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COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
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NONISOLATED SINGULARITIES OF SOLUTIONS TO A QUASILINEAR ELLIPTIC SYSTEM

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Abstract: There is presented an example of a quasilinear elliptic system which has solutions with nonisolated discontinuities.

Key words: Elliptic systems of partial differential equations, weak solutions, regularity.

Classification: 35J60, 35D10

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1. Introduction. Let $\Omega \subset R^n$ be an open set. We consider quasilinear elliptic systems

(1) $D_\alpha A_{ij}^{\alpha\beta}(u)\, D_\beta u^j=0,\ i=1,\dots,m.$

(The summation convention concerning repeated indices is used throughout the paper; $i,j=1,\dots,m$, $\alpha,\beta=1,\dots,n$.) Referring to the system (1) we always assume the coefficients to be bounded uniformly continuous functions on R^m satisfying the ellipticity condition

(2) $A_{ij}^{\alpha\beta}\xi_\alpha^i\,\xi_\beta^j \geq |\xi|^2$ for each $\xi \in R^{mn}$.

By a (weak) solution of (1) we understand a (vector valued) function $u \in W_{loc}^{1,2}(\Omega, R^m)$ such that

(3) $D_\alpha b_i^\alpha=0,\ i=1,\dots,m$

holds in the sense of distributions for

(4) $b_i^\alpha=A_{ij}^{\alpha\beta}(u)D_\beta u^j.$

The counterexample by E. Giusti and M. Miranda [3] shows that for $n \geq 3$ the discontinuous function

$$u:x \mapsto \frac{x}{|x|}$$

solves a system of type (1). Thus one cannot expect full regularity results in this general setting. Typical results estimate the Hausdorff dimension of singularities.

Theorem 1. (E. Giusti [2], see also [1].) Let u be a weak solution of (1). Then there is an open set $\Omega_0 \subset \Omega$ such that u is locally Hölder continuous on Ω_0 and the Hausdorff dimension of $\Omega \setminus \Omega_0$ is less than n-2.

An easy modification of the above mentioned counterexample shows that for every $n \geq 4$ there is a system of type (1) which has a solution discontinuous at every point of

$$\{x \in R^n : x_1 = x_2 = x_3 = 0\}.$$

Thus, it has been known for a long time that the singular set can be nonisolated if $n \geq 4$. We shall prove the following theorem concerning n=3.

Theorem 2. There are a weak solution $s: R^3 \to R^6$ of a system of type (1) and a sequence $\{z_k\}$ of discontinuity points for s such that $z_k \neq 0$, $z_k \to 0$.

2. Reduction to two singularities. Let $z_0 \in R^3$, $z_0 \neq 0$. Let u_0 be a bounded weak solution of (1). For each k=0,1,... denote

$$z_k = z_0/2^k,$$
$$u_k(x) = u_0(2^k x),$$
$$B_k = B(z_k, |z_k|/4)$$

(B(z,r) denotes the open ball with center at z and radius r). By a simple homothety argument we see that u_k also solve (1) (the coefficients are fixed!). Now let us assume that

$$u_1 = u_0 \text{ outside } B_0 \cup B_1.$$

Put

$$s_1 = \begin{cases} u_0 & \text{outside } B_1, \\ u_1 & \text{on } B_1. \end{cases}$$

Of course, $s_1 = u_1$ outside B_0. As the concept of weak solution is local, we see that s_1 solves (1). We define recurrently (k=2,3,...)

$$s_k = \begin{cases} s_{k-1} & \text{outside } B_k, \\ u_k & \text{on } B_k. \end{cases}$$

We see by induction that s_k solve (1) and

$$\|s_k - s_{k-1}\|^2_{W^{1,2}(\Omega)} \leq C(\Omega)\, 2^{-k}$$

for every bounded domain $\Omega \subset R^3$. Hence the sequence $\{s_k\}$ has a limit s in the sense of (strong) $W^{1,2}_{loc}$ convergence. By routine arguments we see that s solves (1), too. If u_0 is discontinuous at z_0, then s is discontinuous at all points z_k (and at the origin).

