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Antonio R. Tineo

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EXISTENCE OF SOLUTIONS FOR A CLASS OF BOUNDARY VALUE PROBLEMS FOR THE EQUATION $x'''=F(t,x,x',x'')$

Antonio TINEO

Abstract: An existence theorem for a boundary value problem $x'''=F(t,x,x',x'')$ is proved.

Key words: Boundary value problem

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0. Introduction. In this paper we will prove an existence theorem for the boundary value problem

$$x'''=F(t,x,x',x''); \quad x\in E \tag{0.1}$$

where $F:[0,1]\times R^3\to R$ is a continuous function and E is a closed subspace of $C^2([0,1],R)$ of codimension two such that for all $x\in E$ there exist $t_0=t_0(x)\in[0,1]$ with

$$|x(t)|\le|x(t_0)| \quad (0\le t\le 1) \text{ and } x'(t_0)=0 \tag{0.2}$$

We will call such subspace E an <u>admissible subspace</u> of C^2.

Our main result is the following.

0.1. Theorem. Suppose that

1) there exist $R>0$ such that

$$F(t,-R,0,0)\le 0\le F(t,R,0,0) \quad (0\le t\le 1).$$

2) There exist $c\in[0,1]$ and $h:[0,\infty)\to(0,\infty)$ continuous such that

$$|F(t,x,y,z)|\le h(|y|)+c|z| \text{ if } |x|\le R\,.$$

$$\int_0^\infty \frac{sds}{h(s)} \quad 2(1-c)^{-1}R \tag{0.3}$$

3) The function $z\to z-F(t,x,y,t)$ is strictly increasing for each fixed $(t,x,y)\in[0,1)\times[-R,R]\times R$.

Then the problem (0.1) has at least one solution u such that $|u(t)|\le R$ $(0\le t\le 1)$.

Remarks

a) When $F(t,x,y,z)=f(t,x,y)$ do not depend on z, our result includes some Granas, Guenther and Lee Theorems [1],[2].

b) Our Theorem is an alternative to Theorem 1.1 of [3]. In fact, it is interesting to compare the hypothesis (i) of this theorem with our hypothesis (3). Moreover, our Theorem covers the principal examples of existence treated in [3] (see § 2 below).

c) The author has classified the admissible subspaces of $C^2([0,1],R)$ which are described by equations of the form

$$a_1\, x(0)+a_2\, x'(0)+a_3\, x(1)+a_4\, x'(1)=0$$

$$b_1\, x(0)+b_2\, x'(0)+b_3\, x(1)+b_4\, x'(1)=0$$

where $a_1,\ldots,a_n$, $b_1,\ldots,b_4$ are fixed real numbers.

1. Proof of the main result. In the following C^0 denotes the space of all continuous functions $u:[0,1]\to R$; with the usual norm $\|u\|_0=\sup\{|u(t)|:\ 0\leq t\leq 1\}$. Moreover, C^2 denotes the space of all C^2-functions $u:[0,1]\to R$ with the norm $\|u\|_2=\max\{\|u\|_0, \|u'\|_0, \|u''\|_0\}$.

1.1. Proposition. Let $\varepsilon>0$ and let us define $L_\varepsilon:C^2\to C^0$ by $L(x)=x''-\varepsilon x$. If E is an admissible subspace of C^2 then the restriction of L_ε to E is an isomorphism onto C^0.

Proof. Let $x\in E$ such that $L_\varepsilon(x)=0$ and choose $t_0\in[0,1]$ satisfying (0.2), then $x(t_0)\, x''(t_0)\leq 0$ and hence $\varepsilon x(t_0)^2\leq 0$. So $x=0$ and $C^2=E\oplus \mathrm{Ker}\ 1$. The proof follows now easily.

Now, using the arguments in § 2 of [1] and Theorem 3.1 of [2], it is easy to prove the following result:

1.2. Proposition. Let $f:[0,1]\times R^2\to R$ be a continuous function and suppose that there are $R>0$ and a continuous function $h_0:[0,\infty)\to(0,\infty)$ such that

1) $f(t,-R,0)\leq 0\leq f(t,R,0)\quad (0\leq t\leq 1)$
2) $|f(t,x,y)|\leq h(|y|)$ if $|x|\leq R$
3) $\int_0^\infty s\, h_0(s)^{-1}ds>2R$.

If E is an admissible subspace of C^2 then the problem $[x''=f(t,x,x'),\ x\in E]$ has at least one solution u such that $\|u\|_0\leq R$.

Proof of Theorem 0.1.

Claim 1. For each $(t_0,x_0,y_0)\in[0,1]\times[-R,R]\times\mathbf{R}$ there is a unique $z_0\in\mathbf{R}$ such that $z_0=F(t_0,x_0,y_0,z_0)$.

Proof. Let us define $\Delta:\mathbf{R}\to\mathbf{R}$ by $\Delta(z)=z-F(t_0,x_0,y_0,z)$; then by hypothesis (2) and (3) of Theorem 0.1 we have that Δ is a bijective function and hence there is z_0 such that $\Delta(z_0)=0$. So the proof of Claim 1 is finished.

