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Karel Čuda

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A CONTRIBUTION TO TOPOLOGY IN AST: ALMOST INDISCERNIBILITIES

K. ČUDA

Abstract: A natural generalization of equivalences of indiscernibility (called here equivalences of almost indiscernibility) is defined and studied. A general form of topological product is introduced and investigated. A question under what conditions an equivalence of almost indiscernibility is a restriction of a suitable equivalence of indiscernibility is considered.

Key words: Equivalences of almost indiscernibility, equivalences of indiscernibility, compact real equivalences, pseudocontinuous functions, quasicontinuous system of equivalences, topological product.

Classification: 03E70, 54J05

Introduction. Equivalences of indiscernibility play an important role in AST from both philosophical and technical point of view. They are e.g. connected with the remarkable notion of a real class. Remember that a class X is real iff there is an equivalence of indiscernibility R such that X is a figure in R (X is saturated on R). Formally: $(\exists R)(R \subseteq V^2 \,\&\, R$ is a π-class $\&\, R$ is an equivalence $\&\,(\forall u)((u$ infinite $\&\, u \subseteq \mathrm{dom}(R)) \Rightarrow (\exists t,v \in u)(t \neq v \,\&\, \&\, \langle t,v\rangle \in R) \,\&\, (\forall t,v)(t \in X \,\&\, \langle t,v\rangle \in R \Rightarrow v \in X))$ (cf. ČV). From the philosophical point of view real classes model those classes, which may be seen when doing some observation. From the technical point of view they are interesting as this system of classes contains the class FN (of finite natural numbers), every Sd_V class and it is closed on Morse's scheme of definitions. Hence every class definable (also by a non-normal formula) from a real class must be a real one.

Equivalences of almost indiscernibility are a natural generalization of equivalences of indiscernibility. The generalization lies in the requirement that we need them to behave as equivalences of indiscernibility only in every sharp view, which is modelled by the property that every their restriction on a subset of the domain is an equivalence of indiscernibility. We also need them to be real classes. Trivial examples of equivalences of almost indiscernibility are equivalences of indiscernibility restricted to real subclasses of their domains. A typical example of an equivalence of almost indiscernibility is the class of set functions which are continuous in irrational monads with the nearness defined pointwise. Functions continuous only on a suitable figure are very natural mathematical objects. The exact description may be found later in the paper. The given typical example has been the main motivation to the study of the problematics. The investigation of the equivalence of indiscernibility defined pointwise on the whole domain is very limiting and it leads only to the power equivalence on the product.

A remarkable property of equivalences of almost indiscernibility is a form of "heredity", namely: If we consider two equivalences of almost indiscernibility $\overset{*}{=}$ and $\overset{+}{=}$ on X and Y respectively and functions from X to Y continuous on X with nearness defined pointwise, we obtain an equivalence of almost indiscernibility, too. We hold the proof of this property for the main assertion of the paper.

Another important contribution of the paper is (by our opinion) the creation of a very general notion of a topological product in AST and its investigation. The importance of equivalences of almost indiscernibility appears here once more, as the product of equivalences of almost indiscernibility is an equivalence of almost indiscernibility, too. On the other hand, the product of equivalences of indiscernibility need not be an equivalence of indiscernibility.

§ 1. Preliminaries. In this section we remind some notions and prove some theorems which we shall use later.

Definition 1.1: A real symmetric relation R is called compact iff for every infinite subset of its domain m there are two different elements t, $u \in m$ such that $\langle t,u \rangle \in R$ (cf. [Č 87]).

Definition 1.2: A compact equivalence which is a π-class is called an equivalence of indiscernibility (cf.[V]).

We differ here from [V] as we do not require the equivalence to be defined on the whole universal class. The reader may easily prove that every equivalence of indiscernibility can be extended on V.

Definition 1.3: $x \underset{\{c\}}{\overset{o}{=}} y \equiv (\forall \varphi \in FL_{\{c\}})(\varphi$ set formula of one free variable $\varphi(x) \equiv \varphi(y))$ (cf. [ČK 82]). In words: x, y are near in the basic equivalence $\underset{\{c\}}{\overset{o}{=}}$ iff they satisfy the same set-formulas of one free variable with the parameter c.

Note that $\overset{\circ}{=}_{\{c\}}$ is an equivalence of indiscernibility and dom($\overset{\circ}{=}_{\{c\}}$)=V.

Let us remember an elegant theorem due to A. Vencovská. This theorem shows an outstanding position of the basic equivalences of indiscernibility.

Theorem 1.4 (A. Vencovská): If $\doteq$ is a compact equivalence which is a figure in $\overset{\circ}{=}_{\{c\}}$ then $\overset{\circ}{=}_{\{c\}} \cap (\mathrm{dom}(\doteq))^2 \subseteq \doteq$.

Proof: See [Č 87].

The author was told the following topological theorem by A. Vencovská. He does not know, however, the contribution of P. Vopěnka to the theorem. The author is also not able to guarantee the originality of the presented proof.

Theorem 1.5 (A. Vencovská, P. Vopěnka): An equivalence defined on V is an equivalence of indiscernibility iff it is a real compact equivalence fulfilling the following separation condition: For every x, y which are not near ($x \not\doteq y$), there are set theoretically definable classes X, Y containing the monads of x, y respectively and such that $\mathit{Fig}(X) \cap \mathit{Fig}(Y)=0$. Formally: $(\forall x,y)(\neg\, x \doteq y \Rightarrow (\exists X,Y \in Sd_V)\ ((\doteq\text{"}\{x\}) \subseteq X \,\&\, (\doteq\text{"}\{y\}) \subseteq Y \,\&\, (\doteq\text{"}X) \cap$ $\cap (\doteq\text{"}Y)=0))$.