Conclusion. Theorem 2 is proved if we construct a system (1) (i.e. coefficients $A_{ij}^{\alpha\beta}$) and its bounded weak solution u such that

(5) u is discontinuous at some point $z \in R^3$, $z \neq 0$,

(6) $u(2x)=u(x)$ for all $x \notin B(z,|z|/4) \cup B(z/2,|z|/8)$.

3. Construction. Fix a decreasing function $\varphi \in C^2([0,1])$ with

$$\varphi'(1\_)=0,\ \varphi(0)=1,\ \varphi(1)=0$$

and denote by ψ its inverse. Now, prolong the functions φ and ψ to $[0,\infty)$ putting

$$\varphi(r)=0,\ \psi(t)=0 \text{ for } r,t \in (1,\infty).$$

Fix a point $z \in R^3$, $|z|=4$. Denote

$$y=x-z \text{ for every } x \in R^3.$$

Let c be a fixed constant,

(7) $c \geq 2 \sup (4+r)|\varphi'(r)|:r \in (0,1)$.

Put

$$(8)\qquad u^i(x)=\begin{cases} c\,\dfrac{x_{i-3}}{|x|} & \text{if } i=4,5,6,\\[2mm] \varphi(|y|)\,\dfrac{y_i}{|y|} & \text{if } i=1,2,3.\end{cases}$$

We have

$$(9)\quad D_\alpha u^i=\begin{cases} \dfrac{c}{|x|}\left(\delta^\alpha_{i-3}-\dfrac{x_{i-3}x_\alpha}{|x|^2}\right) & \text{if } i=4,5,6,\\[2mm] \dfrac{\varphi(|y|)}{|y|}\left(\delta^\alpha_i-\dfrac{y_iy_\alpha}{|y|^2}\right)+\varphi'(|y|)\dfrac{y_iy_\alpha}{|y|^2} & \text{if } i=1,3,3.\end{cases}$$

Denote

$$(10)\quad b^\alpha_i=\begin{cases} \dfrac{c}{|x|}\left(\delta^\alpha_{i-3}+\dfrac{x_{i-3}x_\alpha}{|x|^2}\right) & \text{if } i=4,5,6,\\[2mm] \dfrac{\varphi(|y|)}{|y|}\left(\delta^\alpha_i+\dfrac{y_iy_\alpha}{|y|^2}\right)+\varphi'(|y|)\left(\delta^\alpha_i-\dfrac{y_iy_\alpha}{|y|^2}\right) & \text{if } i=1,2,3.\end{cases}$$

By a routine calculation we obtain the validity of (3). Obviously, the function u satisfies (5) and (6). It remains only to find coefficients $A_{ij}^{\alpha\beta}$ such that (4) holds.

4. Coefficients. In this section we construct coefficients $A^{\alpha\beta}_{ij}(v,w)$ $(v,w\in R^3)$ such that the functions u^i, b^α_i given by (8), (10) satisfy

(11) $b^\alpha_i = A^{\alpha\beta}_{ij}((u^1,u^2,u^3),(u^4,u^5,u^6))\, D_\beta u^j$.

We follow essentially the method due to J. Souček [6]. Denote

$$Y=\psi(|v|),$$

$$X=\begin{cases}|z+Y\frac{v}{|v|}|, & v\neq 0,\\ 4, & v=0,\end{cases}$$

$$h=\frac{cY}{cY+X\varphi(Y)},$$

$$f=\frac{X\varphi(Y)}{cY+X\varphi(Y)},$$

$$g=\frac{XY\varphi'(Y)}{cY+X\varphi(Y)}.$$

Using the conventions

$$\frac{v_i v_\alpha}{|v|^2}=0 \text{ if } v=0,\quad \frac{w_{i-3}w_\alpha}{|w|^2}=0 \text{ if } w=0,$$