By Claim 1 there is a function $f_0:[0,1]\times[-R,R]\times\mathbf{R}\to\mathbf{R}$ such that

$$f_0(t,x,y)=F(t,x,y,\ f_0(t,x,y)) \tag{1.1}$$

Claim 2. f_0 is a continuous function.

Proof. It is easy to prove that

$$|f_0(t,x,y)|\le(1-c)^{-1}\ h(y) \tag{1.2}$$

Suppose now that f_0 is discontinuous at the point (t_0,x_0,y_0) then there are a sequence $\{(t_n,x_n,y_n)\}$ in $[0,1]\times[-R,R]\times\mathbf{R}$ and $\varepsilon>0$ such that $t_n\to t_0$, $x_n\to x_0$, $y_n\to y_0$ and

$$|f_0(t_n,x_n,y_n)-f_0(t_0,x_0,y_0)|\ge\varepsilon\ . \tag{1.3}$$

By (1.2) we conclude that $\{f_0(t_n,x_n,y_n)\}$ is a bounded sequence and hence we can assume, without loss of generality, that $f_0(t_n,x_n,y_n)\to z_0$ for some $z_0\in\mathbf{R}$. But we know that

$$f_0(t_n,x_n,y_n)=F(t_n,x_n,y_n,\ f_0(t_n,x_n,y_n))$$

and hence $z_0=F(t_0,x_0,y_0,z_0)$. So $z_0=f_0(t_0,x_0,y_0)$. On the other hand, by (1.3), one has $|z_0-f_0(t_0,x_0,y_0)|\ge\varepsilon$ and this contradiction proves Claim 2.

Claim 3. $f_0(t,-R,0)\le 0\le f(t,R,0)$ $(0\le t\le 1)$.

Proof. Let us fix $t\in[0,1]$ and define $\Delta:\mathbf{R}\to\mathbf{R}$ by $\Delta(z)=z-f(t,R,0,z)$; we know that Δ is a bijective and increasing function; hence $\Delta(z)\to+\infty$. On the other hand $\Delta(0)=-F(t,R,0,0)\le 0$ and by Bolzano Theorem there is $z_0\ge 0$ such that $\Delta(z_0)=0$; so $z_0=F(t,R,0,z_0)$ and by Claim 1 $f_0(t,R,0)=z_0\ge 0$. Similarly, we can show that $f_0(t,-R,0)\le 0$ $(0\le t\le 1)$ and the proof of Claim 3 is finished.

Now let $f:[0,1]\times\mathbf{R}^2\to\mathbf{R}$ be a continuous extension of f_0; by Claim 2 and 3 we have that f satisfies the hypotheses of Proposition 1.2 with $h_0=(1-c)^{-1}h$; in consequence there is $u\in E$ with $\|u\|_0\le R$ such that $u''(t)=f(t,u(t),u'(t))$. Hence $u''(t)=f_0(t,u(t),u'(t))$ and the proof follows from the relation (1.1).

Remark. Theorem 0.1 remains true if the hypothesis (2) is replaced by $|F(t,x,y,z)| \leq h(|y|)+c(|y|)z$ where $h:[0,\infty) \longrightarrow (0,\infty)$ and $c:[0,\infty) \longrightarrow [0,1)$ are two continuous functions such that

$$\int_0^\infty \frac{(1-c(s))s\, ds}{u(s)} > 2R.$$

For example, the integral above diverges if $h(s)=A+Bs^2$ ($A>0$, $B\geq 0$) and $c(s)= =1-[A_0+B_0 \ln(1+s)]^{-1}$ ($A_0>0$, $B_0\geq 0$).

2. Examples. In this section we apply Theorem 0.1 to some special cases of Problem (0.1). For purposes of comparison we shall consider some particular examples of [3].

2.1. Corollary. Let $H:[0,1]\times R^3 \longrightarrow R$ be a continuous function and let $p \in C^0$. Suppose that

1) there is $R>0$ such that

$$H(t,-R,0,0) \leq \min p \leq \max p \leq H(t,R,0,0).$$

2) There are $A,B,C \geq 0$; $c<1$, such that

$$|H(t,x,y,z)| \leq A+B\, y^2+c|z| \text{ if } |x| \leq R.$$

3) The function $z \longrightarrow z-H(t,x,y,z)$ is strictly increasing for all fixed $(t,x,y) \in [0,1]\times[-R,R]\times R$.

Then the generalized Lienard boundary value problem

$$x''=g(x)\, x'+H(t,x,x',x'')-p(t), \quad x \in E$$

has at least one solution for all continuous functions $g:R \longrightarrow R$ and all admissible subspaces E of C^2.

Proof. It is easy to prove that the function

$$F(t,x,y,z)=g(x)\, y+H(t,x,y,z)-p(t)$$

satisfies the hypotheses of Theorem 0.1 with

$$h(y)=A+By^2+D|y|+\|p\|_0,$$

when $D=\sup\{|g(x)|:|x| \leq R\}$. So the proof is finished.

Remark. Compare with Theorem 3.1 of [3].