Proof: Let us prove, at first, that every equivalence of indiscernibility fulfils the mentioned separation condition. Let $\{R_n : n \in FN\}$ be a generating system of $\doteq$ (see [V]). If $\neg\, x \doteq y$ then there is n such that $\langle x,y\rangle \notin R_n$. If we put $X=R''_{n+2}\{x\}$ and $Y=R''_{n+2}\{y\}$ then $(\doteq)\text{"}X \subseteq (R_{n+2} \circ R_{n+2})\text{"}\{x\} \subseteq R''_{n+1}\{x\}$ and similarly $(\doteq)\text{"}Y \subseteq R''_{n+1}\{y\}$. It follows that $R''_{n+1}\{x\} \cap R''_{n+1}\{y\}=0$ as $R_{n+1} \circ R_{n+1} \subseteq R_n$ and $\langle x,y\rangle \notin R_n$.

Let, on the other hand, $\doteq$ fulfil the separation condition. It suffices to prove that it is a π-class. As $\doteq$ is a real class, it is a figure in an equivalence $\overset{\circ}{=}_{\{c\}}$ (see [Č 87]). Now it remains to prove that $\doteq$ is a closed figure in $\overset{\circ}{=}_{\{c\}}$ (see [V]). Thus for every x, y such that $\neg\, x \doteq y$ we have to find an Sd_V class Z such that $\overset{\circ}{=}_{\{c\}}\text{"}\{\langle x,y\rangle\} \subseteq Z \,\&\, Z \cap \doteq\, =0$. Let X, Y be the classes from the separation condition. As $\overset{\circ}{=}_{\{c\}}$ is finer than $\doteq$ (due to the theorem of Vencovská) and $\overset{\circ}{=}_{\{c\}}\text{"}\{\langle x,y\rangle\} \subseteq (\overset{\circ}{=}_{\{c\}}\text{"}\{x\}) \times (\overset{\circ}{=}_{\{c\}}\text{"}\{y\})$ (e.g. also by the theorem of Vencovská, or see [ČK 82]) we know that for $Z=X \times Y$ the formula $\overset{\circ}{=}_{\{c\}}\text{"}\{\langle x,y\rangle\} \subseteq Z$ holds. $Z \cap \doteq\, =0$ we obtain by rewriting the assumption

$(\overset{*}{\doteq}"X)\cap(\overset{*}{\doteq}"Y)=0$.

The following example proves that the given separation property cannot be substituted by the following one: Two disjoint monads can be separated by set-theoretically definable classes.

Example 1.6: Let $\overset{*}{\doteq}$ be the equivalence obtained from $\overset{\circ}{=}$ by connecting every even finite natural number with its successor. I.e. $x \overset{*}{\doteq} y \equiv x \overset{\circ}{=} y \vee \vee (\exists n \in FN)(x=2n \& y=2n+1)$. $\overset{*}{\doteq}$ is a real compact equivalence such that its monads are σ-classes and hence the weaker form of the separation condition is fulfilled. But $\overset{*}{\doteq}$ is not revealed, as the countable class $X=\{\langle 2n,2n+1\rangle ; n \in FN\}$ is included in $\overset{*}{\doteq}$ and there is no subset of $\overset{*}{\doteq}$ containing X since $\neg\, 2\alpha \overset{\circ}{=} 2\alpha +1$ (one is even, the other is odd).

The third theorem concerns one generalization of the prolongation axiom for real classes. One version of this theorem (for nonstandard models of PA) can be found in [Č 83]. As no version of the theorem was published in the framework of AST, let us do it now.

Theorem 1.7: If F is a real function such that $(\forall x \subset \mathrm{dom}(F))(F \upharpoonright x \in V)$ then there is a function G such that $F=G \upharpoonright \mathrm{dom}(F)$ and G is a σ-class. Moreover, if F is a figure in $\overset{\circ}{=}_{\{c\}}$ then G is composed from functions definable by set formulas with the parameter c.

The theorem is an easy consequence of the following technical lemma.

Lemma 1.8: If F is a function which is a figure in $\overset{\circ}{=}_{\{c\}}$ and $(\forall x \subseteq \subseteq \mathrm{dom}(F))(F \upharpoonright x \in V)$ then for every $t \in \mathrm{dom}(F)$ there is $G \in Sd_{\{c\}}$ such that $t \in \mathrm{dom}(G)$ and $(\forall u \in \mathrm{dom}(G) \cap \mathrm{dom}(F))(F(u)=G(u))$.

Proof: Let G_0 be such $Sd_{\{c\}}$ function that $G_0(t)=F(t)$. (The existence of such a function is an easy consequence of the definition of $\overset{\circ}{=}_{\{c\}}$, the proof can be found in [ČK 83].) Let $\{X_n ; n \in FN\}$ be a decreasing sequence of $Sd_{\{c\}}$ classes such that $\overset{\circ}{=}_{\{c\}}\{t\} = \bigcap\{X_n ; n \in FN\}$. If there is an element (denote it by u_n) in every class X_n such that $F(u_n) \neq G_0(u_n)$ then this property is saved for a prolongation of the sequence $\{u_n ; n \in FN\}$ (due to the assumption $F \upharpoonright x \in V$) contradicting the equality $F(t)=G_0(t)$ which must hold on the whole monad. Hence for some n the equality $G_0(u)=F(u)$ holds for every $u \in X_n$. We may suppose $X_n \subseteq \mathrm{dom}(G_0)$ (as $t \in \mathrm{dom}(G_0)$) and put $G=G_0 \upharpoonright X_n$.

The proof of the theorem 1.7: As there are only countably many functions definable with parameter c and fulfilling the assertion of L.1.8 $((\forall u \in \mathrm{dom}(G) \cap \mathrm{dom}(F))(F(u)=G(u))$, we can enumerate them and suppose that their domains are disjoint (we put $\bar{G}_n = G_n \wedge (\mathrm{dom}(G_n) - \bigcup_{m<n} \mathrm{dom}(G_m))$). Then we define G as the union of this sequence; we know that dom(G) covers dom(F) due to the assertion of L.1.8.