$$v_i=w_i=0 \text{ if } i\notin\{1,2,3\},$$

we define

$$B^\alpha_i=h(\delta^\alpha_{i-3}+\frac{w_{i-3}w_\alpha}{|w|^2})+f(\delta^\alpha_i+\frac{v_i v_\alpha}{|v|^2})+g(\delta^\alpha_i-\frac{v_i v_\alpha}{|v|^2}),$$

$$T^\alpha_i=h(\delta^\alpha_{i-3}-\frac{w_{i-3}w_\alpha}{|w|^2})+f(\delta^\alpha_i-\frac{v_i v_\alpha}{|v|^2})+g\,\frac{v_i v_\alpha}{|v|^2}$$

$(i=1,\dots,6;\ \alpha=1,2,3)$. Finally we put

$$Q=\min(1,|w|)\,(3B_iT_i-T_iT_i)^{-1},$$

$$A_{ij}=\delta^j_i\delta^\beta_\alpha+Q(3B^\alpha_i-T^\alpha_i)(3B^\beta_j-T^\beta_j)$$

$(i,j=1,\dots,6;\ \alpha,\beta=1,2,3)$. By (7) we verify that Q is a nonnegative bounded function on $R^3\times R^3$. Indeed, we have

$$3B_iT_i-T_iT_i=\frac{\frac{3}{4}c^2Y^2+X^2\varphi^2(Y)+3(X\varphi(Y)+2XY\varphi'(Y))^2+\frac{13}{4}Y^2(c^2-4X^2(\varphi'(Y))^2)}{(cY+X\varphi(Y))^2}.$$

The coefficients are continuous: discontinuities $\frac{v_i v_\alpha}{|v|^2}$, $\frac{w_{i-3}w_\alpha}{|w|^2}$, X are always neutralized being multiplied by vanishing continuous functions. We observe that the supremum of $|A_{ij}^{\alpha\beta}|$ as well as the modulus of continuity of $A_{ij}^{\alpha\beta}$ are estimated by the same quantities on $\{v:|v|\le 1\}\times\{w:|w|\le 1\}$. Hence the coefficients are bounded and uniformly continuous. By a direct calculation we see that (11) is satisfied.

5. Some remarks. A) If we admit discontinuous coefficients (Borel measurable only) then Theorem 1 does not hold.

Let $a_{ij}^{\alpha\beta}$ be bounded Borel measurable functions on R^3 satisfying

$a_{ij}^{\alpha\beta}\,\xi_\alpha^i\,\xi_\beta^j \ge |\xi|^2$ for each $\xi\in R^9$.

Let v be a weak solution to the linear system

(12) $D_\alpha a_{ij}^{\alpha\beta}(x)D_\beta\, v^j=0.$

Then the function u defined by

$u^i=v^i$, i=1,2,3, $\quad u^i=x_{i-3}$, i=4,5,6

solves the quasilinear system

$$D_\alpha A_{ij}^{\alpha\beta}(u)D_\beta\, u^j=0$$

where

$$A_{ij}^{\alpha\beta}(u)=\begin{cases} a_{ij}^{\alpha\beta}(u^4,u^5,u^6) & \text{if } i,j\in\{1,2,3\},\\ \delta_i^j\,\delta_\alpha^\beta & \text{otherwise.}\end{cases}$$

However, solutions of (12) can be everywhere discontinuous (see [4]).

B) Our example does not fill the gap between the estimate of Hausdorff dimension of singular sets given in Theorem 1 and n-3-dimensionality of singular sets in the known counterexamples. It is not even clear whether the singular set for n=3 can be uncountable.

C) It would be nice to have counterexamples (or further positive regularity results) in case when the quasilinear system (1) is obtained as a system in variation. The only known counterexamples (see e.g. J. Nečas [5]) have one singular point.

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