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Commentationes Mathematicae Universitatis Carolinae, Vol. 29 (1988), No. 1, 195--196

Persistent URL: `http://dml.cz/dmlcz/106610`

ANNOUNCEMENTS OF NEW RESULTS

(of authors having an address in Czechoslovakia)

MAXIMUM RANK OF A POWER OF A MATRIX OF A GIVEN PATTERN

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Let G be a digraph with possible loops and without multiple edges. A t-walk is a sequence $w=(v_0,e_1,v_1,\dots,v_{t-1},e_t,v_t)$ of (not necessarily distinct) vertices and edges of G such that $e_i=v_{i-1}v_i$ is a directed edge for each i. We say that two t-walks w and w' are vertex (edge) independent if $v_i \neq v_i'$ for $i=0,\dots,t$ ($e_i \neq e_i'$ for $i=1,\dots,t$). A path is a walk with $v_i \neq v_j$ for $i \neq j$, and a cycle is a walk with distinct vertices but $v_0=v_t$. We denote by $|P|$ and $|C|$ the number of vertices of a path or a cycle.

Theorem. For every digraph G and a positive integer p there are mutually vertex disjoint cycles $C_1,\dots,C_l$ and paths $P_1,\dots,P_k$ such that the maximum number of vertex independent p-walks equals $\sum_{i=1}^{l}|C_i| + \sum_{i=1}^{k}(|P_i|-p)$.

The above theorem may be interpreted as follows. Let m be the maximum number of people who could simultaneously walk in digraph G for p time units traversing one edge per a unit so that two or more people never meet in a vertex. Then the optimal schedule can always be organized as follows. The people are divided into several subgroups and each subgroup either walks round a cycle or along a path.

Let $A=(a_{ij})$ be a real matrix of size n by n. The pattern of G is the digraph on vertices $\{1,2,\dots,n\}$ and with an edge ij if $a_{ij} \neq 0$. For a digraph G let $\mathcal{A}(G)$ be the class of matrices of pattern G.

Corollary 1. Maximum possible rank of the p-th power A^p of a matrix A of a given pattern G equals maximum number of vertex independent p-walks in G. For a symmetric digraph G, we denote by $\mathcal{S}(G)$ the class of symmetric matrices of pattern G.

Corollary 2. $\max_{A\in\mathcal{S}(G)} rA^p = \max_{A\in\mathcal{A}(G)} rA^p$ for every symmetric digraph and a positive integer p.

The following Corollary 3 answers a question by J. Holenda who proved the case p=2.

Corollary 3. $\max rA^p = \max r(A_1,\dots,A_p)$ where $A,A_1,\dots,A_p$ are matrices of pattern G.

I thank J. Kratochvíl for a valuable discussion about the problem.

ORDINAL TYPES IN RAMSEY THEORY AND WELL-PARTIAL-ORDERING THEORY

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There is a big gap between the infinite Ramsey theorem

(+) $$\omega \to (\omega)^n_k$$

and its finite version

$$R(n;l_1,\dots,l_k) \to (l_1,\dots,l_k)^n_k$$

The finite Ramsey theorem is much finer. We fill in the gap by defining "Ramsey numbers' $R(n;\gamma_1,\ldots,\gamma_k)$" for arbitrary ordinals $\gamma_1,\ldots,\gamma_k$. These generalized Ramsey numbers are again ordinals, and their estimates provide a quantitative strengthening of the infinite Ramsey theorem.

There are different possible ways of measuring how sparse homogeneous sets are. For example, we may use the arithmetical density (this approach was studied by V. Rödl). Also our Ramsey numbers may be regarded as such measures, in fact, as somehow canohical ones: those of "finite character". In this sense, we are able to define Ramsey numbers in a much broader context than that of the classical Ramsey theorem mentioned above. We use a general axiomatical approach.

The axioms themselves imply some estimates and other facts. For example, there are "non-uniform" and "uniform" Ramsey theorems and there is a broad class of cases when they coincide.

To obtain sharper results, however, we have to consider more concrete situations. Those include beside the classical Ramsey theorem also the Erdös -Szekerés theorem on monotone sequences, the Canonical (selective) Ramsey theorem and the well-partial-ordering (WPO) theory. In the last case, the Ramsey numbers generalize the types of WPO sets, a concept already studied in great detail.

More concretely, in the classical case we obtain upper and lower bounds analogous to (or slightly weaker than) those known in the finite case. This includes also the stepping-up lemma. Upper bounds for the "Canonical Ramsey numbers" are also obtained. The point of considering the Erdös-Szekerés theorem separately is that we have exact values or sharp estimates of the Ramsey numbers by certain product formulas. We also give a survey of the WPO theory from our point of view. Some known theorems with new and simpler proofs are included.

Our Ramsey numbers are also closely connected with independence results in finite combinatorics. It has been already observed in the case of the types of WPO sets. To give a different example, the existence of "$R(n;\omega,\ldots,\omega)$" implies the Paris-Harrington modification of Ramsey theorem. As one might expect from unprovability of this theorem in PA, we have

$$\lim_{n\to\omega} R(n;\omega,\ldots,\omega)=\varepsilon_0.$$

Finally, let us remark that our theory is different from the ordinal Ramsey theorems (like e.g. $\omega^2\longrightarrow(\omega^2,n)$). From the philosophical point of view, our results are directed in the more "intensive" way. Their "extensive" power, for instance, in the classical case, does not exceed (+).