Corollary 1.9: If F is a real semiset function with the property $(\forall x \subseteq \mathrm{dom}(F))(F \wedge x \in V)$, then there is a set function g such that $F \subseteq g$.

Proof: Use T.1.7 and the prolongation axiom.

Next lemma is a version of Robinson's lemma.

Lemma 1.10 (A. Robinson): Let $\{a_\alpha ; \alpha \in \beta\}$, $\{b_\alpha ; \alpha \in \beta\}$ for $\beta \in N-FN$ be two infinite sequences. Let $\doteqdot$ be an equivalence of indiscernibility. If $(\forall n \in FN)(a_n \doteqdot b_n)$, then there is $\gamma \in \beta - FN$ such that $(\forall \alpha \in \gamma)(a_\alpha \doteqdot b_\alpha)$.

Proof: The assertion is an immediate consequence of the revealness of the π-class $\doteqdot$.

The following technical lemma appears to be very useful.

Lemma 1.11: Let $\{a_\alpha ; \alpha < \beta\}$, $\{b_\alpha ; \alpha < \beta\}$, $\{d_\alpha ; \alpha < \beta\}$ be set sequences. If $(\forall i \in FN)(a_i \stackrel{\circ}{=}_{\{c,d_i\}} b_i)$, then there is a $\gamma \in \beta$-FN such that $(\forall \alpha < \gamma)(a_\alpha \stackrel{\circ}{=}_{\{c,d_\alpha\}} b_\alpha)$.

Proof: Remember that $a \stackrel{\circ}{=}_{\{c,t\}} b \equiv \langle a,t \rangle \stackrel{\circ}{=}_{\{c\}} \langle b,t \rangle$ (see [ČK 82]). Now we reformulate our assertion to $(\forall i \in FN)(\langle a_i, d_i \rangle \stackrel{\circ}{=}_{\{c\}} \langle b_i d_i \rangle) \Rightarrow (\exists \gamma \in \beta\text{-FN})$ $(\forall \alpha \in \gamma)(\langle a_\alpha, d_\alpha \rangle \stackrel{\circ}{=}_{\{c\}} \langle b_\alpha, d_\alpha \rangle)$, which is a special case of Robinson's lemma.

§ 2. Almost indiscernibilities. In this section we define and investigate equivalences of almost indiscernibility.

Definition 2.1: A real equivalence $\doteqdot$ is called an equivalence of almost indiscernibility iff $(\forall x \subseteq \mathrm{dom}(\doteqdot))$ ($\doteqdot \cap x^2$ is an equivalence of indiscernibility).

The proof of the first three assertions of the following theorem is quite easy and hence we omit it.

Theorem 2.2: 1) Every equivalence of almost indiscernibility is compact.

2) Every equivalence of indiscernibility is an equivalence of almost indiscernibility.

3) If $\stackrel{*}{=}$, $\stackrel{+}{=}$ are equivalences of almost indiscernibility, then $\stackrel{*}{=}\cap\stackrel{+}{=}$ is an equivalence of almost indiscernibility, too. Especially, if $\stackrel{*}{=}$ is an equivalence of indiscernibility and X is a real class, then $\stackrel{*}{=}\upharpoonright X=\stackrel{*}{=}\cap X^2$ is an equivalence of almost indiscernibility.

4) The power equivalence $\stackrel{*}{=}^{\mathcal{P}}$ of an equivalence of almost indiscernibility $\stackrel{*}{=}$ is an equivalence of almost indiscernibility.

Proof: 4) Remember that $x\stackrel{*}{=}^{\mathcal{P}}y\equiv(x\cup y\subseteq\mathrm{dom}(\stackrel{*}{=}))\,\&\,(\stackrel{*}{=})''x=(\stackrel{*}{=})''y$. $\stackrel{*}{=}^{\mathcal{P}}$ is a real class as it is defined from the real class $\stackrel{*}{=}$. If $z\subseteq\mathrm{dom}(\stackrel{*}{=}^{\mathcal{P}})$ then $\bigcup z\subseteq\mathrm{dom}(\stackrel{*}{=})$ and we have $\stackrel{*}{=}^{\mathcal{P}}\cap z^2=(\stackrel{*}{=}\cap(\bigcup z)^2)^{\mathcal{P}}\cap z^2$. Now it suffices to use the fact that the power equivalence of an equivalence of indiscernibility is an equivalence of indiscernibility, too. (See [V].)

Remark: From this theorem we obtain the trivial examples of equivalences of almost indiscernibility mentioned in the introduction, namely equivalences of indiscernibility restricted to suitable figures.

When studying the equivalences of almost indiscernibility it appears that real compact equivalences (a more general notion) are highly useful. For these relations Theorem 2.2 may be reformulated word by word as it is done in the following theorem for the points 3) and 4).

Theorem 2.3: 1) If $\stackrel{*}{=}$ and $\stackrel{+}{=}$ are real compact equivalences, then $\stackrel{*}{=}\cap\stackrel{+}{=}$ is a real compact equivalence, too.

2) The power equivalence $\stackrel{*}{=}^{\mathcal{P}}$ of a compact real equivalence $\stackrel{*}{=}$ is also a real compact equivalence.

Proof: 1) Using Vencovská's Theorem, we obtain c such that $\stackrel{\circ}{=}_{\{c\}}$ is finer than both $\stackrel{*}{=}$ and $\stackrel{+}{=}$. Hence $\stackrel{\circ}{=}_{\{c\}}$ (being compact) is finer than $\stackrel{*}{=}\cap\stackrel{+}{=}$.