2.2. Corollary. Let $f,g:[0,1]\times R^2 \longrightarrow R$ be continuous functions such that:

1) There are R, $\delta>0$ such that

$$f(t,-R,0) \leq -\delta < \delta \leq f(t,R,0) \quad (0 \leq t \leq 1)$$

2) There are $A,B \geq 0$ such that

$$|f(t,x,y)| \leq A+By^2.$$

3) There is c, $0 \leq c < 1$ such that

$$|g(t,y,z_1)-g(t,y,z_2)| \leq c|z_1-z_2|.$$

4) There are $A_1,B_1 \geq 0$ such that

$$|g(t,y,0)| \leq A_1+B_1\ y^2$$

then the problem

$$x''=f(t,\ x,\ x')+g(t,x',x'')-p(t),\ x \in E$$

has at least one solution if E is an admissible subspace of C^2 and $\|p-p_0\|_0 \leq \delta$, where $p_0(t)=g(t,0,0)$.

Proof. Let us define $F(t,x,y,z)=f(t,x,y)+g(t,y,z)-p(t)$, then F satisfies (0.4) and hence F satisfies the hypothesis (3) of Theorem 0.1. On the other hand

$$|g(t,y,z)-g(t,y,0)| \leq c|z|$$

and hence

$$|F(t,x,y,z)| \leq A+A_1+(B+B_1)y^2+c|z|+\|p\|_0.$$

In consequence F satisfies the hypothesis (2) of Theorem 0.1. Now it is easy to verify that F satisfies also the hypothesis (1) of Theorem 0.1 and so the proof is complete.

Remark. Compare with Proposition 3.3 of [3].

2.3. Proposition. If $c \in [0,1)$ and E is an admissible subspace of C^2 then the problem

$$x''=x^3+x'^2+c\ \sin x''-p(t),\ x \in E$$

has at least one solution for each fixed $p \in C^0$.

Proof. It is easy to verify that the function

$$F(t,x,y,z)=x^3+y^2+c\ \text{sen} z-p(t)$$

satisfies the hypotheses of Theorem 0.1 with $R=\|p\|_0^{1/3}$ and $h(s)=s^2+2R^3$; so the proof is finished.

Remark. Compare with Proposition 3.1 of [3].

3. Uniqueness. In this section we shall prove a unicity Theorem for Problem (0.1) likewise Theorem 2.5 of [3].

3.1. Theorem. Suppose that $F(t,x,y,z)$ has continuous partial derivatives $F_x(t,x,y,z)$, $F_y(t,x,y,z)$ and $F_z(t,x,y,z)$ in $[0,1]\times R$ and suppose that $F_z<1$ and $F_x\geq 0$ in $[0,1]\times R^3$. If E is an admissible subspace of E and u, v are two solutions of Problem (0.1) then u-v is a constant function. In particular, the problem (0,1) has at most one solution in E, if E has no nontrivial constant functions.

If E contains the constant functions then the problem (0,1) has at most one solution in E in the following two cases:

1) There is $t_1\in[0,1]$ such that $F_x(t_1,x,y,z)>0$ for all (x,y,z).
2) $F_x(t,x,0,z)>0$ if $x.z\neq 0$.

Proof. Let us define $x=u-v$, it is easy to prove that there are $a,b,c\in C^0$ such that

$x''(t)=a(t)x(t)+b(t)x'(t)+c(t)x''(t)$, $a(t)\geq 0$, $c(t)<1$.

Now let us fix a positive function $p:[0,1]\rightarrow R$ of the class C^1 such that

$$p'(t)=-\frac{b(t)}{1-c(t)}p(t).$$

Now, considering $g(t)=p(t)x(t)x'(t)$, we have that $g'(t)=p(t)x'(t)^2+a(t)p(t)(1-c(t))^{-1}x(t)^2$, because $x''=a(1-c)^{-1}x+b(1-c)^{-1}x'$. In particular, $g'\geq 0$.

Now, choose $t_0\in[0,1]$ satisfying (0.2), then $x.x'\geq 0$ in $[t_0,1]$ and $x.x'\leq 0$ in $[0,t_0]$, hence $|x(t)|\geq|x(t_0)|$ and in consequence x is a constant function. Suppose now that E contains the constant functions and suppose that $u,v\in E$ are two solutions of (0.1) such that $u\neq v$. We know that $v(t)=u(t)+k$ for some $k\in R$, $k\neq 0$. Hence $F(t,u(t)+k,u'(t),u''(t))=F(t,u(t),u'(t),u''(t))$. In particular we have

i) $0=k\ F_x(t_1,u(t_1)+\theta k, u'(t_1), u''(t_1))$ for some $\theta\in[0,1]$ (a contradiction).

ii) $k\ F_x(t_0,u(t_0)+\theta k,0,u''(t_0))=0$, where $t_0\in[0,1]$ is chosen such that $\|u\|_0=|u(t_0)|$ and $u'(t_0)=0$. Notice that $u''(t_0)\ u(t_0)\leq 0$ (a contradiction).

So the proof is now finished.

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Universidad de Los Andes, Facultad de Ciencias, Departamento de Matematica, Merida - Venezuela 5101

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