains to realize that then S has at least τ^{ω} branches.

The proof of this mirrors the standard proof of the following well-known fact: If a tree of height ω with all levels finite has at least ω^+ branches, then it has 2^{ω} of them.

Since $\tau^{\omega} = \kappa^{\omega}$, the proof is completed.

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