2) Due to Vencovská's Theorem there is a c such that $\stackrel{\circ}{=}_{\{c\}}$ is finer than $\stackrel{*}{=}$. The equivalence $\stackrel{\circ}{=}_{\{c\}}^{\mathcal{P}}$ is then compact (see [V]) and finer than $\stackrel{*}{=}^{\mathcal{P}}$ and hence $\stackrel{*}{=}^{\mathcal{P}}$ is compact, too.

One way, how the domain of $\stackrel{*}{=}^{\mathcal{P}}$ can be extended (and hence $\stackrel{*}{=}^{\mathcal{P}}$ can be generalized), is to drop the assumption $x,y\subseteq\mathrm{dom}(\stackrel{*}{=})$ and to ask only $(\stackrel{*}{=})''x=(\stackrel{*}{=})''y$. Unfortunately, this generalization does not preserve the structure of almost indiscernibility. We now investigate this generalization as it is useful e.g. for compact real equivalences.

Definition 2.4: For an equivalence $\stackrel{*}{=}$ and a class X we define $x \underset{X}{\stackrel{*}{=}} y \equiv$
$\equiv ((\stackrel{*}{=})''x)\cap X=((\stackrel{*}{=})''y)\cap X$. (Cf. [ZG].)

Note that for X=V we obtain the equivalence mentioned before the definition.

Lemma 2.5: 1) If we put $\doteq\;=(\stackrel{*}{=}\cap X^2)\cup(V-X)^2$, then $x \underset{X}{\stackrel{*}{=}} y \equiv x \doteq^{\mathcal{P}} y$.
2) $\stackrel{*}{=}^{\mathcal{P}} = \underset{V}{\stackrel{*}{=}} \cap(\mathcal{P}(\mathrm{dom}(\stackrel{*}{=})))^2$.
Proof: Use the definitions.

Theorem 2.6: If $\stackrel{*}{=}$ is a real compact equivalence and X is a real class, then $\underset{X}{\stackrel{*}{=}}$ is a compact real equivalence, too.

Proof: Use the previous lemma and T.2.3.

It seems to be plausible (by the second equality of L.2.5) that the operation $\underset{V}{\stackrel{*}{=}}$ does not preserve the structure of almost indiscernibility. A counter-example follows.

Example 2.7: Let $\stackrel{*}{=}\;=(FN)^2$. If we put $u=\{\{\alpha\};\alpha\in\beta\}$ (where $\beta\in N-FN$) then $\underset{V}{\stackrel{*}{=}}\cap u^2=(\{\{n\};n\in FN\})^2\cup(\{\{\alpha\};\alpha\in\beta-FN\})^2$ which is not an equivalence of indiscernibility.

The following example describes a trivial (having two monads) equivalence of almost indiscernibility which cannot be a restriction of any equivalence of indiscernibility.

Example 2.8: Let R be the relation of satisfaction for finite set formulas of one free variable without parameters. Hence $(\forall\varphi\in FL;\ \varphi$ set formula with one free variable$)(R''\{\varphi\}=\{x;\varphi(x)\})$. Let $\overline{R}=V\times \mathrm{dom}(R)-R$. R and $\overline{R}$ are σ-classes which cannot be separated by any set-theoretically definable class. If $\stackrel{*}{=}$ is the equivalence $R^2\cup\overline{R}^2$ (having exactly two classes of equivalence R and $\overline{R}$ respectively), then $\stackrel{*}{=}$ is obviously an equivalence of almost indiscernibility. If $\stackrel{*}{=}$ is a restriction of an equivalence of indiscernibility, then R, $\overline{R}$ have to be separable by π-classes (the corresponding monads) and hence by Sd classes (see [V]).

Theorem 2.9: A real equivalence $\stackrel{*}{=}$ is an equivalence of almost indiscernibility iff for every π-class $X\subseteq \mathrm{dom}(\stackrel{*}{=})$, $\stackrel{*}{=}\cap X^2$ is an equivalence of indiscernibility.

Proof: The implication $\Leftarrow$ is obvious (remember that every set is a π-

class). To prove the opposite implication it suffices to show that $\stackrel{*}{=} \cap X^2$ is a π-class. Let $\stackrel{*}{=}$ and X be figures in $\underset{\{c\}}{\stackrel{\circ}{=}}$. If $x \subseteq X$ is a set dense in X (with respect to $\underset{\{c\}}{\stackrel{\circ}{=}}$, i.e. $X = \underset{\{c\}}{\stackrel{\circ}{=}}$ "x; the existence of such x is proved in [V]) then $\stackrel{*}{=} \cap x^2$ is a π-class (due to almost indiscernibility of $\stackrel{*}{=}$). Now it is sufficient to prove the equality $\stackrel{*}{=} \cap X^2 = \underset{\{c\}}{\stackrel{\circ}{=}} \circ (\stackrel{*}{=} \cap x^2) \circ \underset{\{c\}}{\stackrel{\circ}{=}}$. (Remember the equality $S \circ T = \mathrm{dom}_2((S \times V) \cap (V \times T))$, the fact that the intersection of two π-classes is a π-class and that the projection of a π-class is a π-class.) If $t \stackrel{*}{=} u$ and $t,u \in X$, then there are $t_1,u_1 \in x$ such that $t \underset{\{c\}}{\stackrel{\circ}{=}} t_1$ and $u \underset{\{c\}}{\stackrel{\circ}{=}} u_1$ (the density of x). By Vencovská's Theorem we know that $\underset{\{c\}}{\stackrel{\circ}{=}}$ is finer than $\stackrel{*}{=}$ and hence $t_1 \stackrel{*}{=} t \,\&\, u_1 \stackrel{*}{=} u$, hence $t_1 \stackrel{*}{=} u_1$. If, on the other hand, for $\langle t,u \rangle$ there is $\langle t_1,u_1 \rangle \in \stackrel{*}{=} \cap x^2$ such that $t_1 \underset{\{c\}}{\stackrel{\circ}{=}} t \,\&\, u_1 \underset{\{c\}}{\stackrel{\circ}{=}} u$ then we know that $\langle t,u \rangle \in X^2$ (density of x) and using the fact that $\underset{\{c\}}{\stackrel{\circ}{=}}$ is finer than $\stackrel{*}{=}$ we obtain $t \stackrel{*}{=} u$.

Motivated by the classical development, we define the product of relations.

Definition 2.10: Let R be a relation such that dom(R)=X is a semiset and $X \subseteq m$. Let $(\forall i \in X)$ (R"{i} is a relation). We define the relation $\prod R$ as follows: $\langle f,g \rangle \in \prod R \equiv \mathrm{dom}(f)=\mathrm{dom}(g)=m \,\&\, (\forall i \in X)(\langle f(i),g(i) \rangle \in R"\{i\})$. The relation $\prod R$ is called the product of the system of relations R. If $\mathrm{dom}(R) \in V$, we demand the equality dom(R)=m in the definition. In the case |dom(R)|=2 we use ordered pairs instead of functions and we use the notation $R_1 \times R_2$ (not quite correctly) or $\stackrel{1\times 2}{=}$.

For dom(R) being uncountable the product does not generally preserve the compactness (see [ZG]). If on the domain of R an equivalence of almost indiscernibility is defined, the system R fulfils a suitable continuity condition and we consider only the class of continuous functions, then the compactness and the structure of almost indiscernibility are preserved. This is the direction of our next investigations. Let us give, at first, some necessary definitions.

Definition 2.11: Let R be a system of equivalences (i.e. $(\forall i \in \mathrm{dom}(R))$ (R"{i} is an equivalence)). Let $\stackrel{*}{=}$ be an equivalence on dom(R). A function F is called pseudocontinuous (with respect to $\stackrel{*}{=}$ and R) iff

$(\forall i,j \in \mathrm{dom}(R))(i \overset{*}{=} j \Rightarrow \langle F(i),F(j)\rangle \in R''\{i\} \cap R''\{j\})$.

It is obvious that if $\overset{*}{=}$ is trivial (the identity restricted on dom(R)) then every function is pseudocontinuous. If $(\forall i,j \in \mathrm{dom}(R))\ (R''\{i\}=R''\{j\})$, then we are consistent with the common definition (see [ZG]).

Remark: If f is pseudocontinuous and $\langle f,g\rangle \in \Pi R$, then, generally, g need not be pseudocontinuous as $(R''\{i\})''\{g(i)\} \ni g(j)$ has not to hold for $i \overset{*}{=} j$. This is the reason for defining the following notion.

Definition 2.12: A system of equivalences R is called quasicontinuous with respect to $\overset{*}{=}$ (where $\overset{*}{=}$ is an equivalence defined on dom(R)) if it has the following two properties:

1) If $i \overset{*}{=} j$ then $X=(\mathrm{dom}(R''\{i\}) \cap \mathrm{dom}(R''\{j\}))$ is a figure in both $R''\{i\}$ and $R''\{j\}$. $((R''\{i\})''X=X=(R''\{j\})''X.)$

2) $R''\{i\}$ coincides with $R''\{j\}$ on the intersection.

$(R''\{i\} \upharpoonright X=R''\{j\} \upharpoonright X.)$ If it is clear (from the context) what equivalence $\overset{*}{=}$ we keep in mind, we shall omit it from the quasicontinuity notion.

The proof of the following theorem is quite easy and we leave it to the reader.

Theorem 2.13: Let R be a quasicontinuous system of equivalences and let dom(R) be a semiset. If f is pseudocontinuous and $\langle f,g\rangle \in \Pi R$, then g is pseudocontinuous, too.

The following theorem describes the fact that by going to the system of power equivalences the quasicontinuity of the system of equivalences is saved. The easy proof of the theorem is left to the reader.

Theorem 2.14: If R is a quasicontinuous system of equivalences, then the corresponding system of powerequivalences is quasicontinuous, too. (We define $R^{\mathcal{P}}$ on dom(R) by $(R^{\mathcal{P}})''\{t\}=(R''\{t\})^{\mathcal{P}}$.)

Now we prove a nontrivial theorem.

Theorem 2.15: Let R be a quasicontinuous real system of compact equivalences and let dom(R) be a semiset. If $\overset{*}{=}$ is a compact real equivalence on dom(R), then ΠR is a compact real equivalence on the subclass of all pseudocontinuous functions.

Proof: Let R and $\overset{*}{=}$ be figures in $\underset{\{c\}}{\overset{\circ}{=}}$ (such c must exist due to the assumption that R and $\overset{*}{=}$ are real classes). For every $i \in \mathrm{dom}(R)$ we have that

$R''\{i\}$ is a figure in $\overset{\circ}{=}_{\{c,i\}}$ and in accordance with the Vencovská´s Theorem $\overset{\circ}{=}_{\{c,i\}}$ is finer than $R''\{i\}$. Let X be a countable class dense in dom(R) (with respect to $\overset{\circ}{=}_{\{c\}}$). Let us enumerate this class as $X=\{x_i; i\in FN\}$. Let a be an infinite set of pseudocontinuous functions (with respect to $\overset{*}{=}$). We have to prove the existence of two different functions f, g such that $f,g\in a$ and $\langle f,g\rangle\in \Pi R$. In the set a there is an infinite subset a_1 such that $(\forall \bar{f},\bar{g}\in a_1)(\bar{f}(x_1) \overset{\circ}{=}_{\{c,x_1\}} \bar{g}(x_1))$. Similarly there is an infinite subset $a_2\subseteq a_1$ such that $(\forall \bar{f},\bar{g}\in a_2)(\bar{f}(x_2) \overset{\circ}{=}_{\{c,x_2\}} \bar{g}(x_2))$ and we follow by the recursion based on FN. Due to the prolongation axiom we obtain an infinite subset $\bar{a}$ of a such that $(\forall \bar{f},\bar{g}\in\bar{a})(\forall x\in X)(\bar{f}(x) \overset{\circ}{=}_{\{c,x\}} \bar{g}(x))$. Let us choose two different functions $f,g\in\bar{a}$ and prove that $\langle f,g\rangle\in\Pi R$. We prolong the sequence $\{x_i; i\in FN\}$ and apply L.1.11 for $a_i=f(x_i)$, $b_i=g(x_i)$ and $d_i=x_i$. For every given t there is $\beta\in\gamma$ (γ taken from L.1.11) such that $t \overset{\circ}{=}_{\{c\}} x_\beta$, as X is dense in dom(R). We have $f(x_\beta) \overset{\circ}{=}_{\{c,x_\beta\}} g(x_\beta)$ and hence $\langle f(x_\beta),g(x_\beta)\rangle\in R''\{x_\beta\}$ (as $\overset{\circ}{=}_{\{c,x_\beta\}}$ is finer than $R''\{x_\beta\}$). $\langle f(t),g(t)\rangle\in R''\{t\}$ we obtain from $x_\beta \overset{*}{=} t$ and pseudocontinuity of f, g and quasicontinuity of R.

The following example proves that the choice of f, g from the previous theorem (i.e. $(\forall x\in X)(f(x) \overset{\circ}{=}_{\{c,x\}} g(x))$) was substantial. It does not suffice to require only $(\forall x\in X)(\langle f(x),g(x)\rangle\in R''\{x\})$.

Example 2.16: Let $c\in N-FN$ and let us define R as follows: For $\alpha\in c\cap$ $\cap\, Def(\{c\})$ we put $R''\{\alpha\}=V\times V$ and for $\alpha\in c-Def(\{c\})$ we put $R''\{\alpha\}=\overset{\circ}{=}_{\{c\}}$. Let us consider the real compact equivalence $\overset{\circ}{=}_{\{c\}}$ on dom(R)=c. R is a quasicontinuous real system of compact equivalences. If we put f(x)=1 and g(x)=2 (for every $x\in c$) then both f and g are pseudocontinuous functions and for every $x\in Def(\{c\})\cap c$ we have $\langle f(x),g(x)\rangle\in R''\{x\}$. On the other hand, we have $(\forall x\in c-Def(\{c\}))(\langle f(x),g(x)\rangle\notin R''\{x\})$ and hence $\langle f,g\rangle\notin\Pi R$.

The following theorem shows a nice behavior of equivalences of almost indiscernibility to the general product.

Theorem 2.17: Let R be a real quasicontinuous system of equivalences of almost indiscernibility with a semiset domain. If $\overset{*}{=}$ is a real compact equivalence on dom(R), then ΠR restricted on the subclass of pseudocontinuous functions is an equivalence of almost indiscernibility.

Proof: The reality and compactness of the considered equivalence follows from the previous theorem. To prove that any restriction on a subset of the domain (say a) is an equivalence of indiscernibility, we use the theorem of Vopěnka and Vencovská (T.1.5). We extend the considered equivalence on V-a by adding $(V-a)^2$ (to fulfil the assumption that the considered equivalence is defined on V). Now it suffices to prove that for any two functions $f,g \in a$ which are not equivalent there are subsets b, d of a containing the monads of f and g, respectively, and having disjoint figures. Let there be $t \in dom(R)$ such that $\langle f(t),g(t)\rangle \notin R''\{t\}$ and let us fix this t. Put $a_t = \{f(t); f \in a\}$. By our assumption, $R''\{t\} \cap a_t^2$ is an equivalence of indiscernibility (as $R''\{t\}$ is an equivalence of almost indiscernibility). By the theorem of Vopěnka and Vencovská there are subsets b_t, d_t of a_t such that $(R''\{t\})''\{f(t)\} \subseteq b_t$, $(R''\{t\})''\{g(t)\} \subseteq d_t$ and $(R''\{t\})'' b_t \cap (R''\{t\})'' d_t = 0$. Now it suffices to put $b = \{h \in a; h(t) \in b_t\}$ and $d = \{h \in a; h(t) \in d_t\}$.

The following example proves that the assumption of the quasicontinuity of the system R is substantial.

Example 2.18: Let $\alpha \in$ N-FN. Let us consider the following system R of equivalences of indiscernibility on α. $R''\{\beta\} = \{\langle\beta,\beta\rangle\} \cup (\alpha - \{\beta\})^2$ (i.e. on the β-th component the monads are $\{\beta\}$ and $\alpha - \{\beta\}$). For $\doteq$ we take α^2. Constant functions are pseudocontinuous. If we consider the set of all constant functions, then we obtain an infinite set of elements such that no two different elements are near in the product equivalence - the constant functions with the values β, γ differ in $R''\{\beta\}$ and $R''\{\gamma\}$.

Remark: The assertions of Theorems 2.15, 2.17 become much more interesting when they are applied on the system of power equivalences $R^{\mathcal{P}}$ to a given system R. Before doing so we recommend the reader to note the following comments. To every set relation r such that $dom(r) \subseteq m$ a corresponding function f_r such that $dom(f_r)=m$ and $(\forall t \in m)(f_r(t)=r''\{t\})$ may be assigned (by a one-one manner). We may then define the pseudocontinuity of set relations relatively to the system R and the equivalence $\doteq$ as the pseudocontinuity of the corresponding functions with respect to the system of power-equivalences $R^{\mathcal{P}}$ and $\doteq$. Note that set functions are (in this setting) pseudocontinuous iff they are pseudocontinuous as relations; moreover, they are near in the product equivalence iff they are near in the product equivalence of the power-system as relations. (But in monads of functions there are also other relations - e.g. relations which are unions of two near functions.)

The following theorem demonstrates the power of the assumption of the quasicontinuity of the system R.

Theorem 2.19: If F is a pseudocontinuous function with respect to an equivalence $\doteq$ and a quasicontinuous system of equivalences R, then $(\forall x,y)(x \doteq y \Rightarrow ((R''\{x\})''\{F(x)\} = (R''\{y\})''\{F(y)\})$.

Proof: Let $z \in (R''\{x\})''\{F(x)\}$. We have $F(x) \in (R''\{y\})''\{F(y)\}$ (pseudocontinuity of F), hence $F(y) \in \mathrm{dom}(R''\{x\}) \cap \mathrm{dom}(R''\{y\})$ (quasicontinuity of R), $z \in \mathrm{dom}(R''\{x\}) \cap \mathrm{dom}(R''\{y\})$ (the intersection is a figure in both $R''\{x\}$ and $R''\{y\}$), $F(y) \in \mathrm{dom}(R''\{x\}) \cap \mathrm{dom}(R''\{y\})$ and thus $z \in (R''\{y\})''\{F(y)\}$, as $R''\{x\}$ and $R''\{y\}$ coincide on the intersection of domains.

Corollary 2.20: If μ is an equivalence class of $\doteq$, if R is a quasicontinuous system of equivalences and if $x \in \bigcap\{\mathrm{dom}(R''\{t\}); t \in \mu\}$ then $(\forall t,u \in \mu)((R''\{t\})''\{x\} = (R''\{u\})''\{x\})$.

Proof: Use the previous theorem for $R \wedge \mu$ and $F = \{x\} \times V$.

Corollary 2.21: The generalized product of a quasicontinuous system R of equivalences described by the theorem 2.15 (the class of pseudocontinuous functions with the pointwise defined nearness) is the same as the generalized product of the quasicontinuous system $\overline{R} \subseteq R$ obtained from R by the following description: $\overline{R}''\{t\}$ is the equivalence composed from those monads which are the same in all $R''\{u\}$ where $t \doteq u$. Formally:
$\langle\langle x,y\rangle,t\rangle \in \overline{R} \equiv \langle x,y\rangle \in R''\{t\} \,\&\, (\forall u, u \doteq t)((R''\{t\})''\{x\} = (R''\{u\})''\{x\})$. Moreover, we have $u \doteq t \Rightarrow \overline{R}''\{u\} = \overline{R}''\{t\}$.

Proof: Obvious.

Due to the Vencovská's Theorem we know that if R is a system of compact equivalences which is a figure in $\overset{\circ}{=}_{\{c\}}$ then every element of this system $R''\{t\}$ is coarser than $\overset{\circ}{=}_{\{c,t\}}$. The following theorem describes a circumstance implying that $R''\{t\}$ is even coarser than $\overset{\circ}{=}_{\{c\}}$.

Theorem 2.22: If R is a system of compact equivalences which is a figure in $\overset{\circ}{=}_{\{c\}}$ and if μ is a monad in $\overset{\circ}{=}_{\{c\}}$ then $(\forall t,u \in \mu)(R''\{t\} = R''\{u\}) \Rightarrow$

$\Rightarrow (\forall t \in \mu)(\stackrel{\circ}{=}_{\{c\}} \cap (\mathrm{dom}(R''\{t\}))^2 \subseteq R''\ t\ ).$

Proof: $R''\{t\} = \mathrm{rng}(R \restriction \mu)$ and hence it is a figure in $\stackrel{\circ}{=}_{\{c\}}$ as $R \restriction \mu$ is.

The following theorem and its corollary concern the heredity property mentioned in the introduction.

Theorem 2.23: If $\doteq$ is an equivalence of almost indiscernibility and if $\mathrel{\dot{\ast}}$ is a compact real equivalence which is a semiset, then pseudocontinuous functions from dom($\mathrel{\dot{\ast}}$) to dom($\doteq$) with the pointwise defined nearness form an equivalence of almost indiscernibility.

Proof: Use the theorem on product (T.2.17) for the system $R = \doteq \times \mathrm{dom}(\mathrel{\dot{\ast}})$.

Corollary 2.24: If $\doteq$, $\mathrel{\dot{\ast}}$ are equivalences of almost indiscernibility and if $\mathrel{\dot{\ast}}$ is a semiset then pseudocontinuous functions with the nearness defined pointwise form an equivalence of almost indiscernibility.

The following example describes an equivalence of almost indiscernibility which we have called in the introduction as a typical one.

Example 2.25: Let us consider for $\alpha \in N-FN$ an equivalence of indiscernibility on α representing the segment [0,1] of real numbers. We use e.g. $\beta \doteq \gamma \equiv (\forall n \in FN)(|\beta - \gamma|/\alpha < 1/n)$ (where $|\beta - \gamma|$ fenotes the absolute value. We restrict this equivalence on the figure of irrational monads (hence we obtain an equivalence of almost indiscernibility). If we consider the semiset of pseudocontinuous functions to the segment [0,1] (i.e. α with the same nearness $\doteq$) with the nearness defined pointwise, we obtain (due to our theorems) an example of an equivalence of almost indiscernibility.

§ 3. Restrictions of indiscernibilities. We devote the third section to an investigation of the problem under what conditions an equivalence of almost indiscernibility is a restriction of an equivalence of indiscernibility.

Remember that in the example 2.8 we have described an equivalence of almost indiscernibility which is no restriction of any equivalence of indiscernibility. The following theorem proves that in the case of semisets the situation is rather different.

Theorem 3.1: If $\mathrel{\dot{\ast}}$ is a semiset equivalence of almost indiscernibility having only a finite number of monads, then there is a set equivalence of indiscernibility $\doteq$ and a real semiset $\wp$ such that $\mathrel{\dot{\ast}} = \doteq \cap \wp^2$.

Proof: Let us number the monads of $\mathrel{\dot{\ast}}$ by 0,1,...,k-1. Let us define a function F (being a real semiset) with dom(F)=dom($\overset{*}{=}$) by the description F(x)= the number of the monad containing x. Then $(\forall u \subseteq \mathrm{dom}(F))\ (F \upharpoonright u \in V)$ as $\overset{*}{=} \cap u^2$ is an equivalence of indiscernibility having only a finite number of monads and hence a set (see [V]). By C.1.9 there is a set function f such that $F=f \upharpoonright \mathrm{dom}(F)$. We may assume (without loss of generality) that rng(f)=k and define $t \overset{+}{=} v \equiv f(t)=f(v)$.

The following example proves that the usage of a parameter (obtained by applying the axiom of prolongation) in the last theorem is substantial.

Example 3.2: Let $\{a_n; n \in FN\}$ be a sequence of definable sets having a nontrivial monad μ in $\overset{\circ}{=}$ as its limit (i.e. if $\{a_\alpha; \alpha \in \beta\}$ and $\beta \in N\text{-}FN$ is a prolongation of the sequence, then there is a $\gamma \in \beta$ such that $(\forall \alpha, \bar{\alpha} \in \gamma\text{-}FN)(a_\alpha \overset{\circ}{=} a_{\bar{\alpha}})$. On this countable class, let us define an equivalence of almost indiscernibility in such a way that to one monad we put all sets with the even indices and in the second one those sets with the odd ones. This equivalence of almost indiscernibility is a figure in $\overset{\circ}{=}$, but no equivalence of indiscernibility extending it is a figure in $\overset{\circ}{=}$. If it is a figure in $\overset{\circ}{=}$, then it has to be coarser than $\overset{\circ}{=}$ (due to Vencovská´s Theorem). Hence a_α, $a_{\alpha+1}$ would be in the same monad for $\alpha \in N\text{-}FN$ and thus the same is valid for some $n \in FN$ contradicting the definition of the equivalence.

The following theorem is useful for deciding whether a product equivalence of almost indiscernibility is a restriction of an equivalence of indiscernibility.

Theorem 3.3: Let $\overset{*}{=}$ be a compact real equivalence and $\overset{+}{=}$ an equivalence of almost indiscernibility. Let F, G be two Sd pseudocontinuous (w.r.t. $\overset{*}{=}$ and $\overset{+}{=}$) functions such that $\mathrm{dom}(F) \supseteq \mathrm{dom}(\overset{*}{=})$ and $\mathrm{dom}(G) \supseteq \mathrm{dom}(\overset{*}{=})$. Let $\cong$ be an equivalence of indiscernibility which is finer (on $\mathrm{dom}(\overset{*}{=})$) than $\overset{*}{=}$ (e.g. $\overset{\circ}{=}_{\{c\}}$ for a suitable c). If X is a countable class dense in dom($\overset{*}{=}$) with respect to $\cong$,then $(\forall t \in \mathrm{dom}(\overset{*}{=}))(F(t) \overset{+}{=} G(t)) \equiv (\forall t \in X)(F(t) \overset{+}{=} G(t))$.

Proof: $\Rightarrow$ obvious. $\Leftarrow$: Let $t \in \mathrm{dom}(\overset{*}{=})$. From the density of X it follows that there is an infinite sequence $\{x_\alpha; \alpha \in \beta\}$ where $\beta \in N\text{-}FN$ such that $(\forall n \in FN)(x_n \in X) \& (\forall \alpha \in \beta\text{-}FN)(x_\alpha \cong t)$. $\{F(x_\alpha), G(x_\alpha); \alpha \in \beta\}$ is a subset of dom($\overset{+}{=}$) (denote it a) as F, G are defined for every x_α ($\cong$ "$\{t\} \subseteq$ $\subseteq \mathrm{dom}(\overset{*}{=})$). $\overset{+}{=} \cap a^2$ is an equivalence of indiscernibility and $(\forall n \in FN)(F(x_n) \overset{+}{=} G(x_n))$. Hence by Robinson´s Lemma there is a $\gamma \in \beta\text{-}FN$ such that $F(x_\gamma) \overset{+}{=} G(x_\gamma)$. $F(t) \overset{+}{=} G(t)$ follows now from the pseudocontinuity of F, G.

Remark: The previous theorem should be compared with the example 2.16.

Corollary 3.4: If an equivalence of almost indiscernibility is obtained as a product of the system $\doteq \times \operatorname{dom}(\doteq)$, where $\doteq$ is a restriction of an indiscernibility equivalence (say $\triangleq$) and $\doteq$ is a real compact equivalence which is a semiset, then this equivalence is a restriction of a suitable equivalence of indiscernibility.

Proof: Let $m \supseteq \operatorname{dom}(\doteq)$ be the set from the definition of ΠR. Let $X = \{x_i; i \in FN\}$ be the countable class from T.3.3. On the class $Y = \{f; \operatorname{dom}(f) = m \mathbin{\&} \operatorname{rng}(f) \subseteq \operatorname{dom}(\triangleq)\}$ define equivalences $\overset{i}{=}$ $(i \in FN)$ by the formula $f \overset{i}{=} g \equiv f(x_i) \triangleq g(x_i)$. Equivalences $\overset{i}{=}$ are obviously equivalences of indiscernibility and we obtain the required equivalence as the intersection of the countable system $\{\overset{i}{=}; i \in FN\}$ due to the previous theorem. This intersection is an equivalence of indiscernibility due to [V].

Remark: Note that the "typical" equivalence of almost indiscernibility given in the example 2.25 is a restriction of a suitable equivalence of indiscernibility.

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Matematický ústav, Univerzita Karlova, Sokolovská 83, 186 00 Praha 8, Czechoslovakia